DESCRIPTION OF INVARIANT SUBSPACES OF $L^p(\mu)$ BY MULTIPLICATION OPERATORS

per

M. L. REZOLA

ABSTRACT

In this paper we give a description for the closed subspaces of $L^p(X,\mathcal{A},\mu)$, $1 \leq p < \infty$, which are invariant under multiplication by a selfconjugate family of essentially bounded functions. This work is a continuation of [3] and [4] and the results obtained form part of the author's doctoral dissertation [5].

1. Introduction and Notation

In what follows, $(X,\mathcal{A},\mu)$ will be a σ-finite measure space, $L^p(\mu)$, $1 \leq p < \infty$, the classical Banach space associated with the pair (X,μ) and $E_{\mathcal{F}}$ the *conditional expectation* operator (or the averaging projection with respect to $\mathcal{F}$, where $\mathcal{F}$ is a σ-finite subσ-algebra of $\mathcal{A}$.

S will always be a closed subspace of $L^p(\mu)$ and H a selfconjugate family of $L^\infty(\mu)$. We say that S is *H-invariant* when $\varphi S \subset S$ for every $\varphi \in H$. We denote by $\sigma(H)$ the smallest subσ-algebra of $\mathcal{A}$ making all the functions in H measurable and by S° the *polar* of S, i.e.,

$$S^\circ = \left\{ g \in L^{p'}(\mu) \,;\; \int_X fg\,d\mu = 0 \text{ for all } f \in S \right\},$$

$$\frac{1}{p} + \frac{1}{p'} = 1$$

Dedicado al Prof. L. Vigil con motivo de su 70 aniversario.

The H-invariant subspaces S of $L^p(\mu)$ are essentially determined by the σ-algebra $\sigma(H)$. More exactly, if H_1 and H_2 are two different families of $L^\infty(\mu)$ such that $\sigma(H_1) \subseteq \sigma(H_2)$, then, the H_1-invariant subspaces ant the H_2-invariant subspaces are the same if and only if the σ-algebras $\sigma(H_1)$ and $\sigma(H_2)$ are equivalent (i.e., they have the same μ-complection). This is a consequence of the following result.

1.1 ***Lemma.*** (see [3], [5])

If S is H-invariant, then the closure of S in $L^p(\mu)$ is $L^\infty(\sigma(H))$-invariant.

When $\sigma(H)$ is σ-finite, we have a description for the H-invariant subspaces of $L^p(\mu)$ by using the conditional expectation operator, $E_{\sigma(H)}$.

1.2 ***Theorem.***

S is H-invariant if and only if there exist a family $(g_i)_{i\in I}$ of $L^{p'}(\mu)$ such that $S = \bigcap_{i\in I} Sg_i$ where

$$Sg_i = \left\{ f \in L^p(\mu) \; ; \; E_{\sigma(H)}(fg_i) = 0 \quad \mu\text{-a.e.} \right\}$$

See [4] for the proof. The reader can also look at Theorem 3.2 below whose proof is quite similar.

1.3 ***Remarks.***

a) The last result contains Beurling's theorem concerning invariant subspaces of $L^2(T)$ by the bilateral shift. In fact, in this case, $H = \{e^{it}, e^{-it}\}$ and $\sigma(H)$ consists of all Borel subsets of T, so that $E_{\sigma(H)}$ is the identity operator and

$$S = \bigcap_{i\in I} Sg_i = \{ f \in L^2(T) \; ; \; f = 0 \text{ a.e. in } E \}$$

where E is the support of the family $\{g_i\}_{i\in I}$.

b) Theorem 1.2 is also true in $L^\infty(X, \mathcal{A}, \mu)$, if we consider the weak-* topology in $L^\infty(\mu)$ and the subspace S is supposed to be weak-* closed.

c) It is possible to extend theorem 1.2 to a more general situation. For example, if S is a closed subspace of $L^\rho_B(X, \mathcal{A}, \mu)$, (a Köthe function space, see [7]), where ρ is a saturated, absolutely continuous norm and B is a Banach space such that the dual space B^* verifies the Radon-Nikodym property. (Many of the important classical Banach function spaces are contained in this class for suitable ρ's).

2. Application to shift operators.

A natural question arises from the above theorem. How many functions of $L^{p'}(\mu)$ are necessary to obtain the subspace S? Here, this question is solved in a particular, non trivial, situation. When, $X = [0,1)$ and $\sigma(H)$ is the σ-algebra of $\frac{1}{n}$ periodic Borel subsets of $[0,1)$, i.e. $\sigma(H)$ is the σ-algebra

$$\mathcal{B}_n = \left\{ B \subset [0,1);\ B \text{ is a Borel set and } \frac{1}{n} \dot{+} B = B \right\}$$

where $\dot{+}$ stands for addition (mod. 1) in $[0,1)$. We shall need the following tecnichal lemmas.

2.1 ***Lemma.***

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space let $\mathcal{H}$ be a family of measurable functions. Then, there exists a unique (μ-a.e.) measurable subset A of X, such that:

i) $f(x) = 0$ a.e. $x \notin A$, $\forall f \in \mathcal{H}$

ii) there is a countable family of functions $(f_j)_j \subset \mathcal{H}$ with $\sum_j |f_j(x)| > 0$ a.e. $x \in A$

(A will be the *support of* $\mathcal{H}$, supp $\mathcal{H}$, and $\mathcal{H}^{-1}(0)$ the set $X \setminus A$)

Proof.

We consider the family

$\mathcal{C} = \{ (A_j)_{j\in J} : A_j \in \mathcal{A}$ pairwise disjoint with $\mu(A_j) > 0$ and such that for each $j \in J$, there is $f_j \in \mathcal{H}$ with $f_j(x) \neq 0$ a.e. $x \in A_j \}$

(each J must be countable because (X,μ) is σ-finite).

$\mathcal{C} \neq \phi$ and $\mathcal{C}$ is an inductive set under the partial order:

$$(A_j)_{j\in J_1} \propto (B_j)_{j\in J_2} \text{ if } (A_j)_{j\in J_1} \text{ is a subfamily of } (B_j)_{j\in J_2}$$

By Zorn's lemma, we have a maximal element of $\mathcal{C}$, $(A_j)_{j\in J}$. Let f_j , $j \in J$, be the functions corresponding to A_j and $A = \bigcup_{j\in J} A_j$. Then, if $f \in \mathcal{H}$ and $B = \{ x : f(x) > 0 \} \cap A^c$, necessarily $\mu(B) = 0$. $\neq$

2.2 ***Lemma.***

Let H be a selfconjugate family of essentially bounded functions on [0,1) such that $\sigma(H) = \mathcal{B}_n$. If S is an invariant subspace of $L^p([0,1),m)$, (m denotes

Lebesgue measure), $0 < p < \infty$, then there exists $s_1, s_2, \dots, s_n$ belonging to S such that

$$g(x) = \sum_{j=1}^{n} \alpha_j(x) s_j(x) \qquad x \in [0,1)$$

for each $g \in S$ and suitable $\mathcal{B}_n$-measurable functions, $\alpha_1, \alpha_2, \dots, \alpha_n$. (The functions α_j, $j = 1,2,\dots,n$ depend on g , and, in general, they are not in $L^\infty(\mathcal{B}_n)$).

Proof.

$n = 1$: By applying the above lemma to supp S , we obtain a countable pairwise disjoint family $(A_j)_{j\in J}$ and their corresponding functions of S , $(f_j)_{j\in J}$. These functions can be modified so that $\|f_j\|^p < 2^{-j}$, $j \in J$. The function $s(x) = \sum_{j=1}^{\infty} f_j(x)\chi_{A_j}(x)$, belongs to S and verifies the result.

Next, we will give only the proof for $n = 2$, because for $n \geqslant 3$ the ideas are the same although the notation is more complicated.

$n = 2$: We take the following families of functions on $[0,1/2)$

$$F_1 = \{ g(x), g(x+1/2) : g \in S \}$$

$$F_2 = \{ \det \begin{pmatrix} g(x) & g(x+1/2) \\ h(x) & h(x+1/2) \end{pmatrix} ; g,h \in S \}$$

and we denote by N_1 and N_2, the sets $F_1^{-1}(0)$ and $F_2^{-1}(0)$. If A is a Borel subset of $[0,1/2)$ we define $\widetilde{A}$ as the set $\widetilde{A} = A \cup (A \dotplus \frac{1}{2})$.

The result holds in $(N_1)^\sim$, taking $s_1(x) = 0 = s_2(x)$. As $N_2 \setminus N_1 = \bigcup_{j\in J} A_j$, by lemma 2.1 (we suppose that the corresponding functions f_j verify $\|f_j\|^p < 2^{-j}$) the functions

$$s_1(x) = \sum_{j=1}^{\infty} f_j \chi_{\widetilde{A}_j}(x) \qquad \text{a.e. } x \in [0,1)$$
$$s_2(x) = 0$$

belong to S. Moreover, if $h \in S$,

$$\det \begin{pmatrix} h(x) & h(x+1/2) \\ f_j(x) & f_j(x+1/2) \end{pmatrix} = 0 \qquad \text{a.e.} \quad x \in A_j$$

then, there exist $c_{h,j}(x)$ such that

$$\begin{aligned} h(x) &= c_{h,j}(x)\, f_j(x) \\ h(x+1/2) &= c_{hj}(x)\, f_j(x+1/2) \end{aligned} \qquad \text{a.e.} \quad x \in \Lambda_j$$

We can extend c_{hj} to $[0,1)$, by defining them on $\widetilde{\Lambda}_j$ as $c_{hj}(x+1/2) = c_{hj}(x)$. Thus, c_{hj} is $\mathcal{B}_2$-measurable and calling $\alpha_1(x) = \sum_{j=1}^{\infty} c_{hj}(x)\chi_{\widetilde{\Lambda}_j}(x)$, we conclude that

$$h(x) = \alpha_1(x)s_1(x) + \alpha_2(x)s_2(x) \qquad \text{a.e.} \quad x \in \widetilde{N}_2 \setminus \widetilde{N}_1$$

for all α_2, $\mathcal{B}_2$-measurable.

Likewise, $[0,1) \setminus \widetilde{N}_2 = ([0,1/2) \setminus N_2)^{\sim}$ and $[0,1/2) \setminus N_2$ is contained in supp F_2, then there are two families of functions $(f_j)_{j\in J}$, $(g_j)_{j\in J}$ in S such that

$$\det \begin{pmatrix} f_j(x) & f_j(x+1/2) \\ g_j(x) & g_j(x+1/2) \end{pmatrix} \neq 0 \qquad \text{a.e.} \quad x \in \Lambda_j$$

The functions

$$s_1(x) = \sum_{j\in J}^{\infty} f_j(x)\chi_{\widetilde{\Lambda}_j}(x)$$

$$s_2(x) = \sum_{j=1}^{\infty} s_j(x)\chi_{\widetilde{\Lambda}_j}(x)$$

belong to S and beasides, if $h \in S$, there exist a_{hj}, b_{hj} verifiying

$$\begin{aligned} h(x) &= a_{hj}(x)f_j(x) + b_{hj}(x)g_j(x) \\ h(x+1/2) &= a_{hj}(x)f_j(x+1/2) + b_{hj}(x)g_j(x+1/2) \end{aligned} \qquad \text{a.e.} \quad x \in \Lambda_j$$

We define a_{hj} and b_{hj} on $\widetilde{\Lambda}_j$, by an $\frac{1}{2}$-periodic extension and denote

$$\alpha_1(x) = \sum_{j\in J} a_{hj}(x)\, \chi_{\widetilde{\Lambda}_j}(x)$$

$$\alpha_2(x) = \sum_{j\in J} b_{hj}(x)\, \chi_{\widetilde{\Lambda}_j}(x)$$

which are $\mathcal{B}_2$-measurable. Thus

$$h(x) = \alpha_1(x)s_1(x) + \alpha_2(x)s_2(x) \qquad \text{a.e.} \quad x \in [0,1) \setminus \widetilde{N}_2 .$$

If n = 3, we should consider the families of functions on [0,1/3)

$$F_1 = \left\{ g(x)\,,\ g(x+1/3)\,,\ g(x+2/3) : g \in S \right\}$$

$$F_2 = \left\{ \det \begin{pmatrix} g(x) & g(x+1/3) \\ h(x) & h(x+1/3) \end{pmatrix},\ \det \begin{pmatrix} g(x) & g(x+2/3) \\ h(x) & h(x+2/3) \end{pmatrix},\ \det \begin{pmatrix} g(x+1/3) & g(x+2/3) \\ h(x+1/3) & h(x+2/3) \end{pmatrix} \quad g,h \in S \right\}$$

$$F_3 = \left\{ \det \begin{matrix} g(x) & g(x+1/3) & g(x+2/3) \\ f(x) & f(x+1/3) & f(x+2/3) \\ h(x) & h(x+1/3) & h(x+(2/3) \end{matrix} \quad ; \quad f,g,h \in S \right\}$$

and we should continue in the same way as above. =

If $p = +\infty$ the last result is true. It is necessary to take the functions f_j with $|f_j| \leqslant 1$, so that $\sum_{j=1}^{\infty} f_j \chi_{T_j} \in S$.

2.3 ***Theorem.***

Let p and n be fixed, with $1 \leqslant p < \infty$ and $n \in N$. If S is an H-invariant subspace of $L^p([0,1))$, $H = \{e^{2\pi i n t}, e^{-2\pi i n t}\}$, then, there exist $h_1, h_2, \ldots, h_n \in L^{p'}([0,1))$ such that

$$S = \left\{ f \in L^p;\ \sum_{j=1}^{n} f(t + \frac{j-1}{n})\, h_k(t + \frac{j-1}{n}) = 0 \quad a.e.\ t \right.$$

$$\left. k = 1,2,\ldots,n \right\}.$$

Proof.

Since H is selfconjugate, then S and S° are $L^\infty(\sigma(H))$-invariant by using lemma 1.1. Now, $\sigma(H) = \mathcal{B}_n$ and by applying lemma 2.2 to S°, we obtain $h_1, h_2, \ldots, h_n \in S^\circ$ such that $g(x) = \sum_{j=1}^{n} \alpha_j(x) h_j(x)$ for each $g \in S^\circ$ and $(\alpha_j)_{j=1}^{n}$ $\mathcal{B}_n$-measurable functions. Hence, by theorem 1.1, we have:

$f \in S$ if and only if $E_{\sigma(H)}(fg) = \sum_{j=1}^{n} \alpha_j E_{\sigma(H)}(fh_j) = 0$ for all $g \in S^\circ$ or equivalently. $E_{\sigma(H)}(fh_k) = 0$ $k = 1,2,\ldots,n$.

2.4 ***Remarks.***

a) If $p = +\infty$, the theorem holds by considering the weak-* topology in $L^\infty(\mu)$ and a weak-* closed subspace S.

b) If $p = 2$, we have obtained an implicit description for the invariant subspaces by the bilateral shift of finite multiciplicity, in the Hilbert space $L^2([0,1))$, because these subspaces can be seen as the invariant subspaces by the multiplication operators associated to functions $e^{+2\pi int}$ (n is the multiplicity of the shift). If we identify the spaces $L^2([0,1))$ and $L^2_{\mathbb{C}^n}([0,1/n))$ by the map $f \to F = (f_j)_{j=1}^n$ such that $f_j(t) = f(t + \frac{j-1}{n})$, we have obtained in theorem 2.3 that

$$(*) \quad S = \left\{ f \in L^2([0,1)) \;;\; F(t) \,.\, H_k(t) = 0 \quad \text{a.e. } t. \quad k = 1,2,\ldots,n \right\}$$

By denoting as M(t) the subspace of $\mathbb{C}^n$, which is orthogonal to the family $\{ H_1(t), H_2(t), \ldots, H_n(t) \}$ (with $0 \leqslant \dim M(t) \leqslant n$), then (*) is equivalent to the customary explicit description for these subspaces which appears for example in [2].

2.5 ***Theorem.***

Let T^2 be the 2-dimensional torus, and let $H = \{ f_1, f_2 \}$ with $f_1(x,y) = e^{2\pi ix}$ and $f_2(x,y) = e^{-2\pi ix}$. If S is an H-invariant subspace of $L^p(T^2)$, $1 \leqslant p < \infty$, then there exist a countable family $(g_j)_{j\in N}$ of $L^{p'}(T^2)$ such that

$$S = \left\{ f \in L^p(T^2) \;;\; \int_T f(x,y)\, g_j(x,y)dy = 0 \qquad a.e. \quad x,\; j \in N \right\}$$

Proof.

Since H is selfconjugate, theorem 1.2 can be applied, and it suffices to observe that $\sigma(H) = \{ B \times T \;;\; B \text{ Borel subset of } T \}$, and therefore:

$$E_{\sigma(H)} f(x,y) = \int_T f(x,y)dy$$

If $p = 2$, we have got an implicit description for the invariant subspaces by the bilateral shift of countable multiplicity in the Hilbert space $L^2(T^2)$, because the multiplication operator by $e^{2\pi ix}$ transforms $e_{n,m} \to e_{n+1,m}$ ($(e_{n,m})_{n,m \in N} = (e^{2\pi(nx+my)})_{n\in Z}$ is an orthonormal basis of $L^2(T^2)$). Moreover, we can identify $L^2(T^2)$ with $L^2_{L^2(T)}(T)$ by the map : $f \to F$ such that $F(x)(y) = f(x,y)$ and then we have

$$S = \left\{ f \in L^2(T^2) \;;\; < F(x) \,.\, G_j(x) > = 0 \qquad \text{a.e. } \; x,\; j \in N \right\}$$

or equivalently $S = \{ f \in L^2(T^2) ; F(x) \in M(x)$ a.e. $x \}$, where $M(x)$ denotes the orthogonal complement of the family $\{ G_k(x) \}_{k \in N}$, for each $x \in T$ (This characteritation can also be seen in [2]).

3. The case of non σ-finite $\mathfrak{F}$

In this section we shall obtain two results similar to theorem 1.2, when $\sigma(H)$ is not supposed to be σ-finite.

Let G be a σ-compact locally compact abelain group, $d\varphi$ a Haar measure on G, m another measure on G given by $dm(\varphi) = \Delta(\varphi)d\varphi$, where the weight Δ is a multiplicative measurable homomorfisme from G to R^+, and $(X_0, \mathcal{A}_0, \mu_0)$ a σ-finite measure space. Let $(X, \mathcal{A}, \mu)$ be the product space $(X_0 \times G, \mathcal{A}_0 \otimes \mathcal{B}(G), \mu_0 \otimes m)$, ($\mathcal{B}(G)$ is the σ-algebra of Borel subsets of G) , and let $\mathfrak{F}$ be, the subσ-algebra of $\mathcal{A}$,

$$\mathfrak{F} = \{ \pi^{-1}(A_0) : A_0 \in \mathcal{A} = A_0 \times G ; A_0 \in \mathcal{A}_0 \}$$

(π is the canonical proyection from X to X_0). Under these hypothesis, G can be considered as a bijective transformation group on X, which carries $\mathcal{A}$-measurable sets to $\mathcal{A}$-measurable sets and dilates the measure according to Δ , i.e.:

$$\mu(\psi(A)) = \Delta(\psi)\mu(A) \qquad , \qquad \psi \in G , A \in \mathcal{A} .$$

Moreover, the σ-algebra $\mathfrak{F}$ coincides with $\{ A \in \mathcal{A} ; \varphi(A) = A$ for all $A \in \mathcal{A} \}$ and an H-measurable function f on X is $\mathfrak{F}$-measurable if and only if $f(x,\varphi) = f(x,e)$ for all $\varphi \in G$, $x \in X_0$ (e is the unit element on G) .

The following lemma is an inmediate consecuence of Fubini's theorem.

3.1. ***Lemma.***

Let f be a function in $L^1(X)$. Then the function

$$\widetilde{f}(x) = \int_G f(x,\varphi) dm(\varphi)$$

exists μ_0 a.e. and it belong to $L^1(X_0,\mu_0)$. Furthermore,

$$\int_{X_0} \widetilde{f} \, d\mu_0 = \int_X f d\mu \quad \text{and} \quad \|\widetilde{f}\|_{L^1(\mu_0)} \leqslant \|f\|_{L^1(\mu)}$$

The function $\widetilde{f}$ admits a natural extension to X :

$$\widetilde{f}(x,\psi) = \Delta(\psi^{-1})\,\widetilde{f}(x,e)$$

where we identify x with (x,e) . In general $\widetilde{f}$ is not $\mathfrak{F}$-measurable

3.2 *Theorem.*

If S is a closed subspace of $L^p(X, \mathcal{A}, \mu)$ and H is a selfconjugate family in $L^\infty(\mu)$ with $\sigma(H) = \mathfrak{F}$, then: S is H-invariant if and only if there exists a family $(g_i)_{i\in I}$ in $L^{p'}(\mu)$ such that $S = \bigcap_{i\in I} \widetilde{Sg_i}$, where

$$\widetilde{Sg_i} = \{ f \in L^p(\mu)\,;\ (fg_i)^\sim = 0 \quad \mu_o \quad a.e. \}$$

Proof.

Assume first that S is a closed subspace of $L^p(\mu)$. For each $A_0 \in \mathcal{A}$, $g \in L^{p'}(\mu)$ and $f \in L^p(\mu)$

$$\int_{A_0} (fg)^\sim(x)d\mu_0(x) = \int_{X_0} (fg)^\sim(x)\chi_{A_0}(x)d\mu_0(x) =$$

$$= \int_{X_0} (fg\chi_{\pi^{-1}(A_0)})^\sim(x)d\mu_0(x) =$$

$$= \int_X (fg\chi_{\pi^{-1}(A_0)})(x,\varphi)d\mu(x,\varphi)$$

By lemma 1.1, the subspaces S and S° are $L^\infty(\mathfrak{F})$-invariant and thus, $f \in S$ implies $f \in \widetilde{Sg}$ for all $g \in S^\circ$. On the other hand, if $f \in \widetilde{Sg}$ for all $g \in S^\circ$ neccesarily $(fg)^\sim = 0$ μ_0 a.e. for all $g \in S^\circ$ and, by lemma 3.1, $\int_X fgd\mu = 0$ for all $g \in S^\circ$, which implies $f \in S$.

To prove the converse, it suffices to show that $\widetilde{Sg}$ is a closed and H-invariant subspace of $L^p(\mu)$ for every $g \in L^{p'}(\mu)$. But

$$(hfg)^\sim(x) = h(x,e)\,(fg)^\sim(x)$$

for all $f \in L^p(\mu)$, $g \in L^{p'}(\mu)$, $h \in H$, and then, $\widetilde{Sg}$ is H-invariant. Furthermore if $f_n \to f$ in $L^p(\mu)$, then $f_ng \to fg$ in $L^1(\mu)$ for all $g \in L^{p'}(\mu)$ and since the operator: $f \to \widetilde{f}$ is continuous from $L^1(\mu)$ to $L^1(\mu_0)$, it follows that $\widetilde{Sg}$ is closed. =

A comparison between theorem 1.2 and 3.2 shows that the operator: $f \to \widetilde{f}$ is a good substitute for the conditional expectation operator: $f \to E_{\mathfrak{F}}(f)$, which cannot be defined for the general kind of σ-algebras $\mathfrak{F}$ considered here.

When $\sigma(\Pi)$ is σ-finite, the subspace $\widetilde{Sg}$ of theorem 3.2 are the same as those appearing in theorem 1.2, i.e.:

$$\widetilde{Sg} = \{ f \in L^p(\mu) \; ; \; E_{\mathfrak{F}}^{\mu}(fg) = 0 \quad \mu \text{ a.e.} \} \; .$$

In fact:

$$\int_{A_0} (fg)^{\sim} d\mu_0 = \int_X fg\chi_{\pi^{-1}(A_0)} d\mu \quad \text{for all } A_0 \in \mathcal{A}_0 \, .$$

3.3 *Examples.*

We present several examples of σ-algebras $\mathfrak{F}$ and projections: $f \to \tilde{f}$ which fall under the scope of theorem 3.2. More examples are given in [5].

1°) Let $\mathfrak{F}$ be the σ-algebra of all Borel subsets of R^n, which are translation invariant with respect to a vector w and $\tilde{f}(x) = \sum_{n \in \mathbb{Z}} f(x+nw)$, $x \in \mathbb{R}^n$; then $\tilde{f}$ is $\mathfrak{F}$-measurable. Taking: $X_0 = \{ x \in \mathbb{R}^n : 0 \leq x \, . \, w < 1 \}$ and G as the group of translation by kw , $k \in \mathbb{Z}$, with their natural measures, theorem 3.2 can be applied in this context.

2°) Let $\mathfrak{F}$ be the σ-algebra of all Borel subsets of $\mathbb{R}^n$ which are radial and $\tilde{f}(x) = \int_{S_{n-1}} f(rx')d\sigma(x')$, $x \in \mathbb{R}^n$, $r = \| x \|$ ($d\sigma(x')$ denotes Lebesgue measure on $S_{n-1} = \{ x \in \mathbb{R}^n : \| x \| = 1 \}$), S_{n-1} which is $\mathfrak{F}$-measurable. In this case, if we take: $X_0 = [0, +\infty)$ with the measure $d\mu_0(r) = \omega_{n-1} \, r^{n-1} dr$ (ω_{n-1} is the total measure of S_{n-1}), and as G the quotient group $0(n)/K$ ($0(n)$ is the group of all orthogonal transformation on $\mathbb{R}^n$ and K its the normal subgroup which fixes a point x_0' of S_{n-1}) with normalized Haar measure, then theorem 3.2 can be applied again.

3°) Let $\mathfrak{F}$ be the σ-algebra of all dilatation-invariant Borel subsets of $\mathbb{R}^n$ and $\tilde{f}(x) = \int_0^{+\infty} f(rx)r^{n-1}dr$, $x \in \mathbb{R}^n$ $r = \| x \|$, which is not $\mathfrak{F}$-measurable. Now, if $X_0 = S_{n-1}$ with its measure and G is the group of homotecies on $\mathbb{R}^n$ (G can be identified with the group $(0,+\infty)$ with measure $dm(\varphi) = r^n \frac{dr}{r}$) again, we have a good situation for the application theorem 3.2.

The following situation is not included in the theorem 3.2 and we shall now give a theorem for it. Let X be a locally compact abelian group, G a closed subgroup of X and $\tilde{X}$ the quotient group X/G equipped with their respective Haar measures m and m_G. We can take a suitable Haar measure $\tilde{m}$ on $\tilde{X}$ such that Weil's formula holds: If $f \in L^1(X)$ and we define

$$\tilde{f}(\tilde{x}) = \int_G f(\varphi(x))dm_G(\varphi)$$

then $\tilde{f} \in L^1(\tilde{X})$ and $\int_{\tilde{X}} \tilde{f} d\tilde{m} = \int_X f dm$.

Now, $\tilde{\mathcal{F}}$ is the subσ-algebra of Borel subsets of X, then $\tilde{\mathcal{F}} = \{ \pi^{-1}(B) ; B \in \mathcal{B}(\tilde{X}) \}$, where π denotes the canonical projection from X onto $\tilde{X}$. This situation is very similar the one described above, but, in general, it is not clear that the σ-algebras $\mathcal{B}(G) \otimes \mathcal{B}(\tilde{G})$ and $\mathcal{B}(X)$ can be identified.

3.4 ***Theorem.***

Let S be a closed subspace of $L^p(X, \mathcal{B}(X), m)$ and H a selfconjugate family of $L^\infty(\tilde{\mathcal{F}})$ with $\sigma(H) = \tilde{\mathcal{F}}$. Then, S is H-invariant if and only if there exists a family $(g_i)_{i \in I} \subset L^{p'}(m)$ such that $S = \bigcap_{i \in I} \tilde{S}g_i$, where

$$\tilde{S}g_i = \{ f \in L^p(m) : (fg)^\sim = 0 \quad \tilde{m}\text{-}a.e. \}$$

The proof is exactly as in Theorem 3.2, Weil's identity being now the substitute of Lemma 3.1. Finally, we observe that the remarks 1.3 b) and c), remains true (with a suitable formulation) in this context.

4. *An application to Operator Theory in Hilbert spaces.*

Let $\mathcal{H}$ be a separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H})$ the family of bounded linear operators on $\mathcal{H}$, by $\sigma(T)$ the spectrum of T $(T \in \mathcal{L}(\mathcal{H}))$ and by C(T), the algebra of operators commuting with T, $C(T) = \{ Q \in \mathcal{L}(\mathcal{H}) ; QT = TQ \}$.

If T is a normal operator on $\mathcal{H}$, there exists a unique resolution of the identity E on $(\sigma(T), \mathcal{B}(\sigma(T)))$ such that $T = \int_{\sigma(T)} \lambda dE\lambda$, i.e.

$$< Tx,y > = \int_{\sigma(T)} \lambda E_{x,y}(\lambda) \text{ for all } x,y \in \mathcal{H} \text{ (see [2], [6])}.$$

Moreover, $Q \in C(T)$ if and only if $(QE(\omega) = E(\omega)Q$ for all $\omega \in \mathcal{B}(\sigma(T))$ (see [6], pág. 308). Another version of the spectral theorem says that, if T is a normal operator on $\mathcal{H}$, then there is a finite measure space $(X, \mathcal{A}, \mu)$ and function $\varphi \in L^\infty(\mu)$ such that T is unitarily equivalent to the multiplication operator M_φ on $L^2(\mu)$. Furthermore, $\sigma(M_\varphi)$ = essential range of $\varphi = \sigma(T)$. We shall denote by E' the resolution of the identity on $(\sigma(T), \mathcal{B}(\sigma(T)))$ associated to M_φ, which is defined by: $E'(\omega) = M_{\chi_{\varphi^{-1}(\omega)}}$, so that E and E' will be unitarily equivalent.

In what follows, we shall identify the spaces $\mathcal{H}$ and $L^2(X, \mathcal{A}, \mu)$, the operators T and M_φ and the resolutions of the identity E and E'.

4.1 ***Theorem.***

Let S be a closed subspace of $\mathcal{H}$ and let T be a normal operator on $\mathcal{H}$. Then, S is T-invariant and T^*-invariant ($TS \subset S$ and $T^*S \subset S$) if and only if S is the intersection of a family of subspaces S_y of $\mathcal{H}$, where, for each $y \in \mathcal{H}$: $S_y = \{ x \in \mathcal{H};\ E_{xy} = 0 \}$.

Proof.

Since $E'_{f,g}(\omega) = \langle E'(\omega)f,g \rangle = \int_{\varphi^{-1}(\omega)} f\bar{g}\,d\mu$ for all $\omega \in \mathcal{B}(\sigma(T))$, by the theorem 1.2 and the above identification the result follows.

4.2 ***Theorem.***

Let T be a normal operator on $\mathcal{H}$. The following statements are equivalent:

(a) $C(T) = \{ F(T)\,;\ F \in L^\infty(\sigma(T)) \}$

(b) The only subspaces S of $\mathcal{H}$ which are T-invariant and T^*-invariant are the ranges of the spectral projections associated to E, i.e., $S = \operatorname{Im} E(\omega)$ with $\omega \in \mathcal{B}(\sigma(T))$.

Proof.

Observe that $F(T) \in C(T)$, and if $\sigma(\varphi) = \mathcal{F}$ then, $F\circ\varphi$ is $\mathcal{F}$-measurable for all $F \in L^\infty(\sigma(T))$.

We shall show that (a) and (b) are equivalent to (c): $\sigma(\varphi) \sim \mathcal{A}$ (i.e., they have the same μ-complection).

(a) $\Leftrightarrow$ (c).

If $A \in \mathcal{A} \setminus \mathcal{F}$, then $M_{\chi_A} \in C(T)$, and it does not belong to $\{ F(T)\,;\ F \in L^\infty(\sigma(T)) \}$. On the other hand, if $\mathcal{F} \sim \mathcal{A}$, there exists a ciclic vector of T in $\mathcal{H}$, because the span of $M_{\bar\varphi}{}^n M_\varphi{}^m \chi_X$ ($m,n \in N$) is dense in $L^2(\mu)$ (see theorem 2 in [3] or theorem 1.2 in [4]), and then, we can take, $X = \sigma(T)$ and $\varphi(z) = z$ for all $z \in \sigma(T)$, in the spectral representacion, (see [2], pág. 13). Moreover, if $Q \in C(T)$, $Q \in C(F(T))$, i.e., $Q.M_F = M_F.Q$ for all $F \in L^\infty(\sigma(T))$. Since $\{ M_F :\ F \in L^\infty(\sigma(T)) \}$ is a maximal abelian albegra (see [2], pág. 21), then, $Q = M_G$ for some $G \in L^\infty(\sigma(T))$ or equivalently $Q = G(T)$.

(c) $\Leftrightarrow$ (b)

If S is T and T^*-invariant and $\mathcal{F} \sim \mathcal{A}$, by using theorem 1.2 of [4] it follows that $S = L^2(\varphi^{-1}(\omega_0))$, where $\varphi^{-1}(\omega_0)$ is the support of S.

Reciprocally if $A \in \mathcal{A}$, $L^2(A, \mathcal{A}, \mu)$ is a subspace of $\mathcal{H}$ which is φ and $\bar\varphi$-invariant, and then, there exists $\omega \in \sigma(T)$ such that $L^2(A, \mathcal{A}, \mu) = \operatorname{Im} E(\omega) = L^2(\varphi^{-1}(\omega), \mathcal{A}, \mu)$ and thus $A = \varphi^{-1}(\omega)$ μ-a.e., i.e., $\mathcal{A} \sim \mathcal{F}$.

REFERENCES

[1] NACHBIN, L.: "The Haar Integral", Van Nostrand (1965).

[2] RADJAVI, H.; ROSHENTHAL, P.: "Invariant Subspaces", Springer (1973).

[3] REZOLA, M.L.: "Un teorema de aproximación en espacios L^p" Rev. Mat. His. Amer. XL (5-6) (1980), 206-215.

[4] REZOLA, M.L.: "Subespacios Invariantes en $L^p(\mu)$". Actualites mathematiques, Actes 6e Congr. Group. Math. Expr. Latine, Luxembourg 1981, 437-440 (1982).

[5] REZOLA, M.L.: "Subespacios Invariantes y Aproximación en Espacios de Funciones Medibles", Disertation (1982). Univ. Zaragoza. Fac. Ciencias, Dpto. Teoría de Funciones.

[6] RUDIN, W.: "Functional Analysis", Mac Graw Hill (1973).

[7] ZAANEN, A.C.: "Integration", North Holland Pu. Co. (1967).

M. L. Rezola
Departamento de Teoría de Funciones.
Facultad de Ciencias.
Universidad de Zaragoza (SPAIN)