A NOTE ON SPACES OF HOLOMORPHIC VECTOR VALUED FUNCTIONS WITH THE STRICT TOPOLOGY

by

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ABSTRACT:

In this note we prove that if Ω is a simply connected open set in the complex plane and E is a Krein space with separable strong dual then the space of all bounded holomorphic functions of Ω into E, endowed with the strict topology, is a Rosenthal space.

The vector spaces used here are defined over the complex field $\mathbb{C}$. Hereinafter, E will denote a locally convex space, $\hat{E}$ its completion and E' its topological dual. E'_b will stand for the strong dual of E. We say E to be a *Krein* space if the closed absolutely convex cover of each compact subset of E is also compact. Having in mind the characterization of Rosenthal ([5]) for the Banach spaces which do not contain closed subspaces topologically isomorphic to 1^1, we say E to be a *Rosenthal space* if every bounded sequence in E has a weakly Cauchy subsequence. If Ω is an open subset of $\mathbb{C}$, $H^\infty(\Omega; E)$ will be the space of all bounded holomorphic functions of Ω into E. If $E = \mathbb{C}$, we simply write $H^\infty(\Omega)$. Let h be a function of Ω into $\mathbb{C}$ bounded and vanishing at infinity, p a continuous seminorm on E and ϕ a function in $H^\infty(\Omega; E)$. The formula

$$p_h(\phi) = \sup_{z \in \Omega} |h(z)| \, p(\phi(z))$$

defines a seminorm p_h on $H^\infty(\Omega; E)$. The topology β generated by the family of all p_h–type seminorms is called the *strict topology.* We denote by $H^\infty(\Omega; E)_\beta$ the space $H^\infty(\Omega; E)$ endowed with this topology.

Within the class $\mathcal{K}$ of Krein spaces we set the following question: Is $H^\infty(\Omega; E)_\beta$ a Rosenthal space when $E \in \mathcal{K}$ is a Rosenthal space and Ω is a simply connected domain? The answer is affirmative if E is a semirreflexive gDF-space with separable strong dual ($E \in \mathcal{K}$ since by [4], p. 263, it is complete; E is a Rosenthal space since E'_b is separable): In fact, in this case the space

$(H^\infty(\Omega; E)_\beta)'_b$ is topologically isomorphic to $(H^\infty(\Omega)_\beta)'_b \hat{\otimes}_\pi E'_b$ (see [1], Th. 2.11) and since $(H^\infty(\Omega)_\beta)'_b$ is separable (see [3], p. 227), $(H^\infty(\Omega; E)_\beta)'_b$ is separable. Then $H^\infty(\Omega; E)_\beta$ is a Rosenthal space. In particular, $H^\infty(\Omega; H^\infty(\Omega)_\beta)_\beta$ is a Rosenthal space, since $H^\infty(\Omega)_\beta$ is a semi-Montel gDF-space ([6]).

In this note we answer the former question in the case in which the space $E \in \mathcal{K}$ has separable strong dual.

THEOREM: *If* Ω *is a simply connected open subset of the complex plane and* E *is a Krein space with separable strong dual then* $H^\infty(\Omega; E)_\beta$ *is a Rosenthal space.*

Proof: If Ω is the whole $\mathbb{C}$, the space $H^\infty(\Omega; E)_\beta$ is topologically isomorphic to E, and so it is a Rosenthal space. Let us now suppose $\Omega \neq \mathbb{C}$. We first consider the case when Ω is connected. By Riemann's conformal mapping theorem, Ω is conformally equivalent to the unit disc U of $\mathbb{C}$ and so $H^\infty(\Omega; E)_\beta$ is topologically isomorphic to $H^\infty(U; E)_\beta$. It suffices then to prove that $H^\infty(U; E)_\beta$ is a Rosenthal space. We argue as in the proposition 2.9. of [5], p. 237, using that the Mac Laurin's series of a function ϕ in $H^\infty(U; E)$ converges to ϕ uniformly on on compact sets of U (see [4], p. 362), to prove that $H^\infty(U) \otimes E$ is a dense subspace of $H^\infty(U; E)_\beta$. Thus the space $\widehat{H^\infty(U; E)_\beta}$ is topologically isomorphic to $H^\infty(U)_\beta \hat{\otimes}_\epsilon E$. Consider now a bounded sequence (ϕ_n) in $H^\infty(U; E)_\beta$. Let (e'_n) be a dense sequence in E'. The sequence $(e'_1 \circ \phi_n)$ is bounded in $H^\infty(U)_\beta$ and by the proposition 5 of [2] we can find a convergent subsequence $(e'_1 \circ \phi_{1n})$ in $H^\infty(U)_\beta$. By the same argument a convergent subsequence $(e'_2 \circ \phi_{2n})$ can be obtained from $(e'_2 \circ \phi_{1n})$ and so on. We shall prove that the diagonal subsequence (ϕ_{nn}) is weakly Cauchy in $H^\infty(U; E)_\beta$. It is easy to see that for every $e' \in E'$ the sequence $(e' \circ \phi_{nn})$ is convergent in $H^\infty(U)_\beta$, since this space is complete and E'_b is separable. Fix now $\omega \in (H^\infty(U; E)_\beta)'$. We shall show that the sequence $(< \phi_{nn}, \omega >)$ is convergent. Associated to ω, by the topological isomorphism before stated, there are closed absolutely convex neighbourhoods of the origin V and W in $H^\infty(U)_\beta$ and E, respectively, and a Borel regular measure μ on the compact set $V° \times W°$ ($V°$ and $W°$ endowed with their corresponding weak topologies) such that

$$< \phi , \omega > = \int_{V° \times W°} < e' \circ \phi , \nu > \, d\mu (\nu, e')$$

for every $\phi \in H^\infty(U; E)$. For each $(\nu, e') \in V° \times W°$ the sequence $(< e' \circ \phi_{nn}, \nu >)$ is convergent because $(e' \circ \phi_{nn})$ is convergent in $H^\infty(U)_\beta$. Also

$$\sup_{(\nu, e', n) \in V° \times W° \times \mathbb{N}} |< e' \circ \phi_{nn}, \nu >| < \infty$$

since the set $\{ e' \circ \phi_{nn}, (e', n) \in W^\circ \times \mathbb{N} \}$ is bounded in $H^\infty(U)_\beta$ (because the set $\cup \{ \phi_{nn}(U), n \in \mathbb{N} \}$ is bounded in E and W° is bounded in E'_b) and V° is bounded in $(H^\infty(U)_\beta)'_b$. By Lebesgue's dominated convergence theorem the sequence

$$< \phi_{nn}, \omega > = \int_{V^\circ \times W^\circ} < e' \circ \phi_{nn}, \nu > \, d\mu(\nu, e')$$

is convergent. Thus (ϕ_{nn}) is weakly Cauchy and $H^\infty(U; E)_\beta$ is a Rosenthal space.

Let us now examine the case of Ω non connected. The connected components Ω_i, $i \in J \subset \mathbb{N}$, of Ω are simply connected domains. If ϕ is a function defined on Ω, we denote by ϕ^i the function such that $\phi^i(z) = \phi(z)$ if $z \in \Omega_i$ and $\phi^i(z) = 0$ otherwise. Let (ϕ_n) be a bounded sequence in $H^\infty(\Omega; E)_\beta$. We next show that it has a weakly Cauchy subsequence. The sequence $(\phi_n | \Omega_1)$ of the restrictions to Ω_1 is bounded in $H^\infty(\Omega_1; E)_\beta$ and as we have already shown it admits a weakly Cauchy subsequence $(\phi_{1n} | \Omega_1)$ in this space. Now the sequence $(\phi_{1n} | \Omega_2)$ has a subsequence $(\phi_{2n} | \Omega_2)$ which is weakly Cauchy in $H^\infty(\Omega_2; E)_\beta$. If J is infinite, we repeat this process and we consider the diagonal subsequence (ϕ_{nn}). If J is finite we call (ϕ_{nn}) the last sequence obtained. Let us see that (ϕ_{nn}) is weakly Cauchy. Let $\omega \in (H^\infty(\Omega; E)_\beta)'$ be given. There exists a continuous seminorm p on E, a bounded function $h: \Omega \to \mathbb{C}$ vanishing at infinity and a number $C > 0$ such that

$$|< \phi, \omega >| < C \sup_{z \in \Omega} |h(z)| \, p(\phi(z))$$

for every $\phi \in H^\infty(\Omega; E)$. Let $M = \sup \{ p(\phi_n(z)), (z, n) \in \Omega \times \mathbb{N} \}$. Given $\epsilon > 0$ we determine a compact set $K \subset \Omega$ such that $|h(z)| < \epsilon/4MC$ for every $z \in \Omega \sim K$. The compact set K is contained in a finite number N of components of Ω. Let $I = \{ i \in J: K \cap \Omega_i \neq \phi \}$. For each $i \in I$ the sequence (ϕ^i_{nn}) is weakly Cauchy in $H^\infty(\Omega; E)_\beta$ and so there exists an integer n_i such that

$$|< \phi^i_{mm} - \phi^i_{nn}, \omega >| < \epsilon/2N,$$

for every $m,n \geq n_i$. We set $n_0 = \max \{ n_i, i \in I \}$. Then, if $m,n \geq n_0$,

$$|< \phi^i_{mm} - \phi^i_{nn}, \omega >| \leq$$

$$\leq \sum_{i \in I} |< \phi^i_{mm} - \phi^i_{nn}, \omega >| + |< \sum_{i \in J \sim I} (\phi^i_{mm} - \phi^i_{nn}), \omega >| <$$

$$< \sum_{i \in I} \epsilon/2N + C \sup_{z \in \Omega} |h(z)| p(\sum_{i \in J \sim I} (\phi^{i}_{mm}(z) - \phi^{i}_{nn}(z))) <$$

$$< \epsilon/2 + C \frac{\epsilon}{4MC} 2M = \epsilon.$$

Therefore $H^{\infty}(\Omega, E)_{\beta}$ is a Rosenthal space, q.e.d.;

REFERENCES

[1] COLLINS, H.S.; RUESS, W.: Duals os spaces of compact operators. Studia Math. LXXIV (1982), 213-245.

[2] COOPER, J.B.: The strict topology and spaces with mixed topologies. Proc. Amer. Math. Soc. 30/3 (1971), 583-592.

[3] COOPER, J.B.: Saks spaces and applications to functional analysis. Notas de Mat. 28. North Holland P.C. (1978).

[4] JARCHOW, H.: Locally convex spaces. B.G. Teubner, Stuttgart (1981).

[5] ROSENTHAL, H.P.: A characterization of Banach spaces containing 1^1. Proc. Nat. Sci. Acad. U.S.A. 71 (1974), 2411-2413.

[6] RUESS, W.: On the locally convex structure of strict topologies. Math. Z. 153 (1977), 179-192.

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