ON VON NEUMANN REGULAR RINGS, X

by

ROGER YUE CHI MING

INTRODUCTION.

Rings having von Neumann regular classical left quotient rings Q are considered in the first section of this note. Conditions for Q to be strongly regular and semi-simple Artinian are given. Next, we introduce a generalization of injective modules, noted NCI modules, which are proved to be an effective intermediate class of modules between injective and continuous modules. Among the results proved are the following: (1) If A has a von Neumann regular classical left quotient ring, then a finitely generated left ideal of A is essential if, and only if, it contains a non-zero-divisor (this is motivated by a well-known result of A. W. Goldie [4, Theorem 3.34]); (2) If A has a classical left quotient ring Q, the following are equivalent: (a) Q is strongly regular; (b) For every $a \in A$, $a \in a^2 Q$; (c) A is reduced and Q is right p-injective; (3) A is left self-injective regular iff A is left non-singular such that every finitely generated non-singular left A-module is NCI; (4) A NCI left A-module is continuous; (5) A is left and right self-injective strongly regular iff A is a reduced left NCI ring; (6) If A is an ELT left NCI ring which is a direct sum of an ideal and a left ideal C such that ${}_AC$ is non-singular, then C is a left and right self-injective regular left and right V-ring. Certain well-known results of B. Osofsky ([9, Lemma 3] and [10, Theorem]) on right self-injective regular rings hold for right non-singular right NCI rings.

Throughout, A represents an associative ring with identity and A-modules are unitary. Recall that (1) A has a classical left quotient ring Q if Q is a ring containing A such that (a) every non-zero-divisor of A is invertible in Q and (b) every element of Q is of the form $q = b^{-1} a$, $a \in A$, b being a non-zero-divisor of A; (2) A satisfies the left Ore condition if, for any a, $b \in A$, where b is a non-zero-divisor, there exist d, $c \in A$, c non-zero-divisor, such that $ca = db$. It is well-known that A has a classical left quotient ring iff A satisfies the left Ore condition (cf. [2, P. 390]). A well-known theorem [4, Theorem 3.35] asserts that A has a semi-simple Artinian classical left quotient ring iff A is a semi-prime left Goldie ring.

As usual, an ideal of A means a two-sided ideal and A is called left duo if every left ideal is an ideal. A is called reduced if it contains no non-zero nilpotent element. A is fully idempotent (resp. fully left idempotent) iff every ideal (resp. left ideal) of A is idempotent. If N is a left submodule of

$$_AM,\ Cl_M(N)=\Big\{y \in M/Ly \subseteq N$$

for some essential left ideal L of A$\Big\}$ is the usual closure of N in M. M is called singular (rep. non-singular) iff $Z_A(M) = Cl_M(o) = M$ (resp. $Z_A(M) = o$). Z, J will denote respectively the left singular ideal $Z_A(A)$ and the Jacobson radical of A.

§. VON NEUMANN REGULAR CLASSICAL QUOTIENT RINGS.

Lemma 1.1. *If a has a classical left quotient ring Q, then any left ideal of A containing a non-zero-divisor is an essential left ideal.*

Proof. Let L be a left ideal of A containing a non-zero-divisor c. Since c is invertible in Q, Qc = Q. For any $o \neq q \in Q$, $q = pc$, $p \in Q$, and if $p = b^{-1}a$, $a \in A$, b non-zero-divisor of A, then $o \neq bq = b\,(pc) = ac \in L$ which proves that $_AL$ is essential in $_AQ$ and therefore $_AL$ is essential in $_AA$.

Our next result is analogous to [4, Theorem 3.34]. Note that if A has a von Neumann regular classical left quotient ring, then A needs neither be regular nor satisfy the maximum condition on left annihilators.

Theorem 1.2. *Suppose that A has a von Neumann regular classcial left quotient ring Q. Then a finitely generated left ideal F of A is essential iff it contains a non-zero-divisor.*

Proof. If F contains a non-zero-divisor, then F is essential by Lemma 1.1. Conversely, suppose that $_AF$ is essential in $_AA$. If

$$F = \sum_{i=1}^{n} Ay_i, \text{ then } QF = \sum_{i=1}^{n} Qy_i \text{ and } {}_QQF$$

is essential in $_QQ$. Since Q is von Neumann regular, then $_QQF$ is generated by an idempotent which implies QF = Q. If

$$1 = \sum_{i=1}^{n} q_i y_i, \ q_i \in Q,$$

we can find a non-zero-divisor $c \in A$, $a_1, \ldots, a_n \in A$ such that

$$q_i = c^{-1} a_i \ (1 \leqslant i \leqslant n).$$

Then we obtain

$$c = \sum_{i=1}^{n} a_i y_i \in F.$$

A left A-module M is called continuous iff every left submodule which is isomorphic to a complement left submodule of M is a direct summand of ${}_AM$. Then A is a left continuous ring iff ${}_AA$ is continuous (cf. [12]).

Theorem 1.3. *If A has a classical left quotient ring Q, the following are then equivalent:*

(1) *Q is left continuous regular;*

(2) *Every complement left ideal of Q is finitely generated and every finitely generated essential left ideal of A contains a non-zero-divisor.*

Proof. (1) implies (2) by Theorem 1.2.

Assume (2). For any $q = b^{-1} a \in Q$, $a, b \in A$, there exists a complement left ideal K of Q such that $L = Qq \oplus K$ is essential in ${}_QQ$. By hypothesis, we can find a finitely generated left ideal F o A such that $QF = K$. If we set $E = Aa + F$, then $QE = Qq + K$. We now show that E is an essential left ideal of A. Suppose that there exists a non-zero left ideal I of A such that $E \cap I = o$. Then it is easily seen that $L \cap QI = o$, which contradicts the fact that L is essential in ${}_QQ$. Therefore, by hypothesis, E contains a non-zero-divisor which implies $Q = L$, whence Q is von Neumann regular and then every complement left ideal of Q is generated by an idempotent. Thus (2) implies (1).

Corollary 1.3 (a). *If A has a left self-injective classical left quotient ring Q, then Q is regular iff every finitely generated essential left ideal of A contains a non-zero-divisor.*

Remark 1. A is regular iff A has a regular classical left quotient ring such that every principal left ideal of A is a complement left ideal. (Apply Lemma 1.1.).

Lemma 1.4. *If A has a classical left quotient ring Q, the following conditions are equivalent:*

(1) *Q is von Neumann regular;*

(2) *For any $a \in A$, there exists $q \in Q$ such that $a = aqa$.* (The proof is direct.)

It is well-known that A is left non-singular iff A has a regular maximal left quotient ring E, where E is left self-injective and $_AE$ is the injective hull of $_AA$ (cf. for example [4]).

Remark 2. A is left self-injective regular iff A is left non-singular whose principal left ideals are complement left ideals such that the maximal left quotient ring of A is the classical left quotient ring of A.

We now prove an important result on classical quotient rings.

Proposition 1.5. *If A is a reduced ring having a classical left quotient ring Q, then Q is reduced.*

Proof. Let $q \in Q$ such that $q^2 = o$. If $q = b^{-1} a$, $a \in A$, b non-zero-divisor of A, then $ab^{-1} a = o$. Since A satisfies the left Ore condition, there exist $d \in A$, c a non-zero-divisor of A such that $ca = db$. Then $da = dbb^{-1} a = cab^{-1} a = o$ implies $ad = o$ (since A is reduced), whence $aca = adb = o$ and therefore $(ac)^2 = o$ which yields $ac = o$. Since c is a non-zero-divisor, $a = o$ which proves that $q = o$ and establishes the proposition.

Corollary 1.5 (a). *Let A be reduced such that the maximal left quotient ring Q of A is a classical left quotient ring of A. Then Q is left and right self-injective strongly regular.*

Corollary 1.5 (b). *Any left non-singular left duo ring has a reduced classical left quotient ring.*

Proof. Apply [15, Lemma 1] to Proposition 5.

Left CM-ring are studied in [19]. Since a semi-prime ring satisfying the maximum condition on left annihilators is left non-singular, then [19, Remark 2 (2)] implies.

Corollary 1.5 (c). *If A is a semi-prime left CM-ring satisfying the maximum condition on left annihilators, then either A is semi-simple Artinian or A has a classical left quotient ring which is a finite direct sum of division rings.*

It is well-known that A is von Neumann regular iff every left (right) A-module is flat. Also, A is regular iff A is a right p-injective right p.p. ring [6, Corollary 5].

Theorem 1.6. The following conditions are equivalent for a ring A having a classical left quotient ring Q:

(1) *Q is strongly regular;*

(2) *For any $a \in A$, there exists $g \in Q$ such that $a = a^2 g$;*

(3) *A is reduced and Q is a right p-injective ring.*

Proof. Apply [7, Theorem 1], [15, Theorem 1], Lemma 1.4 and Proposition 1.5.

Remark 3. If A is a fully idempotent (resp. (1) fully left idempotent (2) fully right idempotent) ring having a classical left quotient ring Q, then Q is fully idempotent (resp. (1) fully left idempotent (2) fully right idempotent).

We conclude this section with a few equivalent conditions for a ring to be semi-prime left Goldie.

Theorem 1.7. The following conditons are equivalent:

(1) *A is semi-prime left Goldie;*

(2) *A is of left finite Goldie dimension having a von Neumann regular classical left quotient ring;*

(3) *A has a classical left quotient ring Q such that for any essential left ideal L of A, QL = Q;*

(4) *A is left non-singular with a classical left quotient ring Q such that every divisible non-singular left A-module is injective.*

Proof. It is well-known that (1) implies (2) [4, Theorem 3.35].

Assume (2). If U is an essential left ideal of Q, then $L = U \cap A$ is essential in ${}_AA$ and since A is of left finite Goldie dimension, then L contains a finite number of elements

$$a_1, \dots, a_m \text{ such that } I = \sum_{i=1}^{m} Aa_i$$

is essential in ${}_AL$ and hence in ${}_AA$. By Theorem 1.2, I contains a non-zero-divisor c and since $c \in U$, then $Q = U$ which proves that Q is semi-simple Artinian. If E is an essential left ideal of A, E contains a non-zero-divisor [4, Theorem 3.34] which implies that $QE = Q$ and thus (2) implies (3).

Assume (3). For any essential left ideal L of A, since $QL = Q$, then

$$1 = \sum_{i=1}^{n} q_i l_i, q_i \in Q, l_i \in L.$$

Then the proof of Theorem 1.2 shows that L contains a non-zero-divisor. If U is an essential left ideal of Q, then $U \cap A$ is essential in ${}_AA$ and therefore contains a non-zero-divisor, whence $U = Q$ which proves that Q is semi-simple Artinian. Therefore (3) implies (4) by [5, P. 102 ex. 18].

Assume (4). Q is clearly a non-singular left A-module. If U is an essential left ideal of Q, c a non-zero-divisor of A, for any $u \in U$, $u = cc^{-1} u \in cU$ implies $U = cU$ which shows that ${}_AU$ is divisible, non-singular and therefore injective. Now ${}_AQ = {}_AU \oplus {}_AP$ and since ${}_AP$ is divisible (because ${}_AQ$ is divisible), for any $p \in P$, any $q \in Q$, $q = b^{-1} d$, $b, d \in A$, since $dp \in P = bP$, $dp = bv$ for some $v \in P$. We then have $qp = b^{-1} dp = v \in P$ which proves that P is a left ideal of Q and hence the essential left ideal U of Q is a direct summand of ${}_QQ$, yielding $Q = U$. This proves that Q is semi-simple Artinian and (4) implies (1).

Rings satisfying the left Ore condition whose singular left modules are injective need neither be regular nor satisfy the maximum condition on left annihilators.

Corollary 1.7 (a). *If A satisfies the left Ore condition such that all divisible singular and divisible non-singular left A-modules are injective, then A is left Noetherian left hereditary.*

Proof. Since every divisible singular left A-module is injective, then A is left hereditary [18, P. 192]. Since a left hereditary left Goldie ring is left Noetherian, the corollary then follows from Theorem 1.7 (4).

§. 2. INJECTIVE AND NCI MODULES.

We here introduce the following generalization of injectivity.

Definition. A left A-module M is called a NCI module if, for any left submodule P containing a non-zero complement left submodule of M and any left submodule N of M which is isomorphic to P, every left A-homomorphism of N into P extends to an endomorphism of ${}_AM$.

A is called a left NCI ring if ${}_AA$ is NCI. If ${}_AM$ is NCI and $M = N \oplus P$, then ${}_AN$ is NCI.

Remark 4. Any completely reducible left A-module is NCI. Consequently, if every NCI left A-module is injective, then by [3, Corollary 20.3E], A is a left Noetherian, left V-ring. If every simple left A-modulo is p-injective, then A is fully left idempotent [14]. Consequently, NCI modules need not be p-injective.

Remark 5. If A is a left NCI ring, then any non-zero-divisor is invertible. Therefore, every left or right A-module is divisible and A is its own classical left and right quotient ring.

It is well-known that A is left self-injective regular iff A is left non-singular such that every finitely generated non-singular left A-module is injective iff A is regular such that every finitely generated non-singular left A-module is projective (cf. [1, Theorem 2.1] and [20, Corollary 6]).

Theorem 2.1. The following conditions are equivalent:

(1) *A is left self-injective regular;*

(2) *For any finitale generated left A-module M, ${}_AM/Z(M)$ is projective NCI;*

(3) *A is left non-singular such that every finitely generated non-singular left A-module is NCI.*

Proof. (1) implies (2) by [20, Corollary 10].

Assume (2). Then ${}_AA/Z$ is projective which implies $Z = o$ [13, Lemma 3]. Therefore (2) implies (3).

Assume (3). Let M be a non-zero finitely generated non-singular left A-module, E the injective hull of ${}_AM$. Suppose that $M \neq E$ and if $y \in E$, $y \notin M$, set $F = M + Ay$ and $Q = M \oplus F$. Then ${}_AQ$ is finitely generated non-singular and the-the refore NCI by hypothesis. If u: $M \to F$, k: $F \to Q$ are the inclusion maps, $M' = (y, o)/y \in M$, j: $M \to M'$ the isomorphism $y \to (y, o)$, i: $M' \to Q$ the inclusion map, then there exists h: $Q \to Q$ such that $hku(y) = j(y)$ for all $y \in M$. If p: $Q \to M$ is the canonical projection, then $phku(y) = pj(y) = pij(y) = y$ for all $y \in M$, which shows that phk: $F \to M$ such that $(phk)u$ = identity map on M. This proves that ${}_AM$ is a direct summand of ${}_AF$, whence $M = F$ (because ${}_AM$ is essential in ${}_AF$), which contradicts $y \notin M$. Thus $M = E$ is injective and (3) implies (1).

Proposition 2.2. If M is a NCI left A-module, then ${}_AM$ is continuous.

Proof. Let N be a complement left submodule of M, Q a relative complement of N in M such that $L = Q \oplus N$ is an essential submodule of ${}_AM$ (and hence N is a relative complement of Q in M). If p: $L \to N$ is the canonical projection, the set of submodules P of M containing L such that p extends to a left A-homomorphism of P into N has a maximal member K by Zorn's Lemma. If $g: K \to N$ is the extension of p to K, $i: N \to K$ the inclusion map, since ${}_AM$ is NCI, then ig extends to an endomorphism h of ${}_AM$. If $h(M) \not\subseteq N$, then $(h(M) + N) \cap Q \neq o$ and for any $o \neq q \in (h(M) + N) \cap Q$, $q = h(m) + n$, $m \in M$, $n \in N$, K is strictly contained in $E = \{y \in M/h(y) \in L\}$ (because $m \in E$, $m \notin K$). Now if $s: E \to L$ is defined by $s(z) = h(z)$ for all $z \in E$, then $ps: E \to N$ extends p to E, which contradicts the maximality of K. This proves that h is a map of M into N. Now $\ker h \cap N = o$ and for any $m \in M$, $m = h(m) + (m - h(m)) \in N \oplus \ker h$ which proves that $M = N \oplus \ker h$.

Now let C be a left submodule of M which is isomorphic to N. If $u: C \to N$ is an isomorphism, by hypothesis, u extends to an endomorphism v of ${}_AM$. Since N is a direct summand of ${}_AM$, if $t: M \to N$ is the natural projection, then for any $m \in M$, $tv(m) \in N$ implies there exists $c \in C$ such that $u(c) = tv(m)$. But $u(c) = v(c)$ implies $tv(c) = tv(m)$, whence $m = c + k$, where $k \in \ker tv$. Since u is an isomorphism, $C \cap \ker tv = o$ which yields $M = C \oplus \ker tv$.

Applying [12, Lemma 4.1], we get.

Corollary 2.2 (a). *If A is a left NCI ring, then A/Z is von Neumann regular and Z = J.*

Corollary 2.2 (b). *If A is left non-singular, then any quotient module of a NCI left A-module contains its singular submodule as a direct summand.* (cf. [13]).

Applying [11, Proposition 1] to Proposition 2.2, we get.

Corollary 2.2 (c). *A is quasi-Frobeniusean iff A is a left Noetherian left NCI rinq whose minimal one-sided ideals are annhilators.*

Corrollary 2.2 (d). *A left non-sigular left NCI, right self-injective ring is left self-injective regular.* (Apply [4, p. 68 ex. 14]).

Corollary 2.2 (e). *A prime left NCI ring either has zero socle or is primitive left self-injective regular* (cf. [16, Proposition 2.5]).

The converse of Proposition 2.2 is not true (cf. Remark 7 below). However, a continuous uniform left A-module is NCI. (A left A-module is uniform iff every non-zero left submodule is essential).

A well-known result of C. FAITH - Y. UTUMI on ring endomorphisms of quasi-injective modules [4, Theorem 2.16] holds for NCI modules.

Theorem 2.3. *Let M be a NCI left A-module, $E = End_A(M)$, $J(E)$ = the Jacobson radical of E. Then $E/J(E)$ is von Neumann regular and $J(E) = f \in E/ker\ f$ is essential in ${}_AM$.*

Rings whose essential left ideals are idempotent need not be semi-prime.

Lemma 2.4. *Let A be a left NCI ring such that each essential left ideal is an idempotent two-sided ideal containing a non-zero complement left ideal of A. Then A is left self-injective.*

Proof. Let L be an essential left ideal of A, f: L → A a non-zero left A-homomorphism. For any $b \in L$, $b \in L^2$ implies

$$\left[b = \sum_{i=1}^{n} c_i d_i, \text{ where } c_i, d_i \in L, \text{ and } f(b) = \sum_{i=1}^{n} c_i f(d_i) \in L \right]$$

(since L is an ideal of A). This shows that f maps L into L and since A is left NCI, f extends to an endomorphism of ${}_AA$, proving that A is left self-injective.

Remark 6. If A satisfies the hypothesis of Lemma 2.4 and either the maximum or minimum condition on left annihilators, then A is quasi-Frobeniusean.

Theorem 2.5. *The following conditions are equivalent:*

(1) *A is left and right self-injective strongly regular;*

(2) *A is a semi-prime left duo left NCI ring;*

(3) *A is a reduced left NCI ring.*

Proof. Apply [15, Lemma 1], Corollary 2.2 (b) and Lemma 2.4.

Combining Proposition 1.5 with Theorem 2.5, we get.

Corollary 2.5 (a). ***If A is a reduced ring having a left NCI classical left quotient ring Q, then Q is left and right self-injective strongly regular.***

Remark 7. In view of [12, Remark 7.11], Theorem 2.5 shows that NCI modules are, in general, strictly between injective and continuous modules.

Lemma 2.6. Let A be a left NCI ring which contains a non-zero non-singular left ideal I. Then every element of I is von Neumann regular.

Proof. For any $o \neq b \in I$, let K be a non-zero complement left ideal of A such that $L = l(b) \oplus K$ is an essential left ideal. If f: $Kb \rightarrow K$ is the left A-homomorphism defined by $f(kb) = k$ for all $k \in K$, then f is an isomorphism and since A is left NCI, then f extends to an endomorphism h of ${}_AA$. Let h $(1) = d$. Then $k = kbd$ implies $kb = kbdb$ for all $k \in K$, whence $L \subset l(b - bdb)$. Therefore $b - bdb \in Z \cap I = Z(I) = o$, which yields $b = bdb$.

Recall that A is ELT iff every essential left ideal of A is an ideal.

***Proposition** 2.7. Let A be an ELT left NCI ring which is a direct sum of two left ideal B, C, where B is an ideal of A and ${}_AC$ is non-singular. Then C is a left and right self-injective regular left and right V-ring.*

Proof. We may assume that $B \neq o$ (otherwise, $C = A$ has the desired property by [19, Lemma 1.1], Lemmas 2.4 and 2.6). Since A is ELT, then so is the ring C. By Lemma 2.6, for any $c \in C$, $Cc = Ac = Ae = Ce$, where $e = e^2 \in C$. This proves that C is a von Neumann regular ring. Let L be an essential left ideal of C, f: $L \rightarrow C$ a non-zero left C-homomorphism. For any $b \in L$, $Cb = Cu$, where $u = u^2 \in C$. If ${}_CL = {}_CCb \oplus {}_CQ$, then ${}_AL = {}_AAb \oplus {}_AQ$ and if i: $C \rightarrow A$ is the inclusion map, g: ${}_AL \rightarrow {}_AC$ is the map defined by $g(y) = f(y)$ for all $y \in L$, then ig: ${}_AL \rightarrow {}_AA$ which extends to t: ${}_AB \oplus {}_AL \rightarrow {}_AA$ by $t(b + l) = ig(l)$ for all $b \in B$, $l \in L$. Now $B \oplus L$ is an essential left ideal of A containing a non-zero complement left ideal B of A and for any $d \in B \oplus L$, $d = v + w$, $v \in B$, $w \in L$, whence $w = wz$, $z = z^2 \in L$ which implies $t(d) = ig(w) = g(w) = wg(z) \in L$ (since C is ELT). If j: $L \rightarrow B \oplus L$ is the natural injection, then jt: $B \oplus L \rightarrow B \oplus L$ and jt extends to an endomorphism h of ${}_AA$ (since ${}_AA$ is NCI). If r is the restriction of h to C, then for any $y \in L$, $r(y) = h(y) - jt(y) = t(y) \in L \subseteq C$ and if p: $A \rightarrow C$ is the natural projection, then pr is a left A-homomorphism of C into C such that for any $y \in L$, $pr(y) = pt(y) = pig(y) = pg(y) = g(y) = f(y)$. This shows that f extends to a left C-homomorphism q: $C \rightarrow C$, where $q(c) = pr(c)$ for all $c \in C$, which proves that C is left self-injective. Then C is a right self-injective left and right V-ring by [19, Lemma 1.1].

Remark 8. If $A = S \oplus C$, where A is left NCI, S is the left socle of A, C is an ideal of A which is an ELT left non-singular ring, then A is a direct sum of a semi-simple Artinian ring and a left and right self-injective strongly regular ring.

If A is left non-singular, then (1) for any left A-module M and left submodule N, $Cl_M(N)$ is a complement left submodule of M [13] and (2) for any left ideal I of A, $Cl_A(I)$ is the unique maximal essential extension of I in ${}_AA$ and it contains every essential extension of I in ${}_AA$ (cf. [17]). If A is a left NCI ring, then for any direct summands I, P of ${}_AA$ such that $I \cap P = o$, $I \oplus P$ is also a direct summand of ${}_AA$. Then with slight modifications, the proofs of [9, Lemma 3] and [10, Theorem] yield.

Proposition 2.8. Let A be a left non-singular left NCI ring which contains an infinits set of non-zero orthogonal idempotents $\{e_i\}_{i \in I}$. *Then.*

(1) *For any subset U of I, there exists an idempotent* E_U *in A such that* $e_w E_U = e_w$ *for all* $w \in U$, $e_v E_U = E_U e_v = o$ *for all* $v \in I - U$. *Also* $E_I = E_U + E_{I-U}$;

(2) $A/(\sum_{i \in I} Ae_i + \ker h)$ *is not a NCI left A-module, where* $h: A \to \pi_{i \in I} Ae_i$ *is the map given by* $h(a) = < ae_i >$, $a \in A$.

Following [8], a left A-module M is called semi-simple if the intersection of the maximal left submodules of M is zero. Then A is a left V-ring iff every left A-module is semi-simple [8, Theorem 2.1].

Theorem 2.9. The following conditions are equivalent:

(1) *A is semi-simple Artinian;*

(2) *A is an ELT ring whose NCI left modules are injective;*

(3) *Every finitely generated left A-module is NCI;*

(4) *A is a semi-prime left NCI-ring satisfying the maximum condition on left annihilators;*

(5) *A is an ELT ring whose cyclic semi-simple left modules are flat and NCI;*

(6) *A is a left CM-ring whose cyclic semi-simple left modules are flat and NCI;*

(7) *A is a left non-singular ring whose cyclic left modules are NCI.*

Proof. Obviously, (1) implies (2), (4) and (6). (2) implies (3) by [19, Theorem 1.11] and Remark 4.

Assume (3). If M is a cyclic left A-module, E an essential extension of ${}_AM$, $y \in E$, $y \notin M$, $F = M + Ay$, then ${}_AM \oplus {}_AF$ is a finitely generated NCI left A-module. The proof of Theorem 2.1 then shows that we get $M = F$, a contradiction. Thus M has no proper essential extension which proves ${}_AM$ is injective and therefore (3) implies (7). (4) implies (5) by Lemma 2.6.

Assume (5). Since every cyclic semi-simple left A-module is flat, then J = o, which, implies that A is left self-injective regular by Lemmas 2.4 and 2.6. Now A is a (left and right) V-ring by [19, Lemma 1.1] which implies that every cyclic left A-module is semi-simple [8, Theorem 2.1] and hence NCI. Thus (5) implies (7).

Assume (6). Then by Lemma 2.6, A is either strongly regular or semi-simple Artinian. In any case, (6) implies (7).

Finally, since it is well-known that a von Neumann regular ring which contains no infinite set of non-zero orthogonal idempotents is semi-simple Artinian, then (7) implies (1) by Lemma 2.6 and Proposition 2.8.

REFERENCES

1. V. C. CATEFORIS. On regular self-injective rings, Pac. J. Math. 30 (1969), 39-45.

2. C. FAITH. Algebra I: Rings, modules and categories, Vol. 190, Springer Verlag, 1973.

3. C. FAITH. Algebra II: Ring Theory, Vol. 191, Springer-Verlag, 1976.

4. K. R. GOODEARL. Ring Theory: Non-singular rings and modules, Pure and Appl. Math. Ser. 33, Dekker (New York) (1976).

5. K. R. GOODEARL. Von Neumann regular rings, Monographs and Studies in Math. 4, Pitman, London (1979).

6. Y. HIRANO and H. TOMINAGA. Regular rings, V-rings and their generalizations, Hiroshima Math. J. 9 (1979), 137-149.

7. M. IKEDA and T. NAKAYAMA. On some characteristic properties of quasi-Frobenius and regular rings, Proc. Amer. Math. Soc. 5 (1954), 15-19.

8. G. O. MICHLER and O. E. VILLAMAYOR. On rings whose simple modules are injective. J. Algebra 25 (1973), 185-201.

9. B. L. OSOFSKY. Rings all of whose finitely generated modules are injective, Pac. J. Math. 14 (1965), 645-650.

10. B. L. OSOFSKY. Non-injective cyclic modules, Proc. Amer, Math. Soc. 19 (1968), 1383-1384.

11. H. H. STORRER. A note on quasi-Frobeniusean rings and epimorphisms, Canad. Math. Bull. 12 (1969), 288-292.

12. Y. UTUMI. On continuous rings and self-injective rings, Trans. Amer. Math. Soc. 118 (1965), 158-173.

13. R. YUE CHI MING. A note on singular ideals. Tôkohu Math. J. 21 (1969), 337-342.

14. R. YUE CHI MING. On simple p-injective modules, Math. Japonica 19 (1974), 173-176.

15. R. YUE CHI MING. On annihilator ideals, Math. J. Okayama Univ. 19 (1976), 51-53.

16. R. YUE CHI MING. On von Neumann regular rings, III, Monatshefte für Math. 86 (1978), 251-257.

17. R. YUE CHI MING. A remark on decomposable modules, Publ. Inst. Math. (Beograd) 25 (39) (1979), 101-104.

18. R. YUE CHI MING. On injective and p-injective modules, Riv. Mat. Univ. Parma (4) 7 (1981), 187-197.

19. R. YUE CHI MING. On regular rings and self-injective rings, Monatshefte für Math. 91 (1981), 153-166.

20. J. ZELMANOWITZ. Injective hulls of torsionfree modules, Canad. J. Math. 23 (1971), 1094-1101.

Université Paris VII
U. E. R. de Mathématique et Informatique
2, Place Jussieu,
75251 PARIS CEDEX 05
France