ON THE RANGE OF SEMIGROUP VALUED MEASURES

by

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ABSTRACT.

It is proved that every measure with values in a topological semigroup, whose topology is defined by a family of semi-invariant in zero pseudometrics, is bounded (in some sence) if it is s-bounded or it is σ-additive and the pseudometrics are invariant in zero, in the second casi it is also proved that the range of the measure is conditionally compact. Moreover it is stated that the range of a σ-additive measure with values in a topological semigroup (of the last type) is compact if the measure is purely atomic and of bounded variation. Some results about the uniform boundness of a sequence of semigroup valued measures and group valued measures, are proved.

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INTRODUCTION.

It is well known that every σ-additive measure with values in a locally convex vector space, is bounded, being this result false in general as it is proved in Ph. Turpin [12], where it is built a non bounded σ-additive measure, defined on a σ-algebra and with values in a topological vector space. In the same paper it is also proved that every σ-additive measure, defined on a σ-algebra with values in a topological vector space, is additively bounded. K. Musiał proves in [10] that every group valued σ-additive measure satisfying the Countable Chain Condition, is bounded in some sence, which is equivalent with classical ones if the group is locally compact or a locally convex vector space. M. P. Kats has proved in [7] last result without assume that the measure satisfies the Countable Chain Condition.

If S is a commutative topological semigroup with identity, whose topology is defined by a family $\mathfrak{T}$ of semi-invariant in zero pseudometrics, then theorem 6 of this paper states that every measure (non necessarily σ–additive) with values in S, is bounded in some sence if it is s–bounded or if it is σ–additive and the pseudometrics of $\mathfrak{T}$ are invariant in zero. This theorem extends for σ–additive measures (non necessarily defined on a σ–algebra) with values in a topological semigroup (of last type), the result about the additionally boundness of vector measures given in [12], mentionned before. As a consequence of theorem 6 it is obtained that the range of a σ–additive measure with values in a topological semigroup whose topology can be defined by a family of invariant in zero pseudometrics, is conditonally compact, and some results about the uniform boundness of a sequence of semigroup valued measures. If S is a group, then the concept of boundness used in this paper is weaker than the concept used in [10], being equivalents if the pseudometrics of the family $\mathfrak{T}$ are invariant in zero.

From the usual notion of set of null measure (given for instance in M. Sion [11]) are studied the atomic semigroup valued measures obtaining some results which extend the results already known for vector measures and group valued measures, which can be fond for instance in J. Hoffmann-Jorgensen [6] and in K. Musiał [10], respectively. Between these results we can mention that the sum of two purely atomic (respectively non atomic and verifying the Countable Chain Condition) σ–additive measures is purely atomic (respectively, non atomic and satisfies the Countable Chain Condition), and that every non atomic σ–additive semigroup valued measure, can be put like a sum of two unique measures, one of them non atomic and the other purely atomic and such that every atom for this second measure contains an atom for the original measure. At the end, it is proved the compactness of the range of every purely atomic σ–additive measure of bounded variation (defined on a σ–algebra) with values in a topological semigroup whose topology is defined by a family of semi-invariant in zero pseudometrics. The definition given here of p-variation of measures with values in semigroups of this type, is an extension of the usual definition for metrizable topological group valued measures (given for instance in H. Heinich [5]). It is easily proved that if a semigroup valued measure (being the topology of the semigroup defined by a family of semi-invariant in zero pseudometrics) is of bounded variation, then it is s-bounded and so bounded (as it results from theorem 6), which implies that the measure is of bounded semi-variation (definition given from the concept of p-semivariation used in P. Morales [9]).

The interest of studying measures with values in topological semigroups whose topology is defined by a family of semi-invariant in zero pseudometrics, has in part its motivation in the fact proved by H. Weber in [14] which states that the uniformity of an uniform semigroup can be generated by a family of continuous semi-invariant pseudometrics valued in the interval [0,1].

Definition 1. Let S be a commutative semigroup with zero element and p a pseudometric defined on S. We say that p is *semi-invariant* (respectively, *invariant*) in zero when $p(x+y, y) \leq p(x, 0)$ (respectively, $p(x+y, y) = p(x, 0)$) for all $x, y \in S$, and we will say that p is *semi-invariant* if $p(x+y, z+y) \leq p(x, z)$ for all $x, y, z \in S$.

As it is proved in H. Weber [14], the uniformity of an uniform semigroup can be generated by a family of continuous semi-invariant pseudometrics valued in [0,1]. Since now we will denote by S a commutative topological semigroup, whose topology is defined by a family $\mathfrak{T}$ of semi-invariant in zero pseudometrics, and by X an arbitrary commutative topological semigroup with zero element. For every pseudometric $p \in \mathfrak{T}$ we will write $|x|_p = p(x, 0)$ for $x \in S$.

An X-valued map on an algebra Σ of subsets of a set Ω is called a *finite additive measure* or a *measure*, if $m(A \cup B) = m(A) + m(B)$ whenever A, B are disjoint sets in Σ. The map m is called a σ *– additive measure* if $m(\cup_{n=1}^{\infty} A_n) = \Sigma_{n=1}^{\infty} m(A_n)$ whenever $(A_n)_{n=1}^{\infty}$ are mutually disjoint sets from Σ such that $\cup_{n=1}^{\infty} A_n \in \Sigma$. Since now we will denote by m a measure defined on an algebra Σ of parts of a set Ω and by $\Sigma \cap A$ the family $\{B \cap A : B \in \Sigma\}$, for each $A \in \Sigma$.

Definition 2. Let m be an S-valued measure, then for every pseudometric $p \in \mathfrak{T}$ the p-*variation* of m is the map $|m|_p$ defined on Σ by

$$|m|_p(A) = \sup_{\pi} \sum_{H \in \pi} |m(H)|_p$$

for $A \in \Sigma$, where the supremum is taken over all finite partitions π of A into pairwise disjoint sets from Σ.

Given a pseudometric $p \in \mathfrak{T}$ the p–*semivariation* of m (cf. [9]) is the map $\|m\|_p$ defined on Σ by

$$\|m\|_p(A) = \sup \{ |m(E)|_p \cdot E \in \Sigma \cap A \}.$$

It is easily proved that $|m|_p$ is a non decreassing finite additive map on Σ and that the function $\|m\|_p$ is subadditive and non decreassing. Clearly $\|m\|_p \leq |m|_p$. If m is σ–additive then $\|m\|_p$ is σ–subadditive and

$$\|m\|_p(\bigcup_{n=1}^{\infty} E_n) = \sup_n \|m\|_p(\bigcup_1^n E_k)$$

holds for every sequence $(E_k)_{k=1}^{\infty} \subset \Sigma$ such that $\bigcup_1^{\infty} E_k \in \Sigma$.

Definition 3. A subset $S_o \subset S$ is said to be *bounded* when for each element U of the uniformity defined by $\mathcal{T}$, there exists $n \in \mathbb{N}$ and a finite number of points $\{x_i\}_{i \in F} \subset S$ such that

$$S_o \subset [\bigcup_{i \in F} U[x_i]]^n \tag{3.1}$$

(being as usual $A^n = A + \overset{n}{\cdots} + A$ and $U[x] = \{y \in S : (x,y) \in U\}$ for $A \subset S$ and $x \in S$).

An S-valued measure m is said to be *bounded* when its range $m(\Sigma)$ is a bounded subset of S. If $|m|_p(\Omega) < +\infty$ for every $p \in \mathcal{T}$ it is said that m is of *bounded variation* and in the same way it is said m is of *bounded semivariation* when $\|m\|_m(\Omega) < +\infty$ for every pseudometric $p \in \mathcal{T}$.

We say that a measure m is s-bounded (c.f. [6]) if $\lim_n m(E_n) = 0$ for every sequence of mutually disjoint sets from Σ.

Remark 4. If m is an S-valued measure of bounded variation then for every sequence $(E_n)_{n=1}^{\infty}$ of mutually disjoint sets from Σ we have that

$$\sum_{k=1}^{n} |m(E_k)|_p \leqslant \sum_{k=1}^{n} |m(E_k)|_p + |m(\Omega - \bigcup_{k=1}^{n} E_k)|_p$$

$$\leqslant |m|_p(\Omega) < +\infty$$

and so, the series $\sum_{k=1}^{n} |m(E_k)|_p$ converges for all $p \in \mathcal{T}$, from where it follows immediately that m is s-bounded.

Furthermore, if m is a σ-additive S-valued measure and the family $\mathcal{T}$ is invariant in zero (i.e. all the pseudometrics belonging to the family $\mathcal{T}$ are invariant in zero), then for every sequence $(E_n)_{n=1}^{\infty} \subset \Sigma$ of mutually disjoint sets, such that $\bigcup_{n=1}^{\infty} E_n \in \Sigma$ we have that $\lim_n m(E_n) = 0$ because $m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$ and given $p_i \in \mathcal{T}$ and $\epsilon_i > 0$ $(i = 1, \dots, r)$ there exists $n_o \in N$ such that

$$|m(\bigcup_{n+1}^{\infty} E_k)|_{p_i} = p_i(m(\bigcup_{1}^{\infty} E_k), m(\bigcup_{1}^{n} E_k))$$

$$< \epsilon_i/2$$

for all $n \geqslant n_o$, and therefore,

$$|m(E_n)|_{p_i} \leqslant p_i(m(E_n), m(\overset{\infty}{\underset{n}{\cup}} E_k)) + |m(\overset{\infty}{\underset{n}{\cup}} E_k)|_{p_i}$$

$$\leqslant |m(\overset{\infty}{\underset{n+1}{\cup}} E_k)|_{p_i} + |m(\overset{\infty}{\underset{n}{\cup}} E_k)|_{p_i}$$

$$\leqslant \epsilon_i$$

holds for all $n \geqslant n_o$. So, if Σ is a σ–algebra, m is a σ–additive S–valued measure and $\mathfrak{I}$ is invariant in zero, then m is s–bounded.

Moreover let us remark that if m is an S–valued bounded measure then it is of bounded semi-variarion because for every $p \in \mathfrak{I}$ there exists and $n \in \mathbb{N}$ and a finite number of points $\{x_i\}_{i \in F}$ belonging to S such that

$$m(\Sigma) \subset (\underset{i \in F}{\cup} U[x_i])^n,$$

being $U = \{(x, y) \in S \times S : p(x, y) < 1\}$ and, therefore, given $A \in \Sigma$ there exists $i_1, \ldots, i_n \in F$ and $u_{i_j} \in U[x_{i_j}]$ for $j = 1, \ldots, n$, such that

$$m(A) = \sum_{j=1}^{n} u_{i_j}$$

and

$$|m(A)|_p \leqslant \sum_{j=1}^{n} |u_{i_j}|_p$$

$$\leqslant n + \sum_{j=1}^{n} |x_{i_j}|_p$$

$$\leqslant n(1 + \sup_{i \in F} |x_i|_p)$$

$$< +\infty.$$

Proposition 5. *An* S*–valued measure* m *of bounded variation is σ–additive if and only if* $|m|_p$ *is σ–additive for all* $p \in \mathfrak{T}$.

Proof. Let us suppose that $|m|_p$ is σ–additive for all $p \in \mathfrak{T}$, then if $(A_n)_1^\infty$ is a sequence of mutually disjoint sets from Σ and $\bigcup_1^\infty A_n \in \Sigma$, then

$$p\,[m(\bigcup_1^\infty A_k), m(\bigcup_1^n A_k)] \leqslant |m(\bigcup_{n+1}^\infty A_k)|_p$$

$$\leqslant |m|_p(\bigcup_{n+1}^\infty A_k)$$

$$\leqslant \sum_{n+1}^\infty |m|_p(A_k)$$

holds, from where it follows that m is σ–additive because the series $\sum_{k=1}^\infty |m|_p(A_k) \leqslant |m|_p(\Omega) < +\infty$.

Conversely, let us suppose that m is σ–additive and let $(A_n)_1^\infty$ be a sequence of mutually disjoint sets from Σ such that $\bigcup_1^\infty A_k \in \Sigma$ and π a finite partition of $\bigcup_1^\infty A_k$ into pairwise disjoint sets from Σ, then

$$\sum_{H \in \pi} |m(H)|_p = \sum_{H \in \pi} |m(\bigcup_1^\infty H \cap A_k)|_p$$

$$= \sum_{H \in \pi} |\sum_{k=1}^\infty m(H \cap A_k)|_p$$

$$\leqslant \sum_{H \in \pi} \sum_{k=1}^\infty |m(H \cap A_k)|_p$$

$$= \sum_{k=1}^\infty \sum_{H \in \pi} |m(H \cap A_k)|_p$$

$$\leqslant \sum_{k=1}^\infty |m|_p(A_k)$$

holds, and consequently we have that

$$|m|_p(\bigcup_1^\infty A_n) \leqslant \sum_{k=1}^\infty |m|_p(A_k).$$

Since $| m |_p$ is additive it follows that

$$\sum_{k=1}^{n} | m |_p (A_k) = | m |_p \left(\bigcup_1^n A_k \right)$$

$$\leqslant | m |_p \left(\bigcup_1^\infty A_k \right)$$

for all $n \in \mathbb{N}$, and so

$$\sum_{k=1}^{\infty} | m |_p (A_k) \leqslant | m |_p \left(\bigcup_1^\infty A_k \right)$$

holds.

Theorem 6. *Every* S*–valued measure* m *is bounded if it satisfies one of the following conditions:*

6.1. m *is* s*–bounded.*
6.2. m *is σ–additive and $\mathfrak{P}$ is invariant in zero* (1).

Proof. Let us suppose that there exists an element $U = \{ (x, y) \in S \times S : p_i (X, y) < \epsilon_i, \ i = 1, ..., n_1 \}$ belonging to the uniformity defined by the family $\mathfrak{P}$ such that for all $n \in \mathbb{N}$ and every finite family $\{ x_i \}_{i \in F}$ of points belonging to S, (3.1) doesn't hold. Then we are going to build a non increassing sequence $(T_k)_{k=1}^{\infty} \subset \Sigma$ such that $m (T_k - T_{k+1}) \notin U[0] (k \in \mathbb{N})$ and each T_k has the following property: For all $n \in \mathbb{N}$ and every finite family $\{ x_i \}_{i \in F}$ of points belonging to S

$$m (\Sigma \cap T_k) \not\subset \left(\bigcup_{i \in F} U[x_i] \right)^n$$

holds.

(1) As we have proved earlier 6.2 implies 6.1, if Σ is a σ–algebra.

Let it be $T_o = \Omega$ and $y_o = m(T_o)$, then there exists $A \in \Sigma$ such that $m(A) \notin U[0] \cup U[y_o]$. Therefore, $m(T_o - A) \notin U[0]$ because if not

$$p_i(m(T_o), m(A)) \leqslant | m(T_o - A) |_{p_i}$$
$$< \epsilon_i$$

$(i = 1, ..., n_1)$ and $m(A) \in U[y_o]$ which is contrary to the choice of A. Let it be $T_1 = A$ if for all $n \in \mathbb{N}$ and every finite family $\{x_i\}_{i \in F} \subset S$, $m(\Sigma \cap A) \not\subset [\bigcup_{i \in F} U[x_i]]^n$ holds, if not let us take $T_1 = T_o - A$ (remark that in this case, for all $r \in \mathbb{N}$ and every finite family $\{y_j\}_{j \in F'} \subset S$ we have that $m(\Sigma \cap T_1) \not\subset [\bigcup_{j \in F'} U[y_j]]^r$).

If $T_o, T_1, ..., T_k$ are built, them $m(\Sigma \cap T_k) \not\subset U[0] \cup U[y_k]$ with $y_k = m(T_k)$ and, therefore there exists $B \in \Sigma \cap T_k$ such that $m(B) \notin U[0]$ and $m(T_k - B) \notin U[0]$. If for all $n \in \mathbb{N}$ and every finite family $\{x_i\}_{i \in F} \subset S$, $m(\Sigma \cap B) \not\subset [\bigcup_{i \in F} U[x_i]]^n$ holds, then let it be $T_{k+1} = B$, if not let us take $T_{k+1} = T_k - B$. So, if it is assumed that m is not bounded it is possible to build a non increassing sequence $(T_k)_1^\infty \subset \Sigma$ such that $m(T_k - T_{k+1}) \notin U[0]$ for all $k \in \mathbb{N}$ which is in contradiction with 6.1 and 6.2.

Corollary 7. *With the conditions and notations of theorem* 6.2 *and if* S *is locally compact, then* $m(\Sigma)$ *is conditionally compact.*

Proof. An immediate consequence of theorem 6.

Corollary 8. *Let* M *be an uniformly* s–*bounded* (i.e. $\lim_{n \to \infty} m(E_n) = 0$ *uniformly for* $m \in M$, *whenever* $(E_n)_1^\infty$ *is a sequence of mutually disjoint sets from* Σ) *sequence of* S–*valued measures. If* S *is a group and* $\mathfrak{T}$ *is invariant, then* M *is uniformly bounded if and only if it is pointwise bounded.*

Proof. Let us suppose that M is pointwise bonded and let $c_b(S)$ be the semigroup of bounded sequences of elements from S (i.e. $(x_n)_{n=1}^\infty \in c_b(S)$ when $\{x_n : n \in \mathbb{N}\}$ is a bounded subset of S) endowed with the topology defined by the family of pseudometrics $\{\hat{p} : p \in \mathfrak{T}\}$ where

$$\hat{p}[(x_n), (y_n)] = \sup_n p(x_n, y_n)$$

for (x_n), $(y_n) \in c_b$ (S), and consider the measure $\hat{m} : \Sigma \longrightarrow c_b$ (S) defined by $\hat{m}$ (A) $= \{ m(A) : m \in M \}$. It follows from theorem 6 (6.1) that $\hat{m}(\Sigma)$ is a bounded subset of c_b (S) and so, given $\epsilon_i \in \mathfrak{T}$, $p_i > 0, i = 1, ..., r_1$ there exists $r_2, r_3 \in \mathbb{N}$ and $x^i = (x^i_n)^{\infty}_{n=1} \in c_b$ (S) for $i = 1, ..., r_2$ such that

$$\hat{m}(\Sigma) \subset [\bigcup_{i=1}^{r_2} \hat{U}[x^i]]^{r_3} ,$$

being $\hat{U} = \{ (x, y) \in c_b (S) x c_b (S) : \hat{p}_i (x, y) < \epsilon_i /2, i = 1, ..., r_1 \}$. Furthermore, from the boundness of the set $B = \{ x^i_n : n \in \mathbb{N}, i = 1, ..., r_2 \}$ it follows the existence of $r_4, r_5 \in \mathbb{N}$ and $\{ y_j \}_{j=1, ..., r_4} \subset$ S such that

$$B \subset [\bigcup_{j=1}^{r_4} U[y_j]]^{r_5},$$

being $U = \{(x, y) \in SxS : p_i (x, y) < \epsilon_i /2 , i = 1, ..., r_1 \}$. Therefore,

$$\{ m(A) : A \in \Sigma , m \in M \} \subset [\bigcup_{j=1}^{r_4} U[y_j]]^{r_3, r_5}$$

and M is uniformly bounded.

Corollary 9. *Let* M *be a uniformly* s *–Bounded sequence of* S *–valued measures. If* $\mathfrak{T}$ *is semi-invariant and* $\{ m(A) : m \in M \}$ *is a finite subset of* S *for all* $A \in \Sigma$ *, then* M *is uniformly bounded.*

Proof. Let c_b (S) and $\hat{m}$ be like in the last proof and let c_f (S) be the subsemigroup formed by the sequences which have at most a finite number of different elements, endowed with the induced topology by the topology considered for c_b (S) in the last proof. Then it results from theorem 6 (6.1) that $\hat{m}$ (Σ) is a bounded subset of c_f (S) and, consequently, given an element of the uniformity defined by $\mathfrak{T}$, there exists $r_1, r_2 \in \mathbb{N}$ and $x^j \in c_f$ (S) with $j = 1, ..., r_1$ such that

$$\hat{m}(\Sigma) \subset [\bigcup_{j=1}^{r_1} \hat{U}[x^j]]^{r_2}$$

holds (where $\hat{U}$ is defined like in last proof). Therefore,

$$\{ m(A) \ : A \in \Sigma \ , m \in M \} \subset [\bigcup_{k=1}^{r} U[y_k]]^{r_3}$$

holds, being $\bigcup_{j=1}^{r_1} \{ x_n^j \ : n \in \mathbb{N} \} = \{ y_1, ..., y_r \}$ and so, M is uniformly bounded.

Definition 10. Let m be a semigroup valued measure. A set $A \in \Sigma$ is said to be m *–null* (c.f. [11]) if $m(B) = 0$ for all $B \in \Sigma \cap A$. Since now we will denote by N (m) the family of m –null sets. We say that $A \in \Sigma$ is an m *–atom* if $A \notin N(m)$ and for all $B \in \Sigma \cap A$ either $B \in N(m)$ or $A - B \in N(m)$. m is called *atomless* if m has no atoms, and m is called *purely atomic* when there exists a sequences $(A_n)_1^\infty$ of m –atoms such that

$$\Omega - \sum_1^\infty A_n \in N(m).$$

If m_1 and m_2 are two X –valued measures and $N(m_1) \subset N(m_2)$ then every m_1 –atom is an m_2 –atom and if m_1 is purely atomic then so is m_2.

Let m_i be a X_i –valued measure defined on an algebra Σ_i $(i = 1, 2)$. m_1 is m_2 *–continuous* when $\Sigma_1 \subset \Sigma_2$ and for every zero-neighbourhood V_1 in X_1 there exists a zero-neighbourhood V_2 in X_2 such that $m_1(A) \in V_1$ as soon as $A \in \Sigma_1$ and $m_2(\Sigma \cap A) \subset V_2$. Clearly, if a measure $m_1 : \Sigma_1 \to X_1$ is continuous with respect some positive finite σ –additive measure on a σ –algebra $\Sigma (\supset \Sigma_1)$, then m_1 is s –bounded an it follows immediately from theorem 6 (6.1) the next result:

Proposition 11. *If an* S *–valued measure* m *is continuous with respect some positive finite* σ *–additive measure defined on a* σ *–algebra* $\Sigma' (\supset \Sigma)$, *then* m *is bounded.*

Proposition 12. *Let* Σ *be a* σ *–algebra and* m_i $(i = 1, 2)$ *an* X *–valued* σ *–additive measure, then the following assertions hold:*

12.1. *If* m_1 *and* m_2 *are purely atomic then* $m_1 + m_2$ *is purely atomic.*

12.2. *If* m_1 *and* m_2 *verify the Countable Chain Condition (being the definition of this condition here the natural extension of the usual one, see for* *instance* [13]), *then* $m_1 + m_2$ *verifies the Countable Chain Condition and moreover if* m_1 *and* m_2 *are atomless then* $m_1 + m_2$ *is atomless.*

12.3. *If* m *is a non atomless* X *–valued* σ *–additive measure defined on* Σ *which verifies the Countable Chain Condition, then there exists unique* X *–valued* σ *–additive measures* m_1, m_2 *defined on* Σ *such that* $m = m_1 + m_2$, m_1 *is purely atomic,* m_2 *is atomless and every* m_1 *–atom contains an* m *–atom. Moreover* m_1 *and* m_2 *verifies the Countable Chain Condition.*

Proof. 12.1. It is enough to follow the proof of theorem 5 of [6].

12.2. The first part is obvious. For proving the second one we will proceed like in the proof of proposition 1 of [10] . Let us suppose that m_1 and m_2 are atomless and that A is an m–atom, being $m = m_1 + m_2$. From Zorn's axiom it is obtained a maximal sequence of pairwise disjoint sets D_n from $(\Sigma \cap A) \cap N(m)$, such that $D_n \notin N(m_1)$. Consider $D = \bigcup\limits_1^\infty D_n$, then $(A\text{-}D) \cap N(m) \subset N(m_1)$ and since m_2 is atomless we have that $(A\text{-}D) \notin N(m_1)$. Let us see that $B = A-D$ is an m_1 –atom. Clearly if $C \in \Sigma \cap B$ then since A is an m–atom we have either $C \in N(m)$, and then $C \in N(m_1)$, or $(A-C) \in N(m)$, and so $(A-C) \cap B = B - C \in N(m)$. Therefore, B is an m_1 –atom and so we have a contradiction because m_1 is atomless.

12.3. From the Zorn's axiom it is deduced the existence of a countable (or finite) family of pairwise disjoint m–atoms A_n and such that there exists no m–atom contained in $\Omega - \bigcup\limits_{n=1}^{\infty} A_n$. Let it be $A = \bigcup\limits_1^\infty A_n$, $m_1(B) = m(B \cap A)$ and $m_2(B) = m(B-A)$ for all $B \in \Sigma$. Evidently, m_1 is purely atomic, m_2 is atomless, they verify the Countable Chain Condition and $m = m_1 + m_2$. Moreover if C is an m_1 –atom it is immediately proved that $C \cap A$ is an m–atom and, consequently, all m_1 –atom contains an m–atom.

Furthermore if m_1' and m_2' are two σ–additive X-valued measures defined on Σ such that m_1' is purely atomic, m_2' is atomless, every m_1' –atom contains an m–atom and $m = m_1' + m_2'$, let us see that $A \in N(m_2')$ and $A^c \in N(m_1')$. In effect, if $A^c \notin N(m_1')$ then there exists an m_1' –atom C such that $A^c \cap C \notin N(m_1')$ and therefore $A^c \cap C$ is an m_1' –atom and there exists an m–atom contained in A^c which is a contradiction.

If $A = \bigcup\limits_1^\infty A_n \notin N(m_2')$ then there exists $n \in N$ such that $A_n \notin N(m_2')$. Since m_2' is atomless, then there exists $B \in \Sigma \cap A_n$ such that neither $B \notin N(m_2')$ nor $B^c \notin N(m_2')$. Since A_n is an m–atom we have that either $B \in N(m)$ or $B^c \in N(m)$. Let us suppose that $B \in N(m)$. (similarly we would proceed if $B^c \in N(m)$), since $B \notin N(m_2')$ it results that $B \notin N(m_1')$ and B contains some m_1' –atom and therefore, B contains some m–atom which is contradictory to the fact of being $\Sigma \cap B \subset N(m)$.

Theorem 13. *Let* m *be a purely atomic σ–additive* S–*valued measure defined on a σ–algebra Σ . If* m *is of bounded variation then* m (Σ) *is compact.*

Proof. If $(A_n)_1^\infty$ is a sequence of m–atoms such that $\Sigma \cap (\Omega - \overset{\infty}{\cup} A_n) \subset N(m)$, we can define (like in [10], theorem 3 or in [6], theorem 10) a map f from the Cantor's set $C = \{0, 1\}^{\aleph_0}$ onto m (Σ) by

$$f(c) = \sum_{n \in D_c} m(A_n)$$

for $c = (c_n)_1^\infty \in C$, being $D_c = \{ n \in \mathbb{N} : c_n = 1 \}$. The proof will be completed proving that f is a continuous map. Since m is of bounded variation the series $\sum_1^\infty |m(A_n)|_p$ is convergent for all $p \in \mathcal{T}$, and so, given $c \in C$, $p_i \in \mathcal{T}$, $\epsilon_i > 0$ with i = 1, ..., r, there exists $n_o \in \mathbb{N}$ such that

$$\sum_{n > n_o} |m(A_n)|_{p_i} < \epsilon_i/2$$

for all i = 1, ..., r. If $d = (d_n)_1^\infty \in C$ is such that $d_j = c_j$ for j = 1, ..., n_o then

$$p_i(f(d), f(c)) \leqslant \sum_{n \in D'_d} |m(A_n)|_{p_i} + \sum_{n \in D'_c} |m(A_n)|_{p_i}$$

$$< \epsilon_i$$

holds for i = 1, ..., r, being $D'_d = D_d \cap \{ n \in \mathbb{N} : n > n_o \}$ and $D'_c = D_c \cap \{ n \in N : n > n_o \}$. Hence, f is continuous and m (Σ) is compact.

REFERENCES

[1] Diestel J. and Jr. Uhl: *Vector measures.* Mathematical Surveys of A.M.S., 1977.

[2] Drewnowski, L.: *Uniform boundness principle for finitely additive vector measures.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. *21* (1973), 115-118.

[3] Drewnowski, L.: On central submeasures and measures. Studia Math. *50* (1974), 203-224.

[4] Fischer, W. and V. Scholer: *Sur la bornitude d'une mesure vectorielle.* C.R. Acad. Sci. Paris Sér. A, *282* (1976), 519-522.

[5] Heinich, H.: *Mesures dans un groupe topologique.* C.R. Acad. Sci. Paris, *272* A (1971), 16-18.

[6] Hoffmann-Jørgensen, J.: *Vector measures.* Math. Scand., *28* (1971), 5-32.

[7] Kats, M.P.: *On the extension of vector measures* (in Russian). Sib. Mat. Zhurn., *13* (5) (1972), 1158-1168.

[8] Kluvanek, I. and G. Knowles: *Vector measures and control systems.* North Holland, Amsterdam, 1975.

[9] Morales, P.: *Regularity and extension of semigroup-valued Baire measures.* Lect. Notes in Math; 794 (1980), 317-323.

[10] Musiał, K.: *Absolute continuity and the range of group valued measures.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. *21* (1973), 105-113.

[11] Sion, M.: A theory of semigroup valued measures. Lect. Notes in Math., *355*, Springer, Berlin, 1973.

[12] Turpin, Ph.; *Une mesure vectorielle non bornée.* C.R. Acad. Sci. Paris, 280 A (1975), 509-511.

[13] Turpin, Ph.: *On the range of atomless vector measures.* (To appear).

[14] Weber, H.: *Fortsetzung von Massen mit Werten in uniformen* Halbgruppen. Arch. der Math. *27* (1976), 412-423.

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