"ON THE BEHAVIOR OF THE FIRST POLAR AT A SINGULAR POINT"

by

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ABSTRACT

It is well known from an example of B. Segre ([4] p. 29-30) (*) that the first generic polar does not necessarily have the behaviour that was assumed to have by the classical geometers since Max Noether. Recently E. Casas [1] studied exhaustively the case of a singular point, origin of a single branch with only one characteristic exponent. The purpose of the present work is to present the principal results of E. Casas in a simpler way.

INTRODUCTION

We begin by recalling, in §1, Pick's theorem on the area of a simple polygon whose vertexs are points of Z^2. From that we obtain that the strip determined by an edge and the parallel through the opposite vertex of a triangle of area 1/2 does not contain any point of $\mathbb{Z}^2$ in its interior. After specifying, in §2, the form of the equation of the curve $f(x,y) = 0$ and the set K of points of $\mathbb{Z}^2$ corresponding to the terms of its first member, we pass (§3) to the generic polar and the corresponding K_1 set, which coincides with the one of $\partial f/\partial y$. As a corollary we deduce that K_1 is obtained from K by deleting its points on the OX-axis and applying a translation of vector (0, 1). After those preliminaries we determine recursively the Newton's polygon of the singular point of the first polar. We begin by its first segment (§4) and express (§5) (in the general case) the slope and the number of points of $\mathbb{Z}^2$ it contains by the development of the characteristic exponent of the branch curve in continued fraction. In §6 the inductive

* Square parenthesis refer to the bibliograpy at the end. For a simpler example than Segre's, see [3].

step is examined vhile §7 deals with the exceptional case which had been left aside and we obtain the same conclusions as in §5. Finally the theorem of §8 is obtained from Newton's polygon because the coefficients of the equation of the polar are generic.

§ 1. PICK'S THEOREM

One of the main tools we are going to use is Pick's theorem ([2] pag. 209).

Theorem. The surface of a *simple polygon*(1) whose vertexs are points in the lattice $\mathbb{Z}^2$ of points with integer coordinates in the plane $\mathbb{R}^2$ is given by the formula

$$\frac{1}{2}b + c - 1$$

where b stands for the number of points of $\mathbb{Z}^2$ on the edges of the polygon and c is the number of points of the lattice in the interior of the polygon.

If follows from this theorem that the triangles with vertexs on the lattice with no interior points of $\mathbb{Z}^2$ and no points of the lattice on its edges (except for the vertexs) are those whose area is $1/2$. The parallelogram obtained from one of those triangles by taking its symetric with respect to the middle point of one of its edges, generates by traslation a strip of $\mathbb{R}^2$ with no points of $\mathbb{Z}^2$ in its interior. This also can be stated as follows:

Proposition: Let ABC be a triangle whose vertexs are points of $\mathbb{Z}^2$ and whose surface is $1/2$, then the strip determined by the linies BC and the parallel to this one through A does not contains any point of $\mathbb{Z}^2$ in its interior.

§ 2. NEWTON'S POLYGON AND EQUATION OF THE CURVE

Let γ be an algebroid plane curve, generic among those who have a single branch through the origin 0 and with a single characteristic exponent $\frac{m}{n}$, witch $(m, n) = 1$, $m > n > 1$.

By means of an analytic transformation we may assume, if necessary, that the Puiseux development of γ begins with the term $x^{\frac{m}{n}}$.

(1) By simple polygon we mean one whose edges intersect just in its vertexs.

Then the Newton's polygon of the curve has just one segment with vertexs $A(m,0)$, $B(0,n)$ which are the only points of $\mathbb{Z}^2$ on this segment. The points (i, j) of $\mathbb{Z}^2$ that belong to the semiplane determined by the line AB and not containing 0 are those with $ni + mj > mn$.

Therefore the implicit equation of γ may be taken to be of the form

$$f(x,y) = y^n \quad x^m + \sum_{\substack{ni + mj > mn \\ i,j \geqslant 0}} a_{ij}\, x^i y^j = 0 \tag{1}$$

where a_{ij} are supposed to be generic because γ also is.

Let K be the set of points $(i,j) \in \mathbb{Z}^2$ whose coefficient of $x^i y^j$ in (1) is different from zero. It follows that K contains, apart from $A(m,0)$ and $B(0,n)$ the points of non negative integer coordinates which satisfy $ni + mj > mn$.

§3. Some Subsets of the Lattice $\mathbb{Z}^2$ Related with the Polar

Let us call K_1 the set of points of $\mathbb{Z}^2$ which is obtained from the first polar.

$$\lambda \frac{\partial f}{\partial x} + \mu \frac{\partial f}{\partial y} = 0 \tag{2}$$

(where λ, μ are arbitrary constants) in the same way as K was obtained from the curve. That is to say K_1 is the set of points (i,j) of $\mathbb{Z}^2$ whose corresponding term $x^i y^j$ appear effectively in (2).

Similary we shall call K_x, K_y the subsets of $\mathbb{Z}^2$ obtained from the polars $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$ respectively.

With those notations we may state the following.

Proposition: $K_1 = K_y$.

In the first place it is clear that $K_1 = K_x \cup K_y$. Therefore, it sufficies to show that $K_x \subset K_y$.

Let (x,y) be a point of K_x. It comes from a term $a_{ij}x^iy^j$ in (1) with $i > 0$. It follows that

$$(x,y) = (i\text{-}1,j),\ \text{has } i\text{-}1 \geqslant 0 \text{ and } ni + mj > mn.$$

As, by assumption, $m > n$ we have

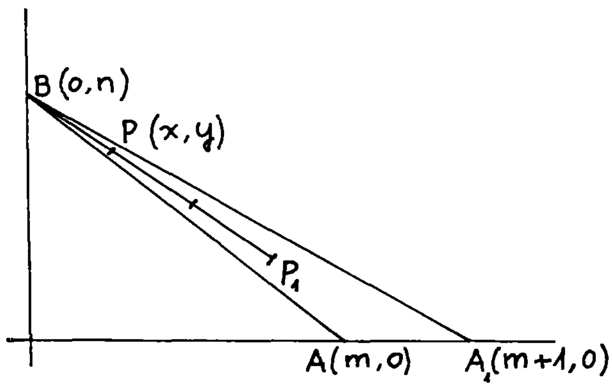
$$n(i-1) + m(j+1) > ni + mj > mn$$

So, it also appears in (1) the term $a_{i-1,\,j+1}x^{i-1}y^{j+1}$.

Corollary. The set K_1 is obtained from K by suppressing those points on the OX-axis and applying to the rest a translation of vector $(0, -1)$.

§4. Determination of the First Segment of the Newton's Polygon of the Polar

We are trying to find the segments of the Newton's polygon of K_1. We are just interesed in the slope and the number of points of $\mathbb{Z}^2$ they contain, so we can substitute K_1 by its translated by the vector $(0, -1)$, that is to say by K minus the points of the OX-axis (according to the corollary in §3). From now on, we shall use this set instead of K_1, but we shall continue to designate it by K_1



Because of the hypothesis $n > 1$ we have $B \epsilon K_1$ and B is a vertex of the first segment BP_1 of the polygon.

Let us call $P(x.y)$ the point of $\mathbb{Z}^2$ on this edge nearer to B. There is no other point of K_1 under the line BP. So, the area of the triangle ABP is 1/2.

Reciprocally, if the area ABP is 1/2, $P \epsilon K_1$ and $y < n$, then P is the point we were looking for. Indeed, in the strip determined by the line BP and its paralell from A there are no points of K_1 because the same is true for $\mathbb{Z}^2$. Thus this property determines P because, granted the existence, the unicity is obvious.

Let us prove that P belongs to the triangle BAA_1, A_1 being the point $(m+1, 0)$.

In fact, the surface of the triangle BAA_1 is $n/2 \geqslant 1$. So $\frac{1}{2} b + c \quad 1 \geqslant 1$, or what is the same $b + 2c \geqslant 4$. Therefore we have either $c > 0$ and the interior of the triangle contains same point of $\mathbb{Z}^2$, or $b > 3$ and then there is same other point on the edbe BA_1 apart from the vertexs. As, in any case, we are dealing with a finite number of points, this proves the existence of P belonging to the triangle BAA_1.

He have seen that $P(x,y)$ is the only point which satisfies

$$\text{area (ABP)} = \frac{1}{2} \text{ ; and } y < n.$$

These conditions can be expressed in the form:

α)

$$1 = \begin{vmatrix} 0 & n & 1 \\ m & 0 & 1 \\ x & y & 1 \end{vmatrix} = nx + my \cdot mn$$

β)

$$y < n$$

Condition α) implies that (x,y) is contained in the semiplane determined by the edge AB which does not contain 0.

Summing up.

Proposition: The conditions α) β) above have a unique solution $P(x,y) \in \mathbb{Z}^2$. Moreover P is the point of Z^2 nearer to B on the first segment of Nexton's polygon of the polar.

§5. Determination of the Slope and the Number of Points of $\mathbb{Z}^2$ on the First Segment (General Case)

Condition α) of §4 may be written as follows

$$\frac{x}{n-y} - \frac{m}{n} = \frac{1}{n(n-y)} \tag{3}$$

In particular, it follows that (being n-y positive after β))

$$(n-y, n) = 1, \qquad (n-y, x) = 1.$$

Let us express m/n as a continued fraction of odd order.

$$\frac{m}{n} = h + \frac{1}{h_1 +}\ \frac{1}{h_2 +}\ \cdots\ \frac{1}{h_{2k}}\ .$$

Them m/n and its last but one convergent satisfy

$$\frac{p_{2k-1}}{q_{2k-1}} - \frac{m}{n} = \frac{1}{q_{2k-1}n}\ ;$$

hence, comparing with (3) and keeping in mind that $q_{2k-1} < n$ and the unicity of the point (x,y), already proved, we obtain

$$x = p_{2k-1}\ , \qquad n - y = q_{2k-1}$$

which allows us to express the slope of BP_1 as

$$\frac{n-y}{x} = -\frac{q_{2k-1}}{p_{2k-1}}$$

Let $a + 1$ be the number of points of $\mathbb{Z}^2$ belonging to the segment BP_1. We have

$$a = \left[\frac{n}{n-y}\right] = \left[\frac{q_{2k}}{q_{2k-1}}\right] = h_{2k}$$

except in the case $n = \dot{\overline{n\text{-}y}}$. We shall consider this case afterward, and condense the results in the following.

Proposition: The slope of the first segment of Newton's polygon of the polar is (q_{2k-1}/p_{2k-1}) and the number of points of $\mathbb{Z}^2$ on this edge is $h_{2k} + 1$.

§6. Determination of the Next Edge in Newton's Polygon

The results of the preceeding paragraph allow us to find the coordinates of the point $P_1(x_1, y_1)$ from those of $B(0, n)$, $P(x,y)$ and the number of points $a + 1$ of Z^2 on BP_1.

$$x_1 = 0 + ax = h_{2k} \quad p_{2k-1}$$

$$y_1 = n + a(y\text{-}n) = q_{2k} \quad h_{2k}q_{2k-1} = q_{2k-2}.$$

In case $q_{2k-2} = 1$, P_1 would be the last vertex of the polygon and we would be through. Then $2k \quad 2 = 0$, $p_{2k-2} = p_0 = \left[\frac{m}{n}\right] = h$.

So, let supose $q_{2k-2} > 1$. To find the next segment it suffices to translate the OY-axis by the vector $(h_{2k}p_{2k-1}, 0)$ and then to apply the method of the preceeding paragraph replacing B by P_1. Taking into account that

$$x_1 \quad h_{2k}p_{2k-1} = 0 \;, \qquad y_1 = q_{2k-2} \;,$$

$$m - h_{2k}p_{2k-1} = p_{2k} \quad h_{2k}p_{2k-1} = p_{2k-2},$$

the coordinates of P_1 and A are P_1 $(0, q_{2k-2})$, $A(p_{2k-2}, 0)$. Therefore proposition in §5 is valid in this case by only diminishing k by one.

This process can obviously be iterated until we reach the last segment.

§7. Analysis of the Special Case

Let us consider now the case excluded in §5, $n - n{\cdot}y$. In this case the line BP cuts the OX axis at the point of $\mathbb{Z}^2$ A_1 $(m + 1, 0)$ because P belongs to the triangle BAA_1 but is not on BA. We have

$$\frac{n}{n - y} = \frac{m + 1}{x}$$

which may be written as

$$nx + my - mn = n - y$$

which compared with α) in §4 implies $n - y = 1$. By substituing in (3) we obtain the following expression of m/n as a continued fraction of the third order (ood):

$$\frac{m}{n} = x \quad \frac{1}{n} = x \quad 1 + \cfrac{1}{1 + \cfrac{1}{n-1}} \qquad ;$$

where

$$h_2 = n\text{-}1 \ , \quad \frac{p_1}{q_1} = \frac{x}{1} \quad .$$

On the other hand, because of $n\text{-}y=1$, BP_1 is the last segment of the polygon. It has n points of $\mathbb{Z}^2$ and slope $- \frac{n\text{-}y}{x} = \frac{1}{x}$. Therefore this case is also contained in proposition in §5.

§8. Composition of the Branches of the Polar

According to the proposition in §5, the first segment of Newton's polygon of the polar BP_1, has slope $\cdot\, q_{2k-1}/p_{2k-1}$ and there are on it $h_{2k} + 1$ points of $\mathbb{Z}^2$, each one corresponding to a coefficient of the equation of the first polar.

We assumed that coefficients a_{ij} in $f(x,y)$ were generic. In consequence, to the segment BP_1 there correspond h_{2k} branches of the first polar only one characteristic exponent equal to p_{2k-1}/q_{2k-1}. We also proved that treating the following segments we obtain the same results by only disminishing k one by one, so long as $k > 1$. And when $k = 1$ we obtain the last segment of the polygon.

This may be condensated in the following:

Theorem: Let us give an albebroid plane curve generic among those who have just one branch through the origin O, and with only one characteristic exponent m/n, $(m,n) = 1$, $m > n > 1$.

Let

$$\frac{m}{n} = h + \frac{1}{h_1 +}\ \frac{1}{h_2 +} \cdots \frac{1}{h_{2k}}$$

be the developpement of m/n as a continued fraction of odd order.

Then the singularity of its first polar in O, has for each value or $r = 1, \ldots, k$, h_{2r} branches with only one characteristic exponent p_{2r-1}/q_{2r-1}.

As $\left[\frac{m}{n}\right] = \left[\frac{p_{2r-1}}{q_{2r-1}}\right] = h$, $r = 1, \ldots, k$ always holds,

all branches go through the h n-uple points of γ infinitely near the origin and also through the next one of smaller multiplicity.

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