A PROJECTIVE DESCRIPTION OF WEIGHTED INDUCTIVE LIMITS OF SPACES OF VECTOR VALUED CONTINUOUS FUNCTIONS

by

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SUMMARY

If ϑ is a regularly decreasing sequence of strictly positive continuous weights on a locally compact Hausdorff space X and E is a complete (gDF) locally convex space, it is proven that $\vartheta_0 C(X, E)$ and $\vartheta C(X, E)$ coincide algebraically and topologically with $CV_0(X, E)$ and $CV(X, E)$ respectively.

The general problem of a projective description of weighted inductive limits of spaces of continuous functions has been treated by Bierstedt, Meise and Summers in (2) and (3) and by the first two authors in (4). We will use the notations of (2). Our terminology for locally convex spaces is standard and can be found in (9) and (10).

To fix some notation we will let X denote an arbitrary completely regular Hausdorff space. In general V will be a Nachbin family on X, and its elements will be written $v \in V$. $\vartheta = (V_n)$ stands for a sequence of Nachbin families on X such that $V_n \geqslant V_{n+1}$. We denote by V the maximal system associated to ϑ. If $V_n = \{ \lambda v_n : \lambda > 0 \}$ for a strictly positive weight v_n, we write $\vartheta = (v_n)$ and suppose that the sequence is decreasing. In what follows E is a separated locally convex space (l.c.s.) and cs(E) in the system of all its continuous seminorms.

In (2) 0.5. the so-called *"basic problem"* is stated as follows: Given a sequence ϑ on X and a l.c.s. E, determine when

$$\text{(i)}\ \vartheta_0 C(X, E) = C\bar{V}_0(X, E) \quad \text{and}$$
$$\text{(ii)}\ \vartheta C(X, E) = C\bar{V}(X, E)$$

hold algebraically and topologically.

Problem (i) is completely solved in (2) if $\vartheta = (v_n)$ is a decreasing sequence of continuous weights on a locally compact space. It is proven that, if E is a Banach

space, (i) holds if and only if the sequence ϑ is regularly decreasing, i.e.,
(RD) Given $n \in N$ there exists $m \geqslant n$ so that, for every $\epsilon > 0$ and every $k \geqslant m$ it is possible to find $\partial(k, \epsilon) > 0$ with $v_k(x) \geqslant \partial(k, \epsilon)\, v_n(x)$ whenever $v_m(x) \geqslant \epsilon\, v_n(x)$. The case of a strongly boundedly retractive inductive limit of Banach spaces E is also treated there.

The article (4) deals with problem (ii). Under some conditions which we specify in section II and which are implied by (RD) if the weights v_n of ϑ are continuous, it is shown that $\vartheta C(X) = C\bar{V}(X)$ holds algebraically and topologically.

In the first section of this article we prove that (i) holds algebraically and topologically if $\vartheta = (v_n)$ is a regularly decreasing sequence on a locally compact space X, and E is a complete (gDF)-space (see(9) Ch 12). To do this we utilize the "Desintegration Theorem", a device discovered by Defant and Govaerts (5).

A l.c.s. E is said to satisfy the countable neighbourhood property (c.n.p.) if given any sequence $(p_n) \subset cs(E)$ there are $\lambda_n > 0$ and $p \in cs(E)$ such that $p_n \leqslant \lambda_n\, p$, $n \in N$. Every (gDF)-space satisfies the c.n.p. In section II we prove that the algebraic equality (ii) holds and that both spaces have the same bounded sets if E satisfies the c.n.p. If moreover $\vartheta = (v_n)$ is a regularly decreasing sequence of continuous weights then (ii) holds also topologically.

Two examples are provided: In the first one we find a Fréchet space E and a regularly decreasing sequence ϑ on X such that $\vartheta_0 C(X, E)$ is not a topological subspace of $CV_0(X, E)$, and in the second we show that $\vartheta C(X, E)$ can be properly contained in $C\bar{V}(X, E)$ for a Fréchet space E, even if ϑ is, again, regularly decreasing.

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I. *On the topological isomorphism* $\vartheta_0 C(X, E) \cong CV_0(X, E)$

We assume in this section that X is *locally compact*. In this case $C_c(X, E)$ is densely contained in $CV_0(X, E)$ for any l.c.s. E and any Nachbin family V.

Our first Lemma is a slight modification of (2) Lemma 1.1.

Lemma 1. *If ϑ is a sequence of Nachbin families on X and E is a l.c.s. satisfysing the c.n.p., then $\vartheta_0 C(X, E)$ and $C\bar{V}_0(X, E)$ induce the same topology on $C_c(X, E)$.*

Proof: Take W a neighbourhood of the origin in $\vartheta_o C(X, E)$. We can find $v_n \in V_n$ and $p_n \in cs(E)$, $n \in N$, such that

$$\Gamma(\bigcup_{n=1}^{\infty} B_n) \subset W \quad , \quad \text{where}$$

$$B_n = \left\{ f \in C(V_n)_o\,(X, E) : \sup_{x \in X} v_n(x)\, p_n(f(x)) \leq 1 \right\}$$

Since E satisfies the c.n.p., we find $\lambda_n > 0$, $n \in N$, and $p \in cs(E)$ with $p_n \leq \lambda_n\, p$, $n \in N$.

Clearly $\bar{v} = \inf\ \{ \lambda_n\, 2^n\, v_n \ : \ n \in N \}$ belongs to V.

Proceeding now as in (2) Lemma 1.1. we prove that

$$\left\{ f \in C_c(X, E) : \sup_{x \in X} \bar{v}(x)\, p(f(x)) < 1 \right\}$$

is included in $W \cap C_c(X, E)$. Since the injection $\vartheta_o C(X, E) \subset CV_o(X, E)$ is continuous, the conclusion follows.

Q.E.D.

Applying (2) Lemma 1.2. we obtain

Theorem 2. *If* $\vartheta = (V_n)$ *is a sequence of Nachbin families on* X *and* E *a l.c.s. satisfying the c.n.p., then* $\vartheta_o C(X, E)$ *is a topological subspace of* $C\bar{V}_o(X, E)$, *and consequently of* $\vartheta C(X, E)$.

Our next theorem will be applied later but it is interesting by itself. It has been also observed by Defant and Hollstein (personal communication). The definition of ϵ-space can be seen in (7).

Theorem 3. *If* F *is a* (DF)*-space (a* (gDF)*-space) and* E *is a normed ϵ-space, then their injective tensor product* $E \otimes_\epsilon F$ *is a* (DF)*-space (a* (gDF)*-space).*

Proof: We supose that F is a (gDF)-space. It is proven in (8) that $c_o(F)$ is then a (gDF)-space (and a (DF)-space if F is a (DF)-space). Every bounded subset of $c_o(F)$ is clearly contained in the closure in $c_o(F)$ of a bounded subset of $c_o \otimes_\epsilon F$.

Therefore this last space is also (gDF). By Defant and Govaerts' Desintegration Theorem (see (5)) there is a surjective topological homomorphism

$$\hat{\phi} : (E \otimes_r l^1) \otimes_\pi (c_0 \otimes_\epsilon F) \longrightarrow E \hat{\otimes}_\epsilon F,$$

where $E \otimes_r l^1$ denote the tensor product of E and l^1 endowed whith the topology the topology of Chevet and Saphar (see (5)), and it is clearly a normed space. Then the projective tensor product $(E \otimes_r l^1) \otimes_\pi (c_0 \otimes_\epsilon F)$ is a (gDF)-space, by (9) 15.6.2. Since every quotient of a (gDF)-space is a (gDF)-space, the conclusion follows.

Corollary 4. *Let* v *be a strictly positive continuous weight on* X. *If* E *is a* (DF)*-space (a* (gDF)*-space), then* $C(v)_0(X, E)$ *is a* (DF)*-space (a* (gDF)*-space).*

Proof: By (7) Proposition 2.3. $C(v)_0(X)$ is a normed ϵ-space. Suppose that E is a (gDF)-space. Applying Theorem 3, $C(v)_0(X) \otimes_\epsilon E$ is a (gDF)-space. The conclusions follows considering the diagram

$$C(v)_0(X) \otimes_\epsilon E \longrightarrow C(v)_0(X, E) \longrightarrow C(v)_0(X, \hat{E}) \cong C(v)_0(X) \hat{\otimes}_\epsilon \hat{E}.$$

The last isomorphisms being a consequence of (1) Theorem 13.

Q.E.D.

Theorem 5. *If* ϑ *is a regularly decreasing sequence of continuous weights on* X *and* E *is a complete* (gDF)*-space, then* $\vartheta_0 C(X, E)$ *is strongly boundedly retractive and therefore complete. Moreover* $\vartheta_0 C(X, E)$ *coincides with* $C\bar{V}_0(X, E)$ *algebraically and topologically, and* $CV_0(X, E)$ *is a complete* (gDF)*-space, which is* (DF) *if* E *is a* (DF)*-space.*

Proof: Given $n \in N$, if we fix $m \in N$ as in the regularly decreasing condition, we have that the spaces $C(v_k)_0(X, E)$, $\vartheta_0 C(X, E)$, $\vartheta C(X, E)$ and $CV_0(X, E)$ induce the same topology on the bounded subsets of $C(v_n)_0(X, E)$ for every $k \geqslant m$. Since $C(v_n)_0(X, E)$ is a complete (gDF)-space for each $n \in N$, it is not difficult to see that this is enough to guarantee that $\vartheta_0 C(X, E)$ is strongly boundedly retractive (see (2) 0.3.). and therefore complete. Now applying Theorem 2,

$\vartheta_0 C(X, E)$ is a dense topological subspace of the complete space $C\bar{V}_0(X, E)$, and thus, they must coincide algebraically and topologically.

Q.E.D.

Remark 6. Under the assumtions of Theorem 5 we have that $\vartheta_0 C(X, E) = \vartheta_0 C(X) \hat{\otimes}_\epsilon E$. This is a consequence of the former Theorem and (1) Theorem 13:

$$\vartheta_0 C(X, E) \cong C\bar{V}_0(X, E) \cong C\bar{V}_0(X) \hat{\otimes}_\epsilon E = \vartheta_0 C(X) \hat{\otimes}_\epsilon E.$$

Example 7. Theorems 2 and 5 are not true in general if we assume that E is a Fréchet space. Consider X the set of natural numbers N endowed with the discrete topology and v_k, $k \in N$, the weights $v_k(n) = n^{-k}$; $n \in N$. In this case $C\bar{V}_0(X)$ is the co-echelon space of Köthe s'_b, the strong dual of the Fréchet space s of rapidly decreasing sequences; and $C(v_k)_0(X, E)$, for any l.c.s. E, is the space

$$\left\{ (x_n) \in E^N \ : \ \lim_{n\to\infty} \ n^{-k} \, p(x_n) \ = 0 \text{ for each } p \in cs(E) \right\}$$

Thus for every Fréchet space E, $\vartheta_0 C(X, E)$ is an (LF)-space, and therefore barrelled. By (6) Ch II, § 4, nº 3, there is a Fréchet space E_0 such that $s'_b \hat{\otimes}_\pi E_0 \cong s'_b \hat{\otimes}_\epsilon E_0 \cong C\bar{V}_0(X) \hat{\otimes}_\epsilon E_0 \cong C\bar{V}_0(X, E_0)$ is not barrelled. Consequently, $\vartheta_0 C(X, E_0)$, which is a dense subspace of $C\bar{V}_0(X, E_0)$, can not be a topological subspace of it. Observe that in this case $\vartheta_0 C(X, E) = \vartheta C(X, E)$ and $C\bar{V}_0(X, E) = C\bar{V}(X, E)$ for every l.c.s. E.

II. *On the topological isomorphism* $\vartheta C(X, E) \cong C\bar{V}(X, E)$.

In this section $\vartheta = (v_n)$ denotes a decreasing sequence of strictly positive weights on a completely regular Hausdorff space X and E is a l.c.s. with the countable neighbourhood property; unless the contrary is specifically stated.

Theorem 8. *If* ϑ *is a decreasing sequence on* X *and* E *satisfies the c.n.p., then*
(a) $\vartheta C(X, E) = C\bar{V}(X, E)$ *algebraically,*
(b) *for any bounded subset* B *of* $C\bar{V}(X, E)$ *there is* $n \in N$ *such that* B *is a bounded subset of* $Cv_n(X, E)$, *and*
(c) $\vartheta C(X, E)$ *is a regular inductive limit.*

Proof: Consider B a bounded subset of CV(X, E) (to prove (a) it is enough to take $B = \{ f \}$ with $f \in CV(X, E)$). Suppose that $\{ v_n(x) f(x) : x \in X, f \in B \}$ is not bounded in E for each $n \in N$. Therefore there exist $p_n \in cs(E)$, $n \in N$, such that

$$\left\{ v_n(x)\, p_n(f(x)) \quad : \quad x \in X\,,\ \ f \in B \right\}$$

is not a bounded subset of IR. Since E satisfies the c.n.p. there exist $p \in cs(E)$ and $\lambda_n > 0$, $n \in N$, with $p_n \leqslant \lambda_n\, p$, $n \in N$.

Proceeding by recurrence we can determine a sequence (x_n) in X with $x_n \neq x_m$, $n \neq m$; and a sequence (f_n) in B such that

$$v_n(x_n)\, p_n(f_n(x_n)) > n\, \lambda_n \quad \text{for each } n \in N.$$

We define $\bar{\bar{v}} : V \longrightarrow R$ by putting $\bar{\bar{v}}(x) = 0$ if $x \neq x_n$, and $\bar{\bar{v}}(x_n) = v_n(x_n)$, $n \in N$. If we fix $n \in N$ and take $k \geqslant n$, we have that

$$\frac{\bar{\bar{v}}(x_k)}{v_n(x_k)} \leqslant \frac{\bar{v}(x_k)}{v_k(x_k)} = 1 = \frac{\ddot{v}(x_n)}{v_n(x_n)}$$

and therefore

$$\sup_{x \in X} \frac{\bar{\bar{v}}(x)}{v_n(x)} = \sup_{k \subset N} \frac{\bar{\bar{v}}(x_k)}{v_n(x_k)} = \max \left\{ \frac{\bar{\bar{v}}(x_j)}{v_n(x_j)} \ : \ j=1,..,n \right\} =: \alpha_n$$

Taking $\bar{v}(x) := \inf\limits_{n \subset N} \left\{ \alpha_n . v_n(x) \right\}$, one has that $\ddot{v} \in \bar{V}$ and $\bar{v} \leqslant v$.

Since B is a bounded subset of $C\dot{V}(X, E)$, there exists $M \geqslant 0$ such that $v(x)\, p(f(x) \leqslant M$ for each $x \in X$ and $f \subset B$. But

$$\ddot{v}\,(x_n)\, p(f_n(x_n)) \geqslant \lambda_n^{-1}\ \ddot{v}(x_n)\, p_n(f_n(x_n)) \geqslant \lambda_n^{-1}\ v_n(x_n)\, p_n(f_n(x_n)) > n,$$

which is a contradiction.
Part (c) follows easily from (b).

Q.E.D.

Corollary 9. *If ϑ is a regularly decreasing sequence on X and E satisfies the c.n.p., then $\vartheta C(X, E)$ is a strongly boundedly retractive inductive limit; and hence complete if $v_n \in \vartheta$ are continuous, E is complete and X is a K_{IR} - space."*

Our previous results extend some propositions of (2)5.

Example 10. Theorem 8 is in general not true for Fréchet spaces E. Consider again $X = N$ endowed with the discrete topology and the weights v_k, $k \in N$, $v_k(n) = n^{-k}$, $n \in N$. Clearly if E is a l.c.s. we have that

$$C(v_k)(X, E) = \left\{ (x_n) \subset E^N \ : \ \sup_{n \in N} n^{-k} p(x_n) < +\infty \text{ for each } p \in cs(E) \right\}$$

Take now $E = s$. A fundamental system of seminorms of E is given by $p_r((\gamma_n)) = \sup \{ n^r |\gamma_n| : n \in N \}$, for each $(\gamma_n) \in E$ and $r \in N$. We define $f: N \to E$, by putting $f(n) = e_n$, the canonical unit vector in s. We claim that $f \in C\bar{V}(X, E) \setminus \vartheta C(X, E)$. Given $v \in \bar{V}$ there exist $\alpha_k > 0$, $k \in N$, with $v(n) \leqslant n^{-k} \alpha_k$, for each $n \in N$. Since $p_r(e_n) = n^r$, $n \in N$, one has that $\sup_{n \in N} v(n)\, p_r(e_n) = \sup_{n \in N} v(n)\, n^r \leqslant \alpha_r$, $r \in N$, and thus $f \in C\bar{V}(X, E)$. Now suppose the existence of $k \in N$ such that $\sup_{n \in N} v_k(n)\, p_r(e_n) < +\infty$ for each $r \in N$. Since $v_k(n)\, p_r(e_n) = n^{r-k}$ for each $r \in N$, we reach a contradiction.

Our next aim is to show that, under some conditions introduced in (4), the topological isomorphism $C\bar{V}(X, E) = \vartheta C(X, E)$ holds if E satisfies the c.n.p.. We follow methods developed in (4).

At this point we need some definitions. For any l.c.s. E and corresponding to a Nachbin family V on X and to an increasing sequence $J = (X_m)$ of subsets of X we associate the space

$$CV_o(X, J, E) = \left\{ f \in CV(X, E) \ : \ \lim_{m \to \infty} \ \sup_{x \in X \setminus X_m} v(x)\, p(f(x)) = 0 \right.$$
$$\left. \text{for each } v \in V \text{ and } p \in cs(E) \right\},$$

endowed with the topology induced by $CV(X, E)$.

If $V = \{ \lambda v \ : \ \lambda > 0 \}$ for a single strictly positive weight v on X, we will write $C(v)_o(X, J, E)$ instead of $CV_o(X, J, E)$.

For a decreasing sequence $\vartheta = (v_n)$ of strictly positive weights on X we put

$$\vartheta_o C(X, J, E) = \operatorname*{ind}_{n \to} C(v_n)_o(X, J, E)$$

$$C_c\vartheta(X, J, E) = \left\{ f \in \vartheta C(X, E) \ : \ f|_{X \setminus X_m} \equiv 0 \text{ for some } m \in N \right\}$$

Clearly $C_c\vartheta(X, J, E)$ is included in $\vartheta_oC(X, J, E)$ and it is dense in $\vartheta_oC(X, J, E)$ if

(*) X is normal and for each $m \in N$ there is $k_m \geqslant m$ with $X_m \subset \mathring{X}_{k_m}$, or

(**) all the weights v_n are continuous. (See (4) 6.2.)

The sequence ϑ is said to satisfy

(N, J) if for each $m \in N$ there is $n_m \geqslant m$ such that

$$\inf_{x \in X_m} \frac{v_k(x)}{v_{n_m}(x)} > 0 \text{ for all } k \geqslant n_m \text{; and}$$

(S, J) if for each $n \in N$ there exists $n' > n$ with

$$\lim_{m \to \infty} \sup_{x \in X \setminus X_m} \frac{v_{n'}(x)}{v_n(x)} = 0$$

If $\vartheta = (v_n)$ is a regularly decreasing sequence of continuous weights on X, then there is an increasing sequence $J = (X_m)$ of subsets of X such that ϑ satisfies (N, J) and (S, J).

Remark 11. *If ϑ is a sequence on X satifying (S, J) for a certain increasing sequence $J = (X_m)$ of subsets of X, then for every l.c.s. E the spaces $\vartheta C(X, E)$ and $\vartheta_oC(X, J, E)$ coincide algebraically and topologically.* Indeed, it is enough to prove that given $n \in N$, if we select n' as in condition (S, J), the space $C(v_n)(X, E)$ is included in $(C(v_{n'})(X, J, E)$. To see this, take $f \in C(v_n)(X, E)$. Given $p \in cs(E)$ there is $M > 0$ with sup $v_n(x)\, p(f(x)) : x \in X$ $< M$. For each $\epsilon > 0$ there exists $m_o \in N$ such that if $m \geqslant m_o$, then

$$\sup_{x \in X \setminus X_m} \frac{v_{n'}(x)}{v_n(x)} < \frac{\epsilon}{M}$$

Therefore if $m \geqslant m_o$, and $x \in X \setminus X_m$

$$v_{n'}(x)\, p(f(x)) = \frac{v_{n'}(x)}{v_n(x)}\, v_n(x)\, p(f(x)) < \epsilon$$

from where if follows that $\lim_{m \to \infty} \sup_{x \in X \setminus X_m} v_{n'}(x)\, p(f(x)) = 0$

Lemma 12. *Let ϑ be a decreasing sequence on* X *satisfying condition* (N, J) *for a certain increasing sequence* $J = (X_m)$ *of subsets of* X. *If conditions* (*) *or* (**) *are satisfied and* E *is a l.c.s. with the c.n.p., then* $\vartheta_0 C(X, J, E)$ *is a topological subspace of* $CV_0(X, J, E)$; *and consequently of* $\vartheta C(X, E)$.

Proff: As a consequence of our previous remarks and (2) Lemma 1.2., since $\vartheta_0 C(X, J, E)$ is continuously embedded in $CV_0(X, J, E)$, it is enought to prove that these spaces induce the same topology on $C_c\vartheta(X, J, E)$. To do this we may assume, replacing (v_m) by (v_{n_m}), that $n_m = m$ in condition (N, J).

Put $$\partial_n := \inf_{x \in X_n} \frac{v_{n+1}(x)}{v_n(x)} > 0 , \quad n \in N$$

Fix $$U = \Gamma(\bigcup_{n=1}^{\infty} B_n)$$

a neighbourhood of the origin in $\vartheta_0 C(X, J, E)$ with

$$B_n = \left\{ f \in C(v_n)_0(X, J, E) \ : \ \sup_{x \in X} v_n(x)\, p_n(f(x)) \leqslant \epsilon_n \right\} , \quad p_n \in cs(E).$$

We can determine $p \in cs(E)$ and $\lambda_n > 0$, $n \in N$, such that $p_n \leqslant \lambda_n p$, $n \in N$. We choose inductively a sequence of positive numbers (α_n) with

$$\alpha_n \geqslant 2^{n+1} \lambda_n \qquad \alpha_{n+1} \geqslant \frac{\alpha_n \epsilon_{n+1}}{\partial_n \epsilon_n} , \quad n \in N$$

Then defining $v = \bigwedge_{n \in N} \frac{\alpha_n}{\epsilon_n} v_n \in \bar{V}$, we conclude (see the proofs of 6.5 and 6.6. in (4) that

$$W := \left\{ f \in C_c\, \vartheta(X, J, E) \ : \ \sup_{x \in X} v(x)\, p(f(x)) < 1 \right\}$$

is contained in U.

Q.E.D.

Theorem 13. *Let ϑ be a decreasing sequence on* X *satisfying* (N, J) *and* (S, J) *for a certain sequence* $J = (X_m)$ *of subsets of* X. *If conditions* (*) *or* (**) *are satisfied, then* $\vartheta(C(X, E) \cong C\bar{V}(X, E)$ *holds topologically foe every l.c.s.* E *with the c.n.p.*

Proof: In general $\vartheta_0 C(X, J, E) \subset CV_0(X, J, E) \subset C\ddot{V}(X, E)$. By remark 11 $\vartheta C(X, E) \cong \vartheta_0 C(X, J, E)$ and by Theorem 8 $\vartheta C(X, E) = C\ddot{V}(X, E)$. Thus these four spaces coincide algebraically. It is enough to apply Lemma 12 to reach the conclusion.

Q.E.D.

Corollary 14. *If ϑ a regularly decreasing sequence of strictly positive continuous weights on* X, *then* $\vartheta C(X, E) \cong C\bar{V}(X, E)$ *holds topologically for every l.c.s.* E *satisfying the c.n.p.*

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