$\alpha\mu$-DUALS AND HOLOMORPHIC (NUCLEAR) MAPPINGS

by

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ABSTRACT

Corresponding to an arbitrary sequence space μ and a sequence α, we introduce the notion of an $\alpha\mu$-dual of a sequence space which, in particular, envelops the concepts of Köthe, β–, γ – duals and the duals of an $\mathcal{G}$-space studied in [4]. Using these concepts, we make a structural study of several subspaces of holomorphic mappings including characterizations of bounded and compact subsets.

1. INTRODUCTION

Depending on a sequence space μ and its $\alpha\mu$-dual which we introduce in this paper, we define a class of weighted holomorphic functions defined on a Banach space, weights being provided by the sequence space μ, and a subclass of this class with the help of the $\alpha\mu$-dual. We study structural properties of these spaces after equipping them with appropriate locally convex topologies; and after having characterized the bounded and relatively compact subsets of the subclass, we investigate conditions under which the subspace topology coincides with various other topologies on bounded sets. In the final section, we make a slight deviation from this study and take up the study of a class of holomorphic (indeed, hypoanalytic) mappings defined on an open subset of a nuclear sequence space, wherein we explore the basis representation of elements in the compact open topology.

2. FUNDAMENTALS

In order to appreciate the subject matter of this paper, the reader is assumed to have a rudimentary familiarity with locally convex spaces, nuclear spaces, Schauder bases, sequence spaces and holomorphic mappings as envisaged in

[12], [23], [16], [17], [15], [19], [21], [24], [2], [3]. However, to facilitate the reader, we mention in brief the salient features of these topics, relevant to the present work.

To begin with, let us denote throughout by (X, T) a Hausdorff locally convex space (l.c. TVS) equipped with a locally convex topology T generated by the family D_T of all T-continuons seminorms; the vector space X being considered over the field IK of reals or complex numbers. The fundamental neighbourhood system at origin for T is denoted by $\mathcal{B}_T$, and the symbols X' and X^* respectively stand for the algebraic and topological duals of X. Further, we write IN $= \{1, 2, \ldots\}$ and $\mathrm{IN}_0 = \{0\} \cup \mathrm{IN}$.

Let p_u be the Minkowski functional corresponding to u in $\mathcal{B}_T$ and $X_u = X/\ker p_u$ where $\ker p_u = \{x \in X: p_u(x) = 0\}$, be equipped with the usual quotient norm $\hat{p}_u$. If $v \in \mathcal{B}_T$ is absorbed by u; that is $v < u$, so that $p_u \leq p_v$, there exists a natural canonical continuous mapping $K_u^v : X_v \to X_u$ with $K_u^v(x_v) = x_u$, $x_v \in X_v$. Let us denote by $\hat{K}_u^v$ the extension of K_u^v from the completion $\hat{X}_v$ of X_v to the completion $\hat{X}_u$ of X_u. Then we have.

Definition 2.1: An l.c. TVS (X, T) is said to be *nuclear* (resp. *Schwartz*) if to each $u \in \mathcal{B}_T$ there corresponds $v \in \mathcal{B}_T$, $v < u$ such that $\hat{K}_u^v : \hat{X}_v \to \hat{X}_u$, is nuclear (resp. precompact); here, a nuclear mapping T from one Banach space E to another Banach space F means a continuous linear mapping having the following representation.

$$Tx = \sum_{n \geqslant 1} \alpha_n f_n(x) y_n \quad , \quad x \in E$$

for some $\{\alpha_n\} \in \ell^1$, $\{f_n\} \subset E^*$ and $\{y_n\} \subset F$ with $\|f_n\| \leq 1$, $\|y_n\| \leq 1, n \geq 1$.

A sequence $\{x_n\}$ in (X, T) is said to be a *base if* each x in X is uniquely expressed as

$$x = \sum_{i \geqslant 1} \alpha_i x_i = \lim_n \sum_{i=1}^{n} \alpha_i x_i$$

where the limit is being considered in the topology T and $\{\alpha_i\}$ is the unique sequence of scalars corresponding to x. Thus we have a sequence $\{f_i\} \subset X'$ such that $f_i(x_j) = \delta_{ij}$, the Kronecker delta and $f_i(x) = \alpha_i, i \geq 1$. If $\{f_i\} \subset X^*$, then $\{x_n\}$ is called a *Schauder base* for (X,T). We shall have occasion to make use of special case of the concept of fully λ-bases. In fact corresponding to an arbitrary sequence space λ (see definition below), a Schauder base $\{x_n; f_n\}$ for an l.c. TVS (X,T) is called a *fully* λ-base (cf. [13]; [18]) if for each p in D_T and x in X, $\{f_n(x)\,p(x_n)\} \in \lambda$ and the mapping $\Psi_p : X \to \lambda$, $\Psi_p(x) = \{f_n(x)\,p(x_n)\}$ is T-$\eta(\lambda, \lambda^x)$ continuous, where $\eta(\lambda, \lambda^x)$ is the normal topology on λ introduced below.

Coming to a brief discussion on sequence spaces, let ω *denote* the vector space of all scalar-valued sequences under usual pointwise addition and scalar multiplication; and φ be the subspace of ω spanned by the set $\{e^n : n \geq 1\}$, where $e^n = \{0, 0, 0, \ldots, 1, 0, \ldots\}$, 1 being placed at the nth-coordinate. In case we consider sequences defined over $\mathbb{N}_0$, the members of our sequence spaces will be indexed from 0 and φ will be span of $\{e^n : n \geq 0.\}$ The letters a, b, c, . . . and e (which denotes the sequence with all its coordinates equal to 1), are used to denote the members of ω, where $a = \{a_n\}$, $b = \{b_n\}$ etc. and ab stands for the sequence $\{a_n b_n\}$. An element a in ω is said to be *positive, written* as $a > 0$, if $a_n \geq 0$ for each $n \geq 1$. The nth section of an element a in ω, *denoted* by $a^{(n)}$, is the sequence

$$a^{(n)} = \sum_{i=1}^{n} a_i e^i = \{a_1, a_2, \ldots, a_n, 0, 0, \ldots\} .$$

A *sequence space* λ is a subspace of ω containing φ. The *Köthe*, β - and γ-*dual* of λ are respectively the spaces λ^x, λ^β and λ^γ defined as

$$\lambda^x = \{b \in \omega : \sum_{n \geq 1} |a_n b_n| < \infty, \forall\, a \in \lambda\} ;$$

$$\lambda^\beta = \{b \in \omega : \sum_{n \geq 1} a_n b_n \text{ converges for each a in } \lambda\} ;$$

and

$$\lambda^\gamma = \{b \in \omega : \sup_{n \geq 1.} |\sum_{i=1.}^{n} a_i b_i| < \infty, \forall\, a \in \lambda\} .$$

For the dual pair $<\lambda, \lambda^x>$, the topology $\eta(\lambda, \lambda^x)$ is the *normal topology* on λ, which is generated by the family $\{p_b : b \in \lambda^x\}$ of seminorms where

$$p_b(a) = \sum_{n \geq 1} |a_n b_n| , a \in \lambda, b \in \lambda^x.$$

Let us also recall the

Definition 2.2: A subset B of a sequence space λ is said to be *normal* if $a \in B$ whenever $|a_n| \leq |b_n|, n \geq 1$ for some $b \in B$; and λ is known to be *perfect* if $\lambda = \lambda^{xx}$. A linear sequence space (λ, T_λ) is called a K-*space* if the co-ordinate maps $P_i : \lambda \to \mathbb{K}$, $P_i(a) = a_i, i \geqslant 1$ are continuous; and a K-space (λ, T_λ) is known as an AK-*space* if each a in λ satisfies the condition

$$(*) \qquad a^{(n)} = \sum_{i=1}^{n} a_i e^i \to a, \text{ as } n \to \infty \text{ in } T_\lambda.$$

A seminorm p on λ is called *solid* if $p(a) \leq p(b)$ for $a, b \in \lambda$ with $|a_n| \leqslant |b_n|, n \geqslant 1$.

Clearly, the normal topology is generated by the family of solid seminorms. For more examples of locally convex topologies generated by solid seminorms, one is referred to [7], [8] and [15].

Following [14] and [15], we have the Schock-Terziöglu criterion and the Grothendieck-Pietsch characterization respectively contained in

Theorem 2.3: A sequence space $(\lambda, \eta(\lambda, \mu))$ is Schwartz if and only if to each $a > 0$ in μ, there exists $b > 0$ in μ with $a_n \leq b_n, n \geq 1$ such that $\{a_n/b_n\} \in c_0$, where μ is a normal subspace of λ^x.

Theorem 2.4: A sequence space $(\lambda, \eta(\lambda, \mu))$ is nuclear if and only if for each positive element a in μ and a positive number s, there exists a $b \in \mu$ with $0 \leq a_n \leqslant b_n, n \geq 1$ such that $\{(a_n/b_n)^s\} \in \ell^1$, where μ is a normal subspace of λ^x.

In the statements of Theorems 2.3 and 2.4, 0/0 means 0.

In what follows, we shall also come across a particular type of perfect sequence spaces known as *Köthe spaces, denoted* by Λ (P). Here P is a *Köthe set* or a *power set* and is a subset of ω satisfying the conditions: (i) each element a in P is positive; (ii) for $a, b \in P$, there exists $c \in P$ with $a_n, b_n \leq c_n$, for each $n \geq 1$; and (iii) for each n in $\mathbb{N}$, there exists $a \in P$ with $a_n > 0$. The *Köthe space* Λ (P) is then defined as

$$\Lambda(P) = \{ b \in \omega : p_a(b) = \sum_{n \geq 1} |b_n| a_n < \infty, \forall a \in P \}.$$

The natural topology on Λ (P), generated by the family $\{p_a : a \in P\}$, is denoted by T_P. It is known that the space $(\Lambda(P), T_P)$ is always complete and it is nuclear if the conclusion of Theorem 2.4 holds with λ^x being replaced by P(cf. [15], p. 98).

In the sequel, we will require the following result which we prove here from [17] for the sake of completeness.

Theorem 2.5: Let $\{ x_n;f_n \}$ be a fully λ-base for an l.c.TVS (X,T), where λ satisfies the (K)-property (i.e. there exists γ in λ^x with $k_\gamma \equiv \inf \gamma_n > 0$). Then there exists a Köthe set P such that $(X,T) \simeq (\delta, T_p | \delta)$ where $\delta = \{ \{ f_n(x) \} : x \in X \}$ is a dense subspace of $(\Lambda(P), T_p)$; in particular, if (X,T) is sequentially complete, then $(X,T) \simeq (\Lambda(P), T_p)$.

Proof. Let $P = \{ \{ p(x_n) \beta_n \} : p \in D_T, \beta \in \lambda^x, \beta > 0 \}$ and $\Lambda(P)$ the corresponding Köthe space equipped with the topology T_p.

Since for each p in D_T, $\{ f_n(x)p(x_n) \} \in \lambda$, $\delta \subset \Lambda(P)$. Consecuently the map $\Psi : X \to \delta$, $\Psi(x) = \{ f_n(x) \}$ is a bijective linear map.

The seminorms generating the topology T_p are given by

$$Q_{p,\beta}(\alpha) = \sum_{n \geq 1} p(x_n) \beta_n | \alpha_n |.$$

Therefore, by the fully λ-character of $\{ x_n;f_n \}$, for every p in D_T and β in λ^x, $\beta > 0$, there exists q in D_T such that $Q_{p,\beta}(\Psi(x)) \leq q(x)$. On the other hand, for p in D_T,

$$p(\Psi^{-1}(\{ f_n(x) \})) \leq \frac{1}{k_\gamma} Q_{p,\gamma}(\{ f_n(x) \})$$

Hence $(X,T) \simeq (\delta, T_p | \delta)$. We next show that $\bar{\delta} = \Lambda(P)$. Let $\alpha \in \Lambda(P)$ but $\alpha \notin \bar{\delta}$. Then by the Hahn-Banach theorem, there exists an f in $(\Lambda(P))^*$ so that $< \alpha, f > = 1$ and $< \beta, f > = 0$ for every β in δ. The last equality yields $< e^n, f > = 0$ for all $n \geq 1$. Thus $< \alpha, f > = 0$, a contradiction and so $\bar{\delta} = \Lambda(P)$.

Finally, if (X,T) is sequentially complete, then $\delta = \bar{\delta}$. For, if $\delta \subsetneq \bar{\delta}$, then we find some α in $\Lambda(P)$ such that $\alpha \notin \delta$.

Now

$$k_\gamma \sum_{n \geq 1} | \alpha_n | p(x_n) \leq \sum_{n \geq 1} | \alpha_n | p(x_n) \gamma_n < \infty, \forall p \in D_T.$$

Therefore $\sum_{n \geqslant 1} \alpha_n x_n$ converges in (X,T); that is, $\alpha = \{ f_n(x) \} \in \delta$, a contradiction again. Now apply the first part to arrive at the desired result.

For Banach spaces E, F and $n \in \mathbb{N}_o$, let *us denote by* $\mathcal{T}(^nE;F)$ the Banach space of all n-homogeneous continuos polynomials from E to F with respect to the norm $||\cdot||$ given by

$$||P|| = \sup_{x \neq o} \frac{||Px||}{||x||^n}, P \in \mathcal{P}(^nE;F).$$

Let us recall from [21] the following

Definition 2.6: A power series from E to F about $x_o \in E$ is a series in $x \in E$ of the form

$$\sum_{n \geq 0} P_n(x-x_o), \tag{2.7}$$

where $P_n \in \mathcal{P}(^nE;F)$, $n \geq 0$, are known as the *coefficients* of the power series.

Proposition 2.8: A necessary and sufficient condition for the power series (2.7) to be convergent is that the sequence $\{(||P_n||/n!)^{1/n} : n \in \mathbb{N}_o\}$ is bounded.

Definition 2.9: A mapping $f:E \to F$ is said to be *holomorphic* at $x_o \in E$ if there exists a unique power series of the form (2.7) such that

$$f(x) = \sum_{n \geq 0} P_n(x-x_o), \tag{2.10}$$

where the series on the right hand side converges uniformly in a neighbourhood of the point x_o, and is termed as the Taylor series of f at x_o.

Let H(E,F) denote the vector space of all holomorphic mappings with usual pointwise addition and scalar multiplication. For $f \in H(E;F)$ having representation (2.10), put

$$\hat{d}^n f(x_o) = n!\, P_n, \; n \geq 0 \tag{2.11}$$

so that

$$f(x) = \sum_{n \geq 0} \frac{\hat{d}^n f(x_o)}{n!} (x-x_o) \tag{2.12}$$

We call the mappings $d^n f (n \geq 0)$ from E to $\mathcal{P}(^nE;F)$, the *differential mappings.* Clearly, the operators $\hat{d}^n$ maps H(E;F) into $H(E, \mathcal{P}(^nE;F))$.

For $f \in H(E;F)$, $x_o \in E$ and $n \geq 0$, the sum

$$(2.13) \qquad \tau_{n,f,x_o}(x) = \sum_{j=0}^{n} \frac{1}{j!} \hat{d}^j f(x_o)(x-x_o), \; x \in E$$

is called the *Taylor polynomial* of f at x_o.

Note: For $F = \mathbb{C}$, the complex plane, we *will write* $\mathcal{P}(^nE)$ and H(E) in place of $\mathcal{P}(^nE;\mathbb{C})$ and $H(E,\mathbb{C})$ respectively.

For $g \in E^*$, one can easily check that $g^n \in \mathcal{P}(^nE)$ and this leads us to

Definition 2.14: The subspace of $\mathcal{P}(^nE)$, spanned by the collection $\{ g^n : g \in E^* \}$, is *denoted by* $\mathcal{P}_f(^nE)$ such that each member of $\mathcal{P}_f(^nE)$ is known as a *polynomial* of *finite type.*

On $\mathcal{P}_f(^nE)$, other than the subspace norm of $\mathcal{P}(^nE)$, we have another stronger norm $\|\cdot\|_N$ known as the *nuclear norm* defined by

$$(2.15) \qquad \|P\|_N = \inf\{ \sum_{i=1}^{m} \|\varphi_i\|^n : P = \sum_{i=1}^{m} \varphi_i^n, \varphi_i \in E^*, i = 1, \ldots m, \}$$

where the infimum is taken over all possible representations of P.

The *completion* of $(\mathcal{P}_f(^nE), \|\cdot\|_N)$ in $\mathcal{P}(^nE)$, is the *Banach space* $\mathcal{P}_N(^nE)$ whose members are called the *nuclear* n-*homogeneous polynomials* on E.

Finally, we follow [2] and [3] in the rest of this section. For a Hausdorff TVS (X, T) and a sequentially complete l.c. TVS (Y, S) let $\mathcal{P}_a(^nX;Y)$ and $\mathcal{P}(^nX;Y)$ *denote respectively* the class of algebraic and continuous n-homogeneous polynomials from X to Y.

For an open subset u of (X, T), we have

Definition 2.16: A function $f: u \to Y$ is to be said G-*holomorphic* in u if for every $x \in u$, there exists a series $\sum_{n \geq 0} f_n$, $f_n \in \mathcal{P}_a(^nX,Y)$, $n \geq 0$ from X to Y such that

$$f(x+h) = \sum_{n \geq 0} f_n(h)$$

for all h in a neighbourhood of 0 in X.

Definition 2.17: A continuous function $f: u \to Y$ is called *holomorphic* in u if for every $x \in u$, there exists a series $\sum_{n \geq 0} fn$, $f_n \in \mathcal{P}(^nX;Y)$ such that

$$f(x+h) = \sum_{n \geq 0} f_n(h)$$

for all h in a neighbourhood of 0 in X.

Definition 2.18: A mapping $f:u \rightarrow Y$ is known as *hypoanalytic* on u if it is G-holomorphic on u and is continuous on compact subsets of u.

We *denote* by H(u) the *class* of all holomorphic mappings from u to $\mathbb{C}$ and by H_{hy} (u) the class of hypoanalytic mappings from u to $\mathbb{C}$. Clearly

(2.19) $$H(u) \subset H_{hy}(u)$$

In the sequel, we use the *symbol* τ_0 to denote the topology on H_{hy} (u) or H(u) of uniform convergence on compact subsets of u.

3. $\alpha\mu$-DUALS AND TOPOLOGIES

In this section we introduce a kind of a dual of a sequence space λ, which in particular includes the notions of α-, β-, γ- and other duals studied earlier in [1], [20], [22], [9] and [4]; indeed, for a given sequence α in ω and a sequence space μ, we define

Definition 3.1: The *$\alpha\mu$ -dual* of a sequence space λ is the subspace λ_α^μ of ω defined as

$$\lambda_\alpha^\mu = \{ b \in \omega : \alpha\, ab \in \mu, \forall a \in \lambda \}$$

If (μ, T_μ) is a locally convex sequence space with T_μ being generated by a family D_μ of seminorms, then we can topologize either of the spaces λ and λ_α^μ with the corresponding locally convex topologies $T_{\alpha\mu}$ and $T^*_{\alpha\mu}$, respectively generated by the families $\{ p_b^\alpha : p \in D_\mu, b \in \lambda_\alpha^\mu \}$ and $\{ p_a^\alpha : p \in D_\mu, a \in \lambda \}$ of seminorms where for $a \in \lambda$, $b \in \lambda_\alpha^\mu$ and $p \in D_\mu$,

(3.2) $$p_b^\alpha(a) = p_a^\alpha(b) = p(\{ \alpha_n a_n b_n \})$$

The topology $T_{\alpha\mu}$ on λ (resp. $T^*_{\alpha\mu}$ on λ_α^μ) is known as the *$\alpha\mu$-topology* on λ (resp. on λ_α^μ).

Note: From now onwards, we shall assume throughout that $\alpha \in \omega$ is such that $\alpha_n \neq$, for each $n \geq 1$. As particular case of α and μ , we have

(a) for $\alpha_n = 1, n \geq 1$, we have the following well known duals:
(i) if $\mu = \ell^1$, λ^μ_α is the Köthe dual λ^x of λ:
(ii) if $\mu = cs$, λ^μ_α is the β-dual λ^β of λ;
(iii) if $\mu = bs$, λ^μ_α is the γ-dual λ^γ of λ; and

(b) for $\alpha_n = \frac{1}{n}$, $n \geq 1$, $\mu = c_o$ and

$$\lambda = \{a \in \omega: \frac{a_n}{n} \to 0 \text{ as } n \to \infty\},$$

λ^μ_α is the space introduced by Boland (cf. [4], Definition 1.3, p. 41).

Thus different values of α and different sequence spaces μ yield various duals of λ. For an arbitrary α with $\alpha_n \neq 0, n \geq 1$, we shall study in general the impact of the structure of (μ, T_μ) on the space $(\lambda, T_{\alpha\mu})$ in this section. Let us begin with

Proposition 3.3: If (μ, T_μ) is a K-space (resp. an AK-space), then $(\lambda, T_{\alpha\mu})$ is also a K-space (resp. an AK-space). Similar result holds for the space $(\lambda^\mu_\alpha, T^*_{\alpha\mu})$.

Proof: For showing the K-property of $(\lambda, T_{\alpha\mu})$, consider a net $\{a^\beta : \beta \in \Lambda\}$ in λ such that $a^\beta \to 0$ in $T_{\alpha\mu}$. Therefore, for $\epsilon > 0$, $p \in D_\mu$ and $b \in \lambda^\mu_\alpha$, there exists an index $\beta_o = \beta_o(\epsilon, p, b)$ in Λ such that

$$p^\alpha_b (a^\beta) = p(\{\alpha_n a^\beta_n b_n\}) < \epsilon, \forall \beta \geq \beta_o.$$

Thus $\alpha_a\beta\, b \to 0$ in (μ, T_μ) for each $b \in \lambda^\mu_\alpha$. Consequently,

$$\alpha_n a^\beta_n b_n \to 0, \forall n \geq 1 \text{ and } \{b_n\} \in \lambda^\mu_\alpha .$$

$$\Longrightarrow \quad a^\beta_n \to 0 \quad , \forall\, n \geq 1.$$

Hence $(\lambda, T_{\alpha\mu})$ is a K-space.

The AK-ness of λ is immediate from the equality

$$p_b^{\alpha}(a^{(n)}-a) = p(\gamma^{(n)} - \gamma),$$

where $a \in \lambda$, $b \in \lambda^{\mu}_{\alpha}$, $\gamma = \alpha\ ab \in \mu$ and $p \in D_{\mu}$.

The result for $(\lambda^{\mu}_{\alpha}, T^{*}_{\alpha\mu})$ follows analogously.

Regarding $\alpha\mu$-duals, let us introduce.

Definition 3.4: A sequence space λ is said to be *$\alpha\mu$-perfect* if $\lambda = \lambda^{\mu\mu}_{\alpha\alpha}$, where $\lambda^{\mu\mu}_{\alpha\alpha} = (\lambda^{\mu}_{\alpha})^{\mu}_{\alpha} = \{ c \in \omega: \alpha bc \in \mu$, for each $b \in \lambda^{\mu}_{\alpha} \}$, the $\alpha\mu$-dual of λ^{μ}_{α}.

Definition 3.4: A sequence space λ is said to be *$\alpha\mu$-perfect* if $\lambda = \lambda^{\mu\mu}_{\alpha\alpha}$, where $\lambda^{\mu\mu}_{\alpha\alpha} = (\lambda^{\mu}_{\alpha})^{\mu}_{\alpha} = \{ c \in \omega: \alpha bc \in \mu$, for each $b \in \lambda^{\mu}_{\alpha}\}$, the $\alpha\mu$-dual of λ^{μ}_{α}.

Remark: The $\alpha\mu$-dual of a sequence space is always $\alpha\mu$-perfect.

For an arbitrary sequence space, we have

Proposition 3.5: Let (μ, T_{μ}) be an AK-space. If $(\lambda, T_{\alpha\mu})$ is sequentially complete, then λ is $\alpha\mu$-perfect.

Proof: For proving the result, we just need show that $\lambda^{\mu\mu}_{\alpha\alpha} \subset \lambda$, for the other inclusion is always true. Let us therefore, take an element c in $\lambda^{\mu\mu}_{\alpha\alpha}$. Then

$$c^{(n)} \in \lambda, \forall\, n \geq 1.$$

Also, for b in λ^{μ}_{α}, $\gamma^{(n)} \to b$ in T_{μ}, where $\gamma = \alpha bc$. Therefore, for $P \in D_{\mu}$ and $m < n$, the equality

$$p_b^{\alpha}\,(c^{(n)} - c^{(m)}) = p\,(\gamma^{(n)} - \gamma^{(m)}),$$

yields that $\{ c^{(n)} \}$ is a Cauchy sequence in $(\lambda, T_{\alpha\mu})$.

Hence there exists an s in λ such that

$$c^{(n)} \to s \ \text{ as } n \to \infty$$

relative to $T_{\alpha\mu}$. But $s = c$ by Proposition 3.2 and the fact that $c_i^{(n)} = c_i$, for each $n \geq i$. Hence $\lambda^{\mu\mu}_{\alpha\alpha} = \lambda$.

Remark. Before we prove a partial converse of Proposition 3.5, let us note that none of the conditions, namely, AK-ness of the space (λ, T_μ) and the sequential completeness of $(\lambda, T_{\alpha\mu})$ is indispensable in the hypothesis of the above result; for we have

Example 3.6. Let μ be the non-AK-space ℓ^∞ of all bounded sequences equipped with the usual supnorm topology and λ be the space c_o of all null sequences. Choose $\alpha = e$. Then one can easily verify

$$\lambda^\mu_\alpha = \ell^\infty \text{ and } \lambda^{\mu\mu}_{\alpha\alpha} = \ell^\infty.$$

Thus λ is not $\alpha\mu$-perfect. However, the space $(c_o, T_{e\ell^\infty})$ is complete; indeed, $T_{e\ell^\infty}$ which is generated by the family of seminorms

$$p^e_b(a) = \sup_{n \geq 1} |a_n b_n|, \ a \in c_o, b \in \ell^\infty$$

is equivalent to the supnorm topology of c_o.

Example 3.7: Let (μ, T_μ) be the AK-space φ equipped with the supnorm $\|\cdot\|_\infty$ and λ be c_o. For $\alpha = e$, we have

$$\lambda^\mu_\alpha = \varphi \quad \text{ and } \lambda^{\mu\mu}_{\alpha\alpha} = \omega.$$

The topology $T_{e\varphi}$ on c_o is $\eta(c_o, \varphi)$ and so the space $(\lambda, T_{\alpha u}) \equiv (c_o, \eta(c_o, \varphi))$ is not sequentially complete [indeed, $\{ e^{(n)} : n \geq 1 \}$, where $e^{(n)} = \{ 1, 1, \ldots, \underset{\text{n-th place}}{1}, 0, 0, \ldots \}$ is a nonconvergent $\eta(c_o, \varphi)$ –Cauchy sequence in c_o]. Observe that c_o is not $\alpha\mu$ –perfect.

On the contrary, the following example illustrates that the AK-ness of the space (μ, T_μ) is not a necessary condition in Proposition 3.5.

Example 3.8: Let (μ, T_μ) be as in Example 3.6 and λ be φ. For $\alpha = e$,

$$\lambda_{\alpha}^{\mu} = \omega \quad \text{and} \quad \lambda_{\alpha\alpha}^{\mu\mu} = \varphi \quad .$$

Thus λ is $\alpha\mu$-perfect. Also, the topology $T_{e\ell^\infty}$ which is generaled by the family of seminorms

$$p_b^e(a) = \sup_{n \geq 1} | a_n b_n | , a \in \varphi, b \in \omega$$

is nothing but the normal topology $\eta(\varphi, \omega)$ [indeed, $T_{e\ell^\infty} \subset \eta(\varphi, \omega)$ is clear; for the other inclusion, use the nuclearity criterion of $(\varphi, \eta(\varphi, \omega))$, cf. [15], p. 288; or equivalently the fact that for each $b \in \omega$, $b_n \geq 0$, there exists $c \in \omega$, $c_n \geq 0$, such that $\{ b_n/c_n \} \in \ell^1$]. Therefore, $(\varphi, T_{e\ell^\infty})$ is complete, cf. [15], p. 83.

Converse of Proposition 3.5 is obtained in the form of.

Proposition 3.9: Let (μ, T_μ) be a complete (resp.sequentially complete) K-space. If λ is $\alpha\mu$-perfect, then $(\lambda, T_{\alpha\mu})$ is complete (resp. sequentially complete).

Proof. Let us prove the result for completeness; the part for sequential completeness follows analogously.

Let $\{ a^\beta : \beta \in \Lambda \}$ be a $T_{\alpha\mu}$-Cauchy net in λ. Then by Proposition 3.2, there exists a sequence $\{ a_n \} \subset$ IK such that

$$(*) \qquad a_n^\beta \to a_n , \forall n \geq 1.$$

For $b \in \lambda_\alpha^\mu$, write $\gamma^\beta = \alpha a^\beta b$, $\beta \in \Lambda$. Then $\{ \gamma^\beta : \beta \in \Lambda \}$ is a Cauchy net in (μ, T_μ) and so for some s in μ,

$$(**) \qquad \gamma^\beta \to s \text{ in } T_\mu.$$

Hence from (*) and (**), $s = \{ \alpha_n a_n b_n \}$ and so $\{ \alpha_n a_n b_n \} \in \mu$. As $b \in \lambda_\alpha^\mu$ is arbitrary, it follows that $a \in \lambda_{\alpha\alpha}^{\mu\mu} = \lambda$.

Also, from (**) we have that $a^\beta \to a$ in $T_{\alpha\mu}$. Thus $(\lambda, T_{\alpha\mu})$ is complete.

Remark. Neither the completeness of (μ, T_μ) nor the $\alpha\mu$-perfectness of λ can be dropped in the above proposition; for we have

Example 3.10. Let $\mu = \ell^1$ and T_μ be the supnorm topology on ℓ^1. Further, take $\lambda = \ell^1$ and $\alpha = e$. Then

$$\lambda_\alpha^\mu = \ell^\infty \text{ and } \lambda_{\alpha\alpha}^{\mu\mu} = \ell^1.$$

Thus λ is $\alpha\mu$-perfect; however $(\lambda, T_{\alpha\mu})$ is not complete, for the topology $T_{\alpha\mu}$, in this case, is the same as T_μ.

Example 3.11. This is the well known example of $\lambda = c_o$ and $(\mu, T_\mu) = (\ell^1, \|\cdot\|_1)$, where $\|\cdot\|_1$ is the usual norm on ℓ^1. For $\alpha = e$,

$$\lambda_\alpha^\mu = \ell^1 \text{ and } \lambda_{\alpha\alpha}^{\mu\mu} = \ell^\infty.$$

Thus c_o is not $\alpha\mu$-perfect. As $T_{\alpha\mu} = \eta(c_o, \ell^1)$, the space $(c_o, T_{\alpha\mu})$ is not complete [cf. also [15], p. 83]

On the other hand, completeness of (μ, T_μ) is not a necessary condition as illustrated in

Example 3.12. Consider the incomplete space $(\varphi, \|\cdot\|_\infty)$ as (μ, T_μ) and $\lambda = \varphi$. For $\alpha = e$,

$$\lambda_\alpha^\mu = \omega \text{ and } \lambda_{\alpha\alpha}^{\mu\mu} = \varphi.$$

Hence λ is $\alpha\mu$-perfect. Also, the space $(\varphi, T_{\alpha\mu})$ is complete since $T_{\alpha\mu} = \eta(\varphi, \omega)$; of Example 3.8.

Since λ_α^μ is always $\alpha\mu$-perfect, a consequence of Proposition 3.9 is contained in.

Corollary 3.13. If (μ, T_μ) is a complete (resp. sequentially complete) K-space, then $(\lambda_\alpha^\mu, T_{\alpha\mu}^*)$ is complete (resp. sequentially complete).

Combining Propositions 3.5 and 3.9, we get a characterization of $\alpha\mu$-perfectness exhibited in

Theorem 3.14. For a sequentially complete AK-space (μ, T_μ), λ is $\alpha\mu$-perfect if and only if $(\lambda, T_{\alpha\mu})$ is sequentially complete.

Boundedness. It is clear that a subset A of λ is $T_{\alpha\mu}$-bounded if and only if the set $Ab\alpha = \{\{a_i b_i \alpha_i\} : a \in A\}$ is bounded in (μ, T_μ) for each $b \in \lambda_\alpha^\mu$. Replacing the singleton set $\{b\}$ in λ_α^μ by a $T_{\alpha\mu}^*$-bounded subset of λ_α^μ, we introduce

Definition 3.15. A subset A of λ is said to be *completely bounded* in λ if for each $T_{\alpha\mu}^*$-bounded subset B of λ_α^μ, the set

$$AB\,\alpha = \{ \{ a_i\, b_i\, \alpha_i \} : a \in A, b \in B \}$$

is bounded in(μ, T_μ).

Remarks. Clearly, every completely bounded set in λ is $T_{\alpha\mu}$-bounded. The converse, given in Proposition 3.17, makes use of the following result reproduced from [10].

Lemma 3.16. A subset B of an l.c. TVS (X, T) is bounded in X if and only if for every sequence $\{ x_n \} \subset B$ and $\{ \alpha_n \} \in \ell^1$, the sequence $\{ \sum_{i=1}^{n} \alpha_i x_i \}$ is a Cauchy sequence in X.

Proposition 3.17. Let (μ, T_μ) be a sequentially complete K-space. Then every $T_{\alpha\mu}$-bounded subset of λ is completely bounded.

Proof. Let us assume that the result is not true. Then there exists a $T_{\alpha\mu}$-bounded subset A of λ, which is not completely bounded. Hence we can find a $T^*_{\alpha\mu}$-bounded set B in λ^μ_α such that $AB\,\alpha = \{ ab\,\alpha\colon a \in A, b \in B \}$ is unbounded in (μ, T_μ). Consequently, for given $\epsilon > 0$ and $p \in D_\mu$, there exists $a^1 \in A$, $b^1 \in B$ with the property that

$$p(a^1 b^1 \alpha) \geq 1 + \epsilon.$$

As A and B are bounded in $(\lambda, T_{\alpha\mu})$ and $(\lambda^\mu_\alpha, T^*_{\alpha\mu})$ respectively, there exist constants $L_1 > 0$ and $M_1 > 0$ satisfying

$$\sup_{a \in A} p(\alpha a b^1) \leq L_1$$

and

$$\sup_{b \in B} p(\alpha a^1 b) \leq M_1 .$$

Choose $m_1 \in \mathbb{N}$ such that

$$2^{-m_1+1} < \epsilon/M_1 .$$

From the unboundedness of AB α, choose $a^2 \in A$ and $b^2 \in B$ such that

$$p(a^2 b^2\, \alpha) \geq 2^{m_1} (L_1 + 2 + \epsilon).$$

Corresponding to the points $a^2 \in A$ and $b^2 \in B$, we can now find constants $L_2 > 0$ and $M_2 > 0$ such that

$$\sup_{a \in A} p(\alpha a b^2) \leq L_2$$

and

$$\sup_{b \in B} p(\alpha a^2 b) \leq M_2.$$

Choose $m_2 > m_1$ such that

$$2^{-m_2+1} < \epsilon/M_2.$$

As above for the constant $2^{m_2}(L_1+2^{-m_1}L_2+3+\epsilon)$, select $a^3 \in A, b^3 \in B$ and then the constants $L_3 > 0, M_3 > 0$ satisfying the relations

$$p(\alpha a^3 b^3) \geq 2^{m_2}(L_1+2^{-m_1}L_2+3+\epsilon),$$

$$\sup_{a \in A} p(\alpha a b^3) \leq L_3$$

and

$$\sup_{b \in B} p(\alpha a^3 b) \leq M_3.$$

Then consider $m_3 > m_2$ with the property

$$2^{-m_3+1} < \epsilon/M_3.$$

Continuing this process, we get sequences $\{a^n\} \subset A$, $\{b^n\} \subset B$, constants $L_n > 0$, $M_n > 0$ and an increasing sequence $\{m_n\}$ of integers such that the following four inequalities hold:

$$p(\alpha a^n b^n) \geq 2^{m_{n-1}} \left(\sum_{i=0}^{n-1} 2^{-m_{i-1}} L_i + n + \epsilon \right), L_o = m_o = 0;$$

$$\sup_{a \in A} p(\alpha a b^n) \leq L_n;$$

$$\sup_{b \in B} p(\alpha a^n b) \leq M_n;$$

and

$$2^{-m_n+1} \leq \epsilon/m_n$$

for n = 1,2,3,...

Now using Lema 3.16 we infer that the sequence $\{\sum_{i=1}^{n} 2^{-m_{i-1}} b^i\}$ is a $T^*_{\alpha\mu}$-Cauchy sequence in λ^μ_α and so by Corollary 3.13, there exists b^o in λ^μ_α such that

$$b^o = T^*_{\alpha\mu} - \lim \sum_{i=1}^{n} 2^{-m_{i-1}} b^i = \sum_{i\geq 1} 2^{-m_{i-1}} b^i.$$

Then for $n \geq 1$,

$$p^\alpha_{b^o}(a^n) = p(\alpha a^n \sum_{i\geqslant 1} 2^{-m_{i-1}} b^i)$$

$$\geq 2^{-m_{n-1}} p(\alpha a^n b^n) - p(\alpha a^n b^1)$$

$$- 2^{-m_1} p(\alpha a^n b^2) - \dots - 2^{-m_{n-2}} p(\alpha a^n b^{n-1})$$

$$- 2^{-m_n} M_n (1 + 2^{m_n - m_{n+1}} + 2^{m_n - m_{n+2}} + \dots)$$

$$\geq (\sum_{i=0}^{n-1} 2^{-m_{i-1}} L_i + n + \epsilon) - L_1 - 2^{-m_1} L_2 - \dots$$

$$- 2^{-m_{n-2}} L_{n-1} - \epsilon$$

$$= n$$

Hence A is $T_{\alpha\mu}$-unbounded. This contradiction proves the result.

Note. Let us observe in the following example that the conclusion of Proposition 3.17 may hold even if (μ, T_μ) is not sequentially complete.

Example 3.18. Let α, λ and (μ, T_μ) be as in Example 3.12. Here $T_{\alpha\mu} = \eta(\varphi, \omega)$ and $T^*_{\alpha\mu} = \eta(\omega, \varphi)$. If A and B are respectively $T_{\alpha\mu}$ - and $T^*_{\alpha\mu}$-bounded sets, then using the characterizations of $\eta(\varphi, \omega)$ and $\eta(\omega, \varphi)$-bounded sets (cf. [15], p. 104 and p. 106), one can easily verify that AB is bounded in $(\varphi, \|\,\|_\infty)$.

Remark. The vector-valued analogues of the results of this section for a fixed α, namely $\alpha = e$, are to be found in [11].

4. SPACES OF HOLOMORPHIC MAPPINGS

In this section we study several subspaces of the class H (E) of holomorphic mappings (cf. Section 2) defined corresponding to an arbitrary normal sequence space μ, a nonzero α in ω and the $\alpha\mu$-dual of a sequence space λ. Indeed, we endow these subspaces with locally convex topologies in order to study their topological behaviour and also to characterize the bounded and relatively compact subsets. In this section, we consider sequences defined over $\mathbb{N}_0$.

To be precise, let us assume throughout that μ *denotes* a *normal sequence space* equipped with a Hausdorff locally convex topology T_μ generated by the family D_μ of *solid seminorms,* α a sequence in ω with $\alpha_n \neq 0$, $n \geqslant 0$ and λ a sequence space. Then we introduce the spaces

(4.1) $$H^\mu(E) = \{ f \in H(E) : \hat{d}^n f(0) \in \mathcal{P}(^nE), n \geq 0 \text{ with } \{ (\frac{\|\hat{d}^n f(0)\|}{n!})^{1/n} \} \in \mu \} ;$$

(4.2) $$H^\mu_N(E) = \{ f \in H(E) : \hat{d}^n f(0) \in \mathcal{P}_N(^nE), n \geqslant 0 \text{ with } \{ (\frac{\|\hat{d}^n f(0)\|_N}{n!})^{1/n} \} \in \mu \} ;$$

(4.3) $$H^\mu_\alpha(E;\lambda) = \{ f \in H^\mu(E) : \{ \|\hat{d}^n f(0)\|^{1/n} \} \in \lambda^\mu_\alpha \};$$

and

(4.4) $$H^\mu_{N\,\alpha}(E;\lambda) = \{ f \in H^\mu_N(E) : \{ \|\hat{d}^n f(0)\|_N^{1/n} \} \in \lambda^\mu_\alpha \}.$$

The *power* $1/n$ wherever it appears for $n = 0$, means 1. Clearly,

$$H^{\mu}_{N}(E;\lambda) \subset H^{\mu}(E)$$

and

$$H^{\mu}_{N\,\alpha}(E;\lambda) = H^{\mu}_{\alpha}(E) \cap H^{\mu}_{N}(E).$$

Let us equip the spaces defined in (4.1), (4.2), (4.3) and (4.4) with the Hausdorff locally convex topologies T_h, T^N_h, $T_{h\alpha}$ and $T^N_{h\,\alpha}$ respectively generated by the families $D_h = \{ Q_p : p \in D_\mu \}$, $D^N_h = \{ Q^N_p : p \in D_\mu \}$, $D^\lambda_h = \{ Q^a_p : p \in D_\mu, a \in \lambda \}$ and $D^{N\lambda}_h = \{ Q^N_{p,a} : p \in D_\mu, a \in \lambda \}$ where for $p \in D_\mu$, $a \in \lambda$,

(4.5) $$Q_p(f) = p\left(\left\{ \left(\frac{\|\hat{d}^n f(0)\|}{n!} \right)^{1/n} \right\} \right), f \in H^{\mu}(E).$$

(4.6) $$Q^N_p(f) = p\left(\left\{ \left(\frac{\|\hat{d}^n f(0)\|_N}{n!} \right)^{1/n} \right\} \right), f \in H^{\mu}_N(E)$$

(4.7) $$Q_{p,a}(f) = p\left(\left\{ \|\hat{d}^n f(0)\|^{1/n} \alpha_n a_n \right\} \right), f \in H^{\mu}_{\alpha}(E;\lambda).$$

(4.8) $$Q^N_{p,a}(f) = p\left(\left\{ \|\hat{d}^n f(0)\|_N^{1/n} \alpha_n a_n \right\} \right), f \in H^{\mu}_{N\alpha}(E;\lambda).$$

Concerning the spaces (4.1) and (4.2), we have

Proposition 4.9: Let (μ, T_μ) be a complete K-space such that $p_o(e^n) = 1$, for each $n \geq 0$ and some $p_o \in D_\mu$. Then the space $(H^{\mu}_N(E), T^N_h)$ [resp. $(H^{\mu}(E), T_h)$] is quasi-complete.

Proof: For proving the quasi-completeness of $(H^{\mu}_N(E), T^N_h)$, consider a T^N_h –bounded Cauchy net $\{ f_\beta : \beta \in \wedge \}$ in $H^{\mu}_N(E)$. Fix $\epsilon > 0$ and $p \in D_\mu$ Then there exists β_o depending on ϵ and p such that

$$p\left(\left\{ \left(\frac{\| \hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0) \|_N}{n!} \right)^{1/n} \right\} \right) < \epsilon, \beta, \gamma \geq \beta_o . \quad (+)$$

Since p is monotone, it follows from (+) that the net $\{ a^\beta : \beta \in \Lambda \}$, where

$$a^{\beta} = \{ (\frac{||\hat{d}^n f_{\beta}(0)||_N}{n!})^{1/n} \}, \beta \in \Lambda$$

is a Cauchy net in μ. Hence there exists an element a in μ such that

$$a^{\beta} \to a \text{ in } T_{\mu}$$

$$a_n^{\beta} \to a_n, \forall \; n \geq 0 \qquad (*)$$

Applying (+) for p = p_o and using the monotone character of p_o, we get

$$p_o\,((\frac{||\hat{d}^n f_{\beta}(0) - \hat{d}^n f_{\gamma}(0)||_N}{n!})^{1/n} e^n) < \epsilon, \beta, \gamma \geq \beta_o.$$

But $p_o(e^n) = 1, n \geq 0$; therefore, the net $\{\hat{d}^n f_{\beta}(0): \beta \in \wedge\}$ is Cauchy in $\mathcal{T}_N(^nE)$, for $n \geq 0$. Hence we can find a sequence $\{P_n\}$ of polynomials, $P_n \in \mathcal{T}_N(^nE), n \geq 0$ such that

$$P_n = \lim_{\beta} \hat{d}^n f_{\beta}(0). \qquad (**)$$

Thus from (*) and (**), we get

$$a_n = (\frac{||P_n||_N}{n!})^{1/n}, \; n \geq 0.$$

Hence $\{(||P_n||_N/n!)^{1/n}\} \in \mu$. Let

$$Q_n(x) = \frac{P_n(x)}{n!}, \; n \geq 0.$$

Clearly, $Q_n \in \mathcal{T}_N(^nE), n \geq 0$. We now show that the sequence $\{(\frac{||Q_n||}{n!})^{1/n}\}$ is bounded so that a function f in $H_N^{\mu}(E)$ could be defined as follows:

$$f(x) = \sum_{n \geq 0} Q_n(x), \quad x \in E. \qquad (***)$$

To prove this, observe that for the p_o of the hypothesis, there exists a cons- .tant $K \equiv K(p_o)$ such that

$$p_o\left(\left\{\left(\frac{\| \hat{d}^n f_\beta(0) \|_N}{n!}\right)^{1/n}\right\}\right) \leq K, \forall \beta \in \Lambda$$

Using the monotonicity of p_o and the fact that $p_o(e^n) = 1, n \geq 0$, we get

$$\left(\frac{\| \hat{d}^n f_\beta(0) \|_N}{n!}\right)^{1/n} \leq K, \forall n \geq 0, \beta \in \Lambda$$

$$==> \qquad \left(\frac{\| P_n \|}{n!}\right)^{1/n} \leq K, \forall n \geq 0$$

$$==> \qquad \left(\frac{\| Q_n \|}{n!}\right)^{1/n} = \left(\frac{\| P_n \|}{(n!)^2}\right)^{1/n} \leq K, \forall n \geq 0$$

Thus $f \in H^\mu(E)$. Since $\{ (\| P_n \|_N / n!)^{1/n} \}$ is in μ, $f \in H^\mu_N(E)$.

Finally, it remains to show that $f_\beta \to f$ in T^N_h. Define a net $\{ b^\beta, \beta \in \Lambda \}$ in μ as follows:

$$b^\beta = \left\{\left(\frac{\| \hat{d}^n f_\beta(0) - \hat{d}^n f(0) \|_N}{n!}\right)^{1/n}\right\}, \beta \in \Lambda$$

Then from the monotonocity of each member of D_μ and the relation (+), it follows that $\{ b^\beta : \beta \in \Lambda \}$ is a Cauchy net in (μ, T_μ). Hence there exists $b \in \mu$ such that

$$b^\beta \to b \text{ in } T_\mu$$

$$==> \qquad b^\beta_n \to b_n, \forall n \geq 0$$

But

$$b_n^\beta = \left(\frac{\| \hat{d}^n f_\beta(0) - \hat{d}^n f(0) \|_N}{n!} \right)^{\frac{1}{n}}$$

$$\to 0 \qquad , \text{for each } n \geq 0.$$

Hence $b_n = 0, \forall\, n \geq 0$. Thus $b^\beta \to 0$ in (μ, T_μ), or equivalently $f_\beta \to f$ in T_h^N. Hence the space $(H_N^\mu(E), T_h^N)$ is quasi-complete.

Proceeding exactly on similar lines, the quasi-completeness of the space $(H^\mu(E), T_h)$ also follows.

For our next result, we need to introduce

Definition 4.10. A sequence space λ is said to have *G-property* if λ contains an element b satisfying the condition

$$|b_n| > \frac{1}{|\alpha_n| \, n!^{1/n}} \quad , \forall n \geq 0.$$

Proposition 4.11: Let (μ, T_μ) be as in Proposition 4.9, λ a sequence space with G-property and $\alpha \in \omega$ with $\alpha_n \neq 0, n \geq 0$. Then the spaces $(H_{N\alpha}^\mu(E;\lambda), T_{h\alpha}^N)$ and $(H_\alpha^\mu(E;\lambda), T_{h\alpha})$ are quasi-complete.

Proof: The proof of this result is not very different from that of Preposition 4.9; however, we outline the same for the sake of completeness. Indeed, we prove the quasi-completeness of $(H_{N\alpha}^\mu(E;\lambda), T_{h\alpha}^N)$; the result for the space $(H^\mu_\alpha(E;\lambda), T_{h\alpha})$ being true on similar lines.

Let $\{ f_\beta : \beta \in \Lambda \}$ be a $T_{h\alpha}^N$–bounded Cauchy net in $H_{N\alpha}^\mu(E;\lambda)$. Then for arbitrarily fixed $a \in \lambda$, $\epsilon > 0$ and $p \in D_\mu$, there exists $\beta_o \equiv \beta_o(\epsilon, p, a)$ such that

$$p(\{ \| \hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0) \|_N^{1/n} \alpha_n a_n \}) < \epsilon . \tag{$*$}$$

for all $\beta, \gamma \geq \beta_o$. Write

$$\delta_a^\beta = \{ \| \hat{d}^n f_\beta(0) \|_N^{1/n} \alpha_n a_n \} \quad , \beta \in \Lambda$$

Then the net $\{ \delta_a^\beta : \beta \in \Lambda \}$ is a Cauchy net in (μ, T_μ) and so we get $\delta_a \in \mu$ such that

$$\delta_a = \lim_\beta \delta_a^\beta$$

$$\delta_{a,n} = \lim_\beta \delta_{a,n}^\beta \quad , \forall\, n \geq 0.$$

On the other hand, choosing $p = p_o$ in $(*)$, we can easily show that $\{ \hat{d}^n f_\beta (0) : \beta \in \wedge \}$ is a Cauchy net in $\mathcal{P}_N ({}^nE)$, for each $n \geq 0$. So, there is sequence $\{ P_n \}$ of nuclear polynomials, $P_n \in \mathcal{P}_N ({}^nE)$ such that

$$P_n = \lim_\beta \hat{d}^n f_\beta(0) \ , \ n \geq 0.$$

Therefore

$$\delta_{a,n} = || P_n ||_N^{1/n} \alpha_n a_n \ , \ \forall\, n \geq 0.$$

As $\delta_a \in \mu$ and $a \in \lambda$ is arbitrary, it follows that $\{ || P_n ||_N^{1/n} \} \in \lambda_\alpha^\mu$.

For $n \geq 0$, define

$$Q_n (x) = \frac{P_n (x)}{n!} \quad ,$$

We now prove the boundedness of the sequence $\{ (\frac{|| Q_n ||}{n!})^{1/n} \}$. Observe that for the given a and p_o as in the hypothesis, there is a constant K depending on a and p_o such that

$$|| \hat{d}^n f_\beta (0) ||_N^{1/n} \ p_o (\alpha_n a_n e^n) \leqslant K, \forall\, n \geqslant 0.$$

Using the G - property of λ , we get

$$\left(\frac{\| \hat{d}^n f_\beta(0) \|_N}{n!}\right)^{1/n} \leqslant K, \forall\, n \geqslant 0$$

$$\Longrightarrow \quad \left(\frac{\| P_n \|}{n!}\right)^{1/n} \leqslant K, \forall\, n \geqslant 0$$

$$\Longrightarrow \quad \left(\frac{\| Q_n \|}{n!}\right)^{1/n} \leqslant K, \forall\, n \geqslant 0.$$

Thus the function f defined by

$$f(x) = \sum_{n \geqslant 0} Q_n(x), x \in E$$

is a member of H (E). Since μ is normal and λ has the G-property, $(\| P_n \|_{N/n!})^{1/n} \} \in \mu$; consequently $f \in H^{\mu}_{N\alpha}(E;\lambda)$.

The convergence of the net $\{ f_\beta : \beta \in \Lambda \}$ to f in the topology $T^N_{h\alpha}$ follows analogously as in the proof of the preceding result; however, in place of the net $\{ b^\beta : \beta \in \Lambda\}$, for $a \in \lambda$ we consider here the net $\{ b^\beta_a : \beta \in \Lambda\}$ in μ , where

$$b^\beta_a = \{ \| \hat{d}^n f_\beta(0) - \hat{d}^n f(0) \|_N^{1/n} \alpha_n a_n \}$$

and show that

$$\lim_\beta b^\beta_a = 0.$$

This establishes the quasi-completeness of the space $(H^{\mu}_{N\alpha}(E;\lambda), T^N_{h\alpha})$.

Remark: If μ, α and λ be as in (b) of the Note given after Definition 3.1, the spaces $H^{\mu}_{N\alpha}(E;\lambda)$ and $H^{\mu}_{N}(E;\lambda)$ are the ones introduced and studied by Boland in [4], Section II; and therefore, our Proposition 4.11 includes his result ([4], Proposition 2.1, p. 49) as a particular case.

Bounded Sets in $H^{\mu}_{N\alpha}(E;\lambda)$ *and* $H^{\mu}_{\alpha}(E;\lambda)$: In this subsection we characterize bounded subsets of the spaces $H^{\mu}_{N\alpha}(E;\lambda)$ and $H^{\mu}_{\alpha}(E;\lambda)$, and obtain results exhibiting the equivalence of two topologies induced on a bounded subset of $H^{\mu}_{N\alpha}(E;\lambda)$ [resp. $H^{\mu}_{\alpha}(E;\lambda)$] by $T^{N}_{h\alpha}$ [resp. $T_{h\alpha}$] and T_h.

In this subsection we consider sequences indexed over $\mathbb{N}_o$. For a subset B of $H^{\mu}_{N\alpha}(E;\lambda)$, we introduce the notation

$$b_o = \sup \{ |f(0)| : f \in B \}$$

$$b_n^n = \sup \{ ||\hat{d}^n f(0)||_N : f \in B \}, n \geq 1. \tag{4.12}$$

In the sequel, the symbol b wherever it is used, will stand for the sequence $\{ b_n \}$ as introduced in (4.12).

Let us begin with

Proposition 4.13: A subset B of $H^{\mu}_{N\alpha}(E;\lambda)$ is bounded if $b \in \lambda^{\mu}_{\alpha}$.

Proof: As D_μ contains monotone seminorms, we have

$$Q^{N}_{p,a}(f) \leq p(ab\alpha)$$

for each f in B, p in D_μ and $a \in \lambda$. Therefore B is bounded.

The situation for the validity of the converse of Proposition 4.13 is not so pleasant in general; however, restriction on μ or on both the spaces μ and λ, leads to the following three different variations of the converse:

Proposition 4.14: Let B be a bounded subset of $H^{\mu}_{N\alpha}(E;\lambda)$. If μ contains e and $p_o(e^n) = 1, n \geq 0$, for some $p_o \in D_\mu$, then $b \in \lambda^{\mu}_{\alpha}$.

Proof: Since B is bounded, for p_o as in the hypothesis and $a \in \lambda$, there exists a constant $K \equiv K(a, p_o)$ such that

$$p_o(\{ ||\hat{d}^n f(0)||_N^{1/n} \alpha_n a_n \}) \leqslant K, \forall f \in B$$

$$\Longrightarrow \quad |\alpha_n a_n b_n| \leq K, \qquad \forall n \geq 0$$

as p_o is monotone and $p_o(e^n) = 1, n \geq 0$. Hence $\alpha a b \in \mu$ for each a in λ; consequently, $b \in \lambda^{\mu}_{\alpha}$.

For our next result, we make use of.

Lemma 4.15: For a normal sequence space λ with $(\lambda^x, \eta(\lambda^x, \lambda))$ nuclear

$$\lambda^x = \{ b \in \omega : \sup_{n \geq 0} |a_n b_n| < \infty, \forall a \in \lambda \}$$

$$= \{ b \in \omega : |a_n b_n| \to 0 \text{ as } n \to \infty, \forall a \in \lambda. \}$$

Proof: Immediate from Theorem 2.4.

Proposition 4.16: Let B be a bounded subset of $H^{\mu}_{N\alpha}(E; \lambda)$. If $(\mu^x, \eta(\mu^x, \mu))$ is a perfect nuclear sequence space, then $b \in \lambda^{\mu}_{\alpha}$.

Proof: Let $b \notin \lambda^{\mu}_{\alpha}$. Then there exists a in λ such that $ab\alpha \notin \mu$. Using Lemma 4 15, we can find c in μ^x such that

$$\sup_n |a_n b_n c_n \alpha_n| = \infty \quad .$$

Therefore, for each $i \geq 0$, there exists a subsequence $\{ n_i \}$ of $\mathbb{N}$ such that

$$| a_{n_i} b_{n_i} c_{n_i} \alpha_{n_i} | > 2^i , \; i \geq 0$$

$$\text{or,} \quad \sup_{f \in B} \| \hat{d}^{n_i} f(0) \|_N^{1/n_i} | a_{n_i} c_{n_i} \alpha_{n_i} | > 2^i, \; i \geq 0.$$

Consequently, there exists a sequence $\{ f_i \}$ in B such that

$$\| \hat{d}^{n_i} f_i(0) \|_N^{1/n_i} | a_{n_i} c_{n_i} \alpha_{n_i} | > 2^i, \; i \geq 0$$

$$\Longrightarrow \quad Q^N_{c,a}(f_i) = \sum_{n \geq 0} \| \hat{d}^n f_i(0) \|_N^{1/n} | a_n c_n \alpha_n |$$

$$> 2^i, \quad i \geq 0.$$

This contradicts the boundedness of B in $H^{\mu}_{N\alpha}(E;\lambda)$.

Hence $b \in \lambda^{\mu}_{\alpha}$

Proposition 4.17: Let λ be a normal sequence space such that $(\lambda^x, \eta(\lambda^x, \lambda))$ is Schwartz and suppose that $\mu = c_o$. Then $b \in \lambda^{\mu}_{\alpha}$ if B is a bounded set in $(H^{\mu}_{N\alpha}(E;\lambda), T^N_{h\alpha})$.

Proof: Assume that $b \notin \lambda^{\mu}_{\alpha}$. Then we can find $a \in \lambda$ such that $ab\alpha \notin c_o$. Hence there is an $\epsilon > 0$ and an increasing sequence $\{n_i\}$ for which

$$|a_{n_i} b_{n_i} \alpha_{n_i}| > \epsilon \quad \forall\, i \geq 0.$$

Consequently, we get a sequence $\{f_i\} \subset B$ satisfying

$$||\hat{d}^{n_i} f_i(0)||_N^{1/n_i} \; |a_{n_i}\, \alpha_{n_i}| > \epsilon \quad \forall\, i \geq 0.$$

By Theorem 2.3, there exists a sequence $c \in \lambda$ such that $\{a_n/c_n\} \in c_o$. Write $\beta_n = a_n/c_n$, $n \geq 0$. Then for each $i \geq 0$,

$$\begin{aligned} Q_{||\cdot||,c}(\beta_{n_i} f_i) &= \sup_{j \geq 0} ||\hat{d}^j(\beta_{n_i} f_i)||_N^{1/j} \; |\alpha_j c_j| \\ &\geq ||\hat{d}^{n_i}(f_i)||_N^{1/n_i} \; |\beta_{n_i}\, \alpha_{n_i} c_{n_i}| \\ &> \epsilon \; . \end{aligned}$$

Hence $\beta_{n_i} f_i \nrightarrow 0$ in $T^N_{h\alpha}$ This contradicts the boundedness of B and so the result holds good.

To characterize bounded subsets of $H^{\mu}_{\alpha}(E;\lambda)$, for a subset D of $H^{\mu}_{\alpha}(E;\lambda)$ let us write

$$d_o = \sup \{ |f(0)| : f \in D \}$$

$$d_n^n = \sup \{ \| \hat{d}^n f(0) \| : f \in D \}, \quad n \geq 1.$$

Let us fix the symbol d to denote the sequence $\{ d_n : n \geq 0 \}$ as defined above. Then we have

Proposition 4.18: A subset D of $H^\mu_\alpha (E;\lambda)$ is bounded if $d \in \lambda^\mu_\alpha$.

Proposition 4.19: Let D be a bounded subset of $H^\mu_\alpha (E;\lambda)$. If μ contains e and $p_o (e^n) = 1, n \geq 0$, for some $p_o \in D_\mu$, then $b \in \lambda^\mu_\alpha$.

Proposition 4.20: Let $(\mu^x, \eta (\mu^x, \mu))$ be a perfect nuclear sequence space. Then for a bounded subset D of $H^\mu_\alpha (E;\lambda)$, $d \in \lambda^\mu_\alpha$.

Proposition 4.21: Let λ and μ be as in Proposition 4.17. Then $d \in \lambda^\mu_\alpha$ if D is a bounded set in $(H^\mu_\alpha (E;\lambda), T_{h\alpha})$.

The proofs of these results are analogous to the corresponding ones for the space $H^\mu_{N\alpha} (E;\lambda)$ and so omitted.

Since the spaces $H^\mu_\alpha (E;\lambda)$ and $H^\mu_{N\alpha} (E;\lambda)$ are respectively contained in $H^\mu (E)$ and $H^\mu_\alpha (E)$, it is natural to inquire the relationship between the original and induced topologies. In this direction, we have.

Proposition 4.22: Let λ possess G-property. Then the induced topology $T_h \mid H^\mu_\alpha (E;\lambda)$ on $H^\mu_\alpha (E;\lambda)$ [resp. $T_{h\alpha} \mid H^\mu_{N\alpha} (E;\lambda)$ on $H^\mu_{N\alpha} (E;\lambda)$] is weaker than the topology $T_{h\alpha}$ [resp. $T^N_{H\alpha}$].

Proof: Straightforward.

On the other hand, on bounded subsets we have.

Proposition 4.23: Let B [resp. D] be a bounded subset of $(H^\mu_\alpha (E;\lambda), T_{h\alpha})$ [resp. $(H^\mu_{N\alpha} (E;\lambda), T^N_{h\alpha})$]. Then the topologies induced on B [resp. on D] by $T_{h\alpha}$ and T_h [resp. $T^N_{h\alpha}$ and T^N_h] coincide provided λ contains G-property and one of the following two conditions hold:

(i) $\mu = c_o$, λ is normal and $(\lambda^x, \eta(\lambda^x, \lambda))$ is Schwartz;

(ii) The space $(\mu^x, \eta (\mu^x, \mu))$ is a perfect nuclear sequence space.

Proof: We prove the result for $(H^\mu_\alpha (E;\lambda), T_{h\alpha})$; the result for the bracketed space follows analogously.

In view of Proposition 4.22, we need prove

$$T_{h\alpha} \mid B \subset T_h \mid B . \qquad (+)$$

Let us consider the cases corresponding to the conditions (i) and (ii) separately.

(i) For proving (+), let us take a net $\{ f_\delta : \delta \in \Lambda \}$ and f such that $f_\delta \to f$ in T_h. Then for given $\epsilon > 0$, there exists δ_o in Λ such that

$$\sup_n \left(\frac{\| \hat{d}^n f_\delta (0) - \hat{d}^n f(0) \|}{n!} \right)^{1/n} < \epsilon \ , \delta \geq \delta_o .$$

$$(*) \Longrightarrow \hat{d}^n f_\delta (0) \to \hat{d}^n f(0) \text{ in } \mathcal{P}(^nE), \ \forall \ n \geq 0.$$

Also, for given a in λ, $ab\alpha \in c_o$ by Proposition 4.17 and so there exists an integer $n_o \equiv n_o(\epsilon, a)$ such that

$$|a_n b_n \alpha_n| < \epsilon/2 \quad , n \geq n_o$$

Hence for $\delta \in \Lambda$,

$$Q_{\|.\|,a}(f_\delta - f) \leq \sup_{n \leq n_o - 1} \{ \| \hat{d}^n f_\delta (o) - \hat{d}^n f(o) \|^{1/n} |\alpha_n a_n| , \epsilon \}$$

Consequently, $f_\delta \to f$ in $T_{h\alpha}$ by using (*). This proves (+).

(ii) For this case, use Proposition 4.16 to infer $b \in \lambda^\mu_\alpha$ and proceed exactly as in (i).

Relatively compact sets in $H^\mu_{N\alpha}(E;\lambda)$ *and* $H^\mu_\alpha(E;\lambda)$:

Using the above characterizations of bounded sets in the spaces $H^\mu_{N\alpha}(E;\lambda)$ and $H^\mu_\alpha(E;\lambda)$, we characterize in this subsection the relatively compact subsets of these spaces. We have

Proposition 4.24: Let $(\mu^x, \eta(\mu^x, \mu))$ be a perfect nuclear sequence space and λ possess the G-property. Then a set B [resp. D] in $(H^\mu_{N\alpha}(E;\lambda), T^N_{h\alpha})$ [resp. in $(H^\mu_\alpha(E;\lambda), T_{h\alpha})$] is relatively compact if and only if B [resp. D] is bounded and the set $\{ \hat{d}^n f(o) : f \in B \}$ [resp. $\{ \hat{d}^n f(o) : f \in D$] is relatively compact in $\mathcal{P}_N(^nE)$ [resp. in $\mathcal{P}(^nE)$] for each $n \geq 0$.

Proof: We prove the result for the space $(H^\mu_{N\alpha}(E;\lambda), T^N_{h\alpha})$; the result for the space $(H^\mu_\alpha(E;\lambda), T_{h\alpha})$ follows on similar lines.

Assume that B is not relatively compact. As the space $(H^{\mu}_{N\alpha}(E;\lambda), T^{N}_{h\alpha})$ is quasi-complete by Proposition 4.11, B is not relatively compact. Hence there exist $a \in \lambda, d \in \mu^{x}$, $\epsilon > 0$ and a sequence $\{f_n\} \subset B$ such that

$$\sum_{n \geq 0} \| \hat{d}^n f_i(0) - \hat{d}^n f_j(0) \|_N^{1/n} \, |\alpha_n a_n d_n| > \epsilon \,, \forall\, i,j \geq 0. \quad (*)$$

Also, $b \equiv \{b_n\} \in \lambda^{\mu}_{\alpha}$ by Proposition 4.16, therefore, for the above $\epsilon > 0$, there exists an integer $n_o \equiv n_o(\epsilon)$ such that

$$\sum_{n \geq n_o+1} |\alpha_n a_n b_n d_n| < \epsilon/4.$$

Consequently,

$$\sum_{n \geq n_o+1} \| \hat{d}^n f_i(0) - \hat{d}^n f_j(o) \|_N^{1/n} \, |\alpha_n a_n d_n|$$

$$\leq 2 \sum_{n \geq n_o+1} |a_n b_n \alpha_n d_n| < \epsilon/2, \forall\, i,j \geq 0.$$

Hence we have from (*),

$$\sum_{n=0}^{n_o} \| \hat{d}^n f_i(0) - \hat{d}^n f_j(0) \|_N^{1/n} \, |\alpha_n a_n d_n| \, \epsilon > \epsilon/2, \forall\, i,j \geqslant 0.$$

Thus for each pair $(i, j) \subset \mathbb{N}_o \times \mathbb{N}_o$, there exists an integer n_{ij} lying between 0 and n_o such that

$$\| \hat{d}^{n_{ij}} f_i(0) - \hat{d}^{n_{ij}} f_j(0) \|_N^{1/n} \, |\alpha_n a_n d_n| > \frac{\epsilon}{2(n_o+1)} \,. \qquad (+)$$

Consequently, the inequality (+) is satisfied for infinitely many i,j' s corresponding to the same $n \equiv n_{ij}$ lying between 0 and n_o. This contradicts the relative compactness of the set $\{ \hat{d}^n f(0) : f \in B \}$ for each $n \geq 0$.

Conversely, if B is relatively compact, it is clearly bounded. For $n \geq 0$, define linear maps $\Psi_n : H^{\mu}_{N\alpha}(E;\lambda) \rightarrow \mathcal{P}_N(^nE)$ as follows:

$$\Psi_n(f) = \hat{d}^n f(0), n \geq 0$$

Clearly, each Ψ_n is continuous. Hence the sets $\{ \hat{d}^n f(0) : f \in B \} = \Psi_n(B), n \geq 0$ are relatively compact. This completes the proof.

Proposition 4.25: Let λ be a normal sequence space with G-property such that $(\lambda^x, \eta(\lambda^x, \lambda))$ is Schwartz and $\mu = c_0$. Then $B \subset H^{\mu}_{N\alpha}(E;\lambda)$ [resp. $D \subset H^{\mu}_{\alpha}(E;\lambda)$] is $T^N_{h\alpha}$ – [resp. $T_{h\alpha}$ –] relatively compact if and only if B is $T^N_{h\alpha}$ – [resp. $T_{h\alpha}$ –] bounded and the set $\{ \hat{d}^n f(o) : f \in B \}$ [resp. $\{ \hat{d}^n f(o) : f \in D \}$] is relatively compact in $\mathcal{T}_N(^nE)$ [resp. $\mathcal{T}(^nE)$] for each $n \geq 0$.

Proof: For proving this result, proceed as in the proof of the precedint proposition.

5. HOLOMORPHIC MAPPINGS ON NUCLEAR SEQUENCE SPACES:

The holomorphic functions on nuclear spaces have considerably been studied in [5] and [6]. This section is a continuation of this study to a class of holomorphic mappings defined on an open subset of a nuclear sequence space. The main result is contained in Theorem 5.6 whose proof makes use of

Proposition 5.1: Let λ be a normal sequence space with $(\lambda^x, \eta(\lambda^x, \lambda))$ nuclear and u be a neighbourhood of zero in $(\lambda^x, \eta(\lambda^x, \lambda))$. Then there exists an absolutely convex neighbourhood v of zero and a sequence $\delta = \{\delta_n\}$ with $\delta_n > 1$, for each $n \geqslant 1$ such that $\{1/\delta_n\} \in \ell^1$ and

$$\delta v = \{ \{\delta_n b_n\} : \{b_n\} \in v \text{ and } \{\delta_n b_n\} \in \lambda \} \subset u.$$

Proof: In view of Lemma 4.15, we may assume

$$u = \{ b \in \lambda^x : \sup_n | b_n c_n | < \epsilon \},$$

for some positive $c = \{c_n\} \in \lambda$ and $\epsilon > 0$. Using Theorem 2.4, we can

find $d = \{d_n\}$ in λ with $0 \leq c_n \leq d_n$ such that $\{c_n/d_n\} \in \ell^1$
Define

$$v = \{a \in \lambda^x : \sup_n |a_n d_n| < \epsilon\}$$

and

$$\delta_n = \begin{cases} 2^n & \text{if } c_n = 0 \\ d_n/c_n & \text{if } c_n \neq 0 \end{cases}$$

Clearly, $\{1/\delta_n\} \in \ell^1$ and $\delta_n > 1$ for each $n \geq 1$. Further, one can easily check that $\delta v \subset u$.

Proposition 5.2: Let $(\lambda, \eta(\lambda, \lambda^x))$ be a barrelled nuclear sequence space and u be a normal open set in $(\lambda^x, \beta(\lambda^x, \lambda))$. If K is a compact subset of u, then there exists a sequence $\delta = \{\delta_n\}$ such that $\delta_n > 1$ for each $n \geq 1$, $\{1/\delta_n\} \in \ell^1$ and

$$\delta K = \{\{\delta_n a_n\} : a = \{a_n\} \in K\}$$

is a relatively compact subset of u.

Proof: We may assume without loss of generality that K is normal. Since $(\lambda, \eta(\lambda, \lambda^x))$ is barrelled, there exists a positive sequence $b = \{b_n\} \in \lambda^x$ such that

$$K \subset \{a \in \lambda : \sum_{n \geq 0} |a_n b_n| \leq 1\}^o$$

$$= \{a \in \lambda^x : |a_n| \leq |b_n|, \forall n \geq 1\}$$

Let c correspond to b in λ^x such that $\{b_n/c_n\} \in \ell^1$ (cf. Theorem 2.4). Define

$$v = \{a \in \lambda : \sum_{n \geq 0} |a_n c_n| \leq 1\}$$

Since K is compact, we can find $\epsilon > 0$ such that

$$K + \epsilon v^o \subset u.$$

As v is a $\eta(\lambda, \lambda^x)$ –neighbourhood of zero, it follows that $K + \epsilon v^o$ is a relatively compact subset of u. Now define $\delta = \{\delta_n\}$ as follows:

$$\delta_n = \begin{cases} 1 + \epsilon \dfrac{c_n}{b_n} & , \text{if } b_n \neq 0 \\ 2^n & , \text{otherwise.} \end{cases}$$

Clearly, $\delta_n > 1$ for each $n \geq 1$ and $\{1/\delta_n\} \in \ell^1$. Since

$$v^o = \{ a \in \lambda^x : |a_n| \leq |c_n| , \forall n \geq 1 \} ,$$

it follows from the normality of K that

$$\delta K \subset K + \epsilon \{ a \in \lambda^x : |a_n| \leq |c_n| , n \geq 1 \} \subset u.$$

This completes the proof.

The main result: We need make some preparation for the main and the last result of this section. For $r \in \mathbb{N}$, write

$$\mathbb{N}^r = \{ m \equiv \{ m_1, m_2, \ldots, m_r, 0, 0, \ldots \} : m_i \in \mathbb{N}_o, i = 1, 2, \ldots, r \} \quad (5.3)$$

and

$$\mathbb{N}^{(\mathbb{N})} = \bigcup_{r \geq 1} \mathbb{N}^r \quad (5.4)$$

Further, for $m \in \mathbb{N}^{(\mathbb{N})}$ with $m = \{ m_1, m_2, \ldots, m_r, 0, 0, \ldots \}$, define a mapping $f^m : \omega \to \mathbb{C}$ by

$$f^m(a) = \prod_{n=1}^{\infty} a_n^{m_n} \equiv a^m. \quad (5.5)$$

Then the mappings f^m, $m \in \mathbb{N}^{(\mathbb{N})}$ are known as the *monomials*.

We are now prepared to state and prove the main result contained in

Theorem 5.6: Let $(\lambda, \eta(\lambda, \lambda^x))$ be a barrelled nuclear space such that $(\lambda^x, \beta(\lambda^x, \lambda))$ is an AK-space. Then the class $H_{hy}(u)$ of hypoanalytic function defined on a normal open subset u of $(\lambda^x, \beta(\lambda^x, \lambda))$, equipped with the topology τ_o of uniform convergence on compact subsets of u, is a complete nuclear space and the set $\{f^m : m \in \mathbb{N}^{(\mathbb{N})}\}$ of monomials forms a fully ℓ^1-base for $(H_{hy}(u), \tau_o)$.

Proof: We prove this result in two parts. Whereas in Part I we prove the fully ℓ^1-basis character of the monomials, Part II exhibits that the space $(H_{hy}(u), \tau_o \ell)$ is complete and nuclear.

I. Clearly, the set $\{f^m : m \in \mathbb{N}^{(\mathbb{N})}\}$ is a countable subset of $H_{hy}(u)$.

Consider now an f in $H_{hy}(u)$ and a compact subset K of u. For $b \in u$ and $r \in \mathbb{N}$, define.

$$[b]_r = \{s \in \omega : |s_i| \leq |b_i|, 1 \leq i \leq r \text{ and } s_i = 0, i > r\}$$

As u is normal, $[b]_r$ is a finite dimensional polydisc in u. Therefore using the theory of analytic functions of several variables, we get for $s = \{s_i\} \in K$,

$$f(t) = \sum_{m \in \mathbb{N}^r} a_m t^m, \forall t \in [s]_r$$

where

$$a_m = \frac{1}{(2\pi i)^r} \int \underset{T}{\cdots} \int \frac{f(u_1, ..., u_r, 0, 0, ...)}{u_1^{m_1+1} \cdots u_r^{m_r+1}} du_1 \ldots du_2$$

$$T = \{(u_1, ..., u_r) : |u_i| = |s_i|, i = 1, ..., r\}$$

Consequently, for $m \in \mathbb{N}^r$

$$| a_m | \leq \frac{\| f \|_{[s]_r}}{| s^m |} \leq \frac{\| f \|_K}{| s^m |} \tag{5.7}$$

where for $A \subset u$, $\| f \|_A = \sup \{ | f(x) | : x \in A \}$ and $| s^m | = | s_1^{m_1} | \ldots | s_r^{m_r} |$.

Since K is compact, we have by Proposition 5.2 a $\delta = \{ \delta_n \}$ with $\delta_n > 1, n \geq 1$ and $\{ 1/\delta_n \} \in \ell^1$ such that δK is a relatively compact subset of u. Applying (5.7) to δK, we get

$$| a_m | < \frac{\| f \|_{\delta K}}{| (\delta s)^m |} \quad , \forall m \in \mathbb{N}^r$$

$$| a_m s^m | \leq \frac{\| f \|_{\delta K}}{\delta^m} \quad , \forall m \in \mathbb{N}^r.$$

As the above inequality is true for each s in K, we get

$$\sup_{s \in K} | a_m s^m | \leq \frac{\| f \|_{\delta K}}{\delta^m} \quad , \forall m \in \mathbb{N}^r \tag{5.8}$$

$$\sum_{m \in \mathbb{N}(\mathbb{N})} \sup_{s \in K} | a_m s^m | \leq \| f \|_{\delta K} \sum_{m \in \mathbb{N}(\mathbb{N})} \frac{1}{\delta^m}. \tag{5.9}$$

Since $\mathbb{N}^r \subset \mathbb{N}^{r+1}, r \geq 1$ and $\{ \frac{1}{\delta_n} \} \in \ell^1$, we have

$$\sum_{m \in \mathbb{N}(\mathbb{N})} \frac{1}{\delta^m} = \frac{1}{\prod_{n=1}^{\infty} (1 - \frac{1}{\delta_n})} = C$$

where C is a finite constant. Hence

$$\sum_{m \in \mathbb{N}(\mathbb{N})} \sup_{s \in K} | a_m s^m | \leq C \| f \|_{\delta K} . \tag{5.10}$$

Consequently, the series $\sum_{m \in \mathbb{N}(\mathbb{N})} a_m s^m$ converges in the field $\mathbb{K}$ for each s in u. Define

$$\tilde{f}(s) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m s^m \quad , s \in u \tag{5.11}$$

Then $\tilde{f}$ is clearly an hypoanalytic function on u. Moreover,

$$\tilde{f}\ (s) = f(s),\ \forall\ s \in D$$

where $D = \bigcup_{r \geq 1} D_r$, $D_r = \bigcup \{ [s]_r : s \in u \}$, is a dense subset of u. Since both the functions f and $\tilde{f}$ are continuous on compact subsets of u, it follows that $f = \tilde{f}$ on u. Hence

$$f(s) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m s^m, \forall\ s \in u \tag{5.12}$$

$$= \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m f^m (s), \forall\ s \in u.$$

In order to show that $\{ f^m : m \in \mathbb{N}^{(\mathbb{N})} \}$ is a Schauder base for $(H_{hy}(u), \tau_o)$, it suffices to prove that the series $\sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m f^m$ converges to f in the topology τ_o, the Schauder character is immediate from (5.8). Therefore consider a compact subset K of u and an $\epsilon > 0$. Then for $\delta = \{ \delta_n \}$ as above we can find a finite subset J_o of $\mathbb{N}^{(\mathbb{N})}$ such that

$$\sum_{\mathbb{N}^{(\mathbb{N})} \setminus J_o} \frac{1}{\delta^m} < \frac{\epsilon}{||f||_{\delta K}} \tag{5.13}$$

Hence for any finite subset J of $\mathbb{N}^{(\mathbb{N})}$ with $J \supset J_o$, we have

$$||f - \sum_{m \in J} a_m f^m ||_K = \sup_{s \in K} | \sum_{m \in \mathbb{N}^{(\mathbb{N})} \setminus J} a_m s^m | \tag{5.14}$$

from (5.12), (5.8) and (5.13). Thus (5.14) yields the unordered convergence of the series $\sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m f^m$ to f in the topology τ_o.

The fully ℓ^1 –character of the base $\{ f^m : m \in \mathbb{N}^{(\mathbb{N})} \}$ is immediate from (5.10) which can be written as

$$\sum_{m \in \mathbb{N}^{(\mathbb{N})}} \| a_m f^m \|_K \leq C \| f \|_{\delta K} < \infty .$$

II. Let us first prove the completeness of the space $(H_{hy}(u), \tau_o)$ and so consider a τ_o –Cauchy net $\{ f_\alpha : \alpha \in \Lambda \}$ in $H_{hy}(u)$. If

$$f_\alpha (s) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m^\alpha s^m , s \in u, \alpha \in \Lambda$$

where $a_m^{\alpha'}$ s are uniquely determined scalars in the basis expansion of f_α ' s, then $\{ a_m^\alpha : \alpha \in \Lambda \}$ is a Cauchy net in IK for each $m \in \mathbb{N}^{(\mathbb{N})}$. Hence there exists a set $\{ a_m : m \in \mathbb{N}^{(\mathbb{N})} \} \subset$ IK such that

$$a_m = \lim_\alpha a_m^\alpha , m \in \mathbb{N}^{(\mathbb{N})} \qquad (*)$$

For given $\epsilon > 0$ and a compact subset K of u, let $\alpha_o \equiv \alpha_o(\epsilon, K)$ in Λ be such that

$$\| f_\alpha - f_\beta \|_{\delta K} \leq \epsilon , \quad \alpha, \beta \geq \alpha_o .$$

where $\delta = \{ \delta_n \}$ is the one as obtained in Proposition 5.2.

Using (5.8) and (*), we get

$$| (a_m^\alpha - a_m) s^m | \leq \frac{\epsilon}{\delta^m} , \alpha \geq \alpha_o$$

for s in K and $m \in \mathbb{N}^{(\mathbb{N})}$. Hence for s in K and $\alpha \geq \alpha_o$,

$$\sum_{m \in \mathbb{N}^{(\mathbb{N})}} | a_m s^m | \leq C \epsilon + \sum_{m \in \mathbb{N}^{(\mathbb{N})}} | a_m^\alpha s^m | < \infty$$

where

$$C = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} \frac{1}{\delta^m} = \frac{1}{\prod_{n=1}^{\infty} (1-1/\delta_n)}$$

Consequently, we can define a function f on u as follows:

$$f(s) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m s^m , \quad s \in u.$$

Further,

$$\sup_{s \in K} \Big| \sum_{m \in \mathbb{N}^{(\mathbb{N})}} (a_m^{\alpha} - a_m) s^m \Big| \leq C \epsilon , \quad \alpha \geq \alpha_o$$

yields that $f_\alpha \to f$ uniformly on K. Hence f is in $H_{hy}(u)$ and it is the required τ_o –limit of $\{ f_\alpha \}$

For nuclearity, observe that the space $(H_{hy}(u), \tau_o)$ can be made topologically isomorphic to the Köthe sequence space $(\Lambda(P), T_p)$, cf. Theorem 2.5; where

$$P = \{ \{ \| f^m \|_K \}_{m \in \mathbb{N}^{(\mathbb{N})}} : K \text{ varies over compact subsets of } u \}$$

Since from (5.8) we have,

$$\| f^m \|_K \leq \frac{1}{\delta^m} \| f^m \|_{\delta K} , \quad m \in \mathbb{N}^{(\mathbb{N})}$$

where $\sum_{m \in \mathbb{N}^{(\mathbb{N})}} \frac{1}{\delta^m} < \infty$ and δK is a relatively compact subset of u, the space $(H_{hy}(u), \tau_o)$ is nuclear. This establishes the result completely.

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