ORTHOGONAL BASES IN $l^\infty(X)$

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ABSTRACT

We compute the dimension of $l^\infty(X,K)$ for any complete field K endowed with a discrete valuation. As an application we characterize all Banach spaces over K which have a predual.

As a corollary of a result of VAN DER PUT [4], it is easy to verify (see our proposition 1) that the Banach space $l^\infty(X,K)$ of bounded functions of an infinite set X into a complete non archimedean field K endowed with the supremum norm has an orthogonal basis whose cardinality is $2^{\# X}$ if K is local (locally compact). On the other hand, it is well-known that, for any infinite set X , $l^\infty(X,K)$ does not have a basis if the valuation of K is dense. In this note we are going to compute the dimension (cardinality of a basis) of $l^\infty(X,K)$ for every complete field K endowed with a discrete (and non-trivial) valuation. As an aplication we characterize all Banach spaces over the latter kind of fields which have a predual.

Throughout this note X will be an infinite set, and K denotes a complete discretely valuated field with a non-trivial valuation. For a Banach space E over K the symbol dim(E,K) or dim E indicates the cardinality of a basis of E ([5] p. 53). Let $\overline{\pi}$ be the clousure in K of this prime field. The symbol dim(E,π) refers to the cardinality of an orthogonal basis of E regarded as a Banach space over π if one such basis exists. Notice that d = dim(K,π) is always defined (see theorem 1 of [1]).

If E,F are Banach spaces by $E \sim F$ we mean that there exists a linear isometry of E onto F. For other notations we refer to [5].

PROPOSITION 1.– Let K be either trivially valued and countable or local. For every infinite set X, dim $l^\infty(X,K) = 2^{\# X}$.

Proof: For K trivially valued, $l^\infty(X,K) = K^X$ and every orthogonal basis in $l^\infty(X,K)$ is an algebraic basis. So, we have dim $l^\infty(X,K) = 2^{\# X}$ for every infinite set X.

For K local and X endowed with the discrete topology, we have $PC(X,K) = l^\infty(X,K)$. Now apply corollary 5.23 and theorem 5.6 of [5] to deduce dim $l^\infty(X,K) = 2^{\# X}$.

Let $\{E_i\}_{i \in I}$ be a family of Banach spaces over K. By $X_i E_i$ we denote the set of all elements a of $\Pi_i E_i$ for which, the set $\{ \| a_i \| : i \in I \}$ is bounded. The space $X_i E_i$ can be normed by $\| a \| = \sup \{ \| a_i \| : i \in I \}$. The elements a of $\Pi_i E_i$ for which, for every $\epsilon > 0$, the set $\{ i \in I : \| a_i \| \geqslant \epsilon \}$ is finite form a closed linear subspace of $X_i E_i$ denoted by $\oplus_i E_i$.

LEMMA 2.– Let E be a Banach space over π and let $(e^i)_{i \in I}$ be an orthogonal basis of E. If J is an index set and, for every $j \in J$, we put $E_j = E$ and $F = \oplus_j E_j$, then $(e^{ik})_{(i,k) \in I \times J}$ is an orthogonal basis of F, where e^{ik} is defined by

$$(e^{ik})_j = e^i \text{ if } k = j \text{ and } (e^{ik})_j = 0 \text{ if if } k \neq j .$$

Proof: It is straightforward to verify that $(e^{ik})_{(i,k) \in I \times J}$ is an orthogonal subset of F.

Now, take $x = (x_j) \in F$ and put $x_j = \sum_i < x_j , e^i > e^i$. To finish the proof it is enough to verify that $x = \sum_{i,j} < x_j , e^i > e^{ij}$.

Choose $\epsilon > 0$ and consider the finite set $J_0 = \{ j \in J : \| x_j \| \geqslant \epsilon \}$. Moreover, there exists a finite subset I_0 of I such that for every $j \in J_0$, $\| x_j - \sum_{i \in I_f} < x_j , e^i > e^i \| < \epsilon$ for all finite subset I_f of I which contains I_0.

Let H be a finite subset of IxJ which contains $I_0 x J_0$. For a fixed element $j \in J$, put $H_j = \{ i \in I : (i,j) \in H \}$. Then,

$$\| x - \sum_{(i,j) \in H} < x_j , e^i > e^{ij} \| = \sup_j \| x_j - \sum_{i \in H_j} < x_j , e^i > e^i \| .$$

For $j \in J_0$, we have $H_j \supset I_0$ and so $\| x_j - \sum_{i \in H_j} < x_j , e^i > e^i \| < \epsilon$.

Now consider $j \notin J_0$. Since $\| x_j \| = \sup_{i \in I} \| < x_j , e^i > e^i \|$ and $\| x_j \| < \epsilon$, we also have $\| x_j - \sum_{i \in H_j} < x_j , e^i > e^i \| < \epsilon$.

We conclude that $\| x - \sum_{(i,j) \in H} < x_j , e^i > e^{ij} \| < \epsilon$ and consequently $x = \sum_{i,j} < x_j , e^i > e^{ij}$.

THEOREM 3.– For every infinite set X, $\dim l^\infty(X,K) = 2^{(\#X).d}$

Proof: As usual, we denote by $C_0(X,K)$ the closed linear subspace of $l^\infty(X,K)$ of all functions $y : X \to K$ such that $|y(x)|$ converges to zero in the Fréchet filter of X. Notice that, with the above notations, $C_0(X,K) = \bigoplus_X K_x$ where $K_x = K$ for every $x \in X$. Now consider a set I such that $K \sim C_0(I,s,\pi)$ (as Banach spaces over $\bar{\pi}$) where $s : I \to (0,+\infty)$, $\#I = d$ and $C_0(I,s,\pi)$ indicates the Banach space of all functions $y : I \to \pi$ such that $|y(i)|s(i)$ converges to zero in the Fréchet filter of I, endowed with the norm $\|y\| = \max |y(i)|s(i)$. Moreover, if the valuation of $\bar{\pi}$ is discrete we can choose s such that $s(i) = 1$ for every $i \in I$, and if the valuation of π is trivial we choose s such that $s(I) = \{|\lambda| : \lambda \in K \setminus \{0\}\}$.

From lemma 2 we deduce that, if $H = X \times I$, then for an adequate function $t : I \to (0,+\infty)$ verifying the same properties as s, we have $C_0(X,K) \sim C_0(H,t,\bar{\pi})$ as Banach spaces over π. Consequently, we have $l^\infty(X,K) \sim l^\infty(H,t^{-1},\bar{\pi})$ over π (even for π trivially valued). Now we take a set $A \subset H$ such that $\#A = \#H$ and $t^{-1}(A)$ is reduced to be a point. It follows from proposition 1 that $\dim l^\infty(A,\bar{\pi}) = 2^{\#H}$. Thus,

$$\dim(l^\infty(X,K),\bar{\pi}) \geq 2^{\#H} = 2^{(\#X).d} .$$

But, on the other hand,

$$\#l^\infty(X,K) = \#l^\infty(H,t^{-1},\pi) \leq (\#\pi)^{\#H} = 2^{(\#X)d} ,$$

and we have $\dim(l^\infty(X,K),\pi) = 2^{(\#X).d}$.

Now, if we consider a set J such that $l^\infty(X,K) \sim C_0(J,K)$ (i.e. $\dim l^\infty(X,K) = \#J$), lemma 2 allows us to conclude that

$$\dim(l^\infty(X,K),\pi) = (\#J).d .$$

From the formula $2^{(\#X).d} = \dim l^\infty(X,K).d$, we finally deduce that

$$\dim l^\infty(X,K) = 2^{(\#X).d} .$$

COROLLARY 4.–

(a) K is local if and only if d is finite.

(b) K is separable if and only if $d \leq \aleph_0$.

Proof: (a) If $d = n$, then K is a finite product of π n times. Therefore, π cannot have the trivial valuation and K is local.

Conversely, since K is a Banach space over π, it cannot be locally compact unless d is finite (this argument also works for trivially valueds fields [3]).

(b) Assume K to be separable. Since $\# K \leqslant c$ (in fact they are qual), then $\# l^\infty(X,K) \leqslant (\# K)^{\# X} = 2^{\# X}$. We deduce from theorem 3 that dim $l^\infty(X,K) = 2^{(\# X).d} = 2^{\# X}$ for every infinite set X, and consequently, $d \leqslant \aleph_0$.

Conversely, assume $d \leqslant \aleph_0$. Thern, K is a Banach space of countable type over π. Since π is always separable, we deduce the same property for K.

In the latter proof we have extended proposition 1 to the case of separable fields. In fact, we have:

COROLLARY 5.– If K is separable, then dim $l^\infty(X,K) = 2^{\# X}$.

For non-separable fields, we are going to compute dim $l^\infty(X,K)$ in a different way. For this, we need the concept of cofinal (notation cf) of a cardinal number (see [2] p. 26). It is relevant to notice that for cardinal numbers x with predecessor, we have cf $x = x$.

COROLLARY 6.– If K is not separable and cf $d > \aleph_0$, then dim $l^\infty(X,K) = 2^{(\# X).(\# K)}$.

Proof: Let $(y_i)_{i \in I}$, with $\# I = d$, be an orthogonal basis of K as a Banach space over π. By $P_c(I x \pi)$ we denote the set of countable subsets of $I x \pi$. If $x = \sum_i < x , y_i > y_i$ for every $x \in K$, the function $T : K \to P_c(I x \pi)$ defined by $T(x) = \{ (i, < x , y_i >) : < x , y_i > \neq 0 \}$ is inyective. Thus, we have $\# K \leqslant \# P_c(I x \pi) = \#(I x \pi)^{\aleph} = \# I^{\aleph}$ (because $\# I \geqslant c$). Since cf $d > \aleph_0$, we have $\# I^{\aleph} = \# I$ and finally we conclude that $\# K = \# I = d$. The rest follows from theorem 3.

A Banach space E over K is said to have a predual if there exists a Banach space F over K such that F' and E are linearly homeomorphic.

THEOREM 7.– Let E be an infinite dimensional Banach space over K.

(a) If dim $E \leqslant d$, then E has no predual.

(b) If dim $E > d$, then E has a predual if and only if dim E has a cardinal predecessor. Moreover, if the latter property holds, all preduals of E are linearly homeomorphic.

Proof: (a) Let F be a Banach space over K. From theorem 3 we have dim $F' = 2^{(\dim F).d} >$ dim E. So, F' and E cannot be linearly homeomorphic.

(b) If F is a predual of E, then $2^{(\dim F).d} =$ dim E, and dim E has a predecessor.

Conversely, if X is a set such that $2^{\# X} = \dim E$, it is obvious that $C_0(X,K)$ is a predual of E.

COROLLARY 7. – The following conditions for K are equivalent:

(a) K is separable.

(b) There exists a predual for every Banach space E over K such that dim E has a predecessor.

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