JUSTIFICATION OF THE AVERAGING METHOD FOR MULTIPOINT BOUNDARY VALUE PROBLEMS FOR A CLASS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH "MAXIMUMS"

par

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ABSTRACT

The averaging method is justified for a class of multipoint boundary value problems with non-linear boundary condition for functional-differential equations with "maximums".

A number of processes in biology and economics [1] are governed by a special class of strongly non-linear functional-differential equations, the so called differential equations with "maximums" [2]. The study of the fundamental and qualitative properties of these equations is subject to specific difficulties. At the same time the preliminary application of the methods for theoretical approximation of the solutions may considerably simplify the problem.

In this paper the averaging method is justified for a class of multipoint boundary value problems for functional-differential equations with maximums.

Consider the system of functional-differential equations

$$\begin{aligned} \dot{x}(t) &= \epsilon X\Big(t, x(t), \max\{x(s): s \in [t-h, t]\}, \\ &\qquad \max\{\dot{x}(s): s \in [t-h, t]\}\Big), \quad t > 0, \qquad (1) \\ x(t) &= \varphi(t), \ \dot{x}(t) = \dot{\varphi}(t), \ -h \leqslant t \leqslant 0 \end{aligned}$$

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with a boundary condition

$$\sum_{i=0}^{N} A_i x(t_i) = \Gamma(x(t_o),\ldots,x(t_N),\dot{x}(t_o),\ldots,\dot{x}(t_N), \varepsilon), \tag{2}$$

where $x \in \mathbb{R}^n$, h is a positive constant,

$\max \{ x(s): s \in [t-h, t] \} =$

$= (\max \{ x^{(1)}(s): s \in [t-h, t] \},\ldots,\max \{ x^{(n)}(s): s \in [t-h, t] \}),$

$\max \{ \dot{x}(s): s \in [t-h, t] \} =$

$= (\max \{ \dot{x}^{(1)}(s): s \in [t-h, t] \},\ldots,\max \{ \dot{x}^{(n)}(s): s \in [t-h, t] \}),$

$\varphi(t)$ is the initial function, $A_i = (a_{jk}^{(i)})_1^n$, $t_i = \alpha_i T$, $i = \overline{0, N}$, $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_N = 1$, $T = L\varepsilon^{-1}$, $L = \text{const} > 0$ and $\varepsilon > 0$ is a small parameter.

Suppose that there exists the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x, x, 0)\, dt = \overline{X}(x). \tag{3}$$

Then the average boundary problem, corresponding to (1), (2), is

$$\dot{\xi}(t) = \varepsilon \overline{X}(\xi(t)) \tag{4}$$

$$\sum_{i=0}^{N} A_i \xi(t_i) = \Gamma(\xi(t_o),\ldots,\xi(t_N), \dot{\xi}(t_o),\ldots,\dot{\xi}(t_N), \varepsilon). \tag{5}$$

For $x = (x^{(1)},\ldots,x^{(n)})$ and $A = (a_{jk})_{m,n}$ we set by definition

$$\| x \| = [\sum_{i=1}^{n} (x^{(i)})^2]^{\frac{1}{2}}, \quad \| A \| = [\sum_{k=1}^{n} \sum_{j=1}^{m} a_{jk}^2]^{\frac{1}{2}},$$

while $\overline{0,N}$ denotes the set of integers $\{ 0, 1, \ldots, N \}$.

The following theorem for proximity between the solutions $x(t)$ and $\xi(t)$ of the boundary value problems (1), (2) and (4), (5) is valid

THEOREM. Let the following conditions be fulfilled:

1. The function $X(t,x,y,z)$ is continuous in the domain

$$\Omega(t,x,y,z) = \Omega(t) \times \Omega(x) \times \Omega(y) \times \Omega(z),$$

where $\Omega(t) = [0, \infty)$ and $\Omega(x) \equiv \Omega(y)$, $\Omega(z)$ are open domains in $\mathbb{R}^n$. The function $\dot{\varphi}(t)$ is continuous in $[-h, 0]$ and $\varphi(t) \in \Omega(x)$, $\dot{\varphi}(t) \in \Omega(z)$. The function $\Gamma(w, \varepsilon)$, where $\omega = (w_0, ..., w_{2N+1})$ is defined in the domain

$$\Omega(w, \varepsilon) = \Omega(w_0) x \ldots x\ \Omega(w_{2N+1})\ x\ \Omega(\varepsilon),$$

$$\Omega(w_i) \equiv \Omega(x),\ i = 0, 2N+1,\ \Omega(\varepsilon) = (0, \bar{\varepsilon}],\ \bar{\varepsilon} = \text{const} > 0.$$

2. The functions $X(t,x,y,z)$ and $\Gamma(w, \varepsilon)$ satisfy the following conditions in the domains $\Omega(t,x,y,z)$ and $\Omega(w, \varepsilon)$

$$\| X(t,x,y,z) \| \leqslant M,$$

$$\| X(t,x,y,z) - X(t,x',y',z') \| \leqslant \lambda (\| x-x' \| + \| y-y' \| + \| z-z' \|)$$

$$\| \Gamma(w,\varepsilon) - \Gamma(w',\varepsilon) \| \leqslant \sum_{i=0}^{N} \mu_i \| w_i - w_i' \| + \nu_i \| w_{N+1+i} - w_{N+1+i} \|,$$

where M, λ, μ_0, ν_i $(i = \overline{0, N})$ are positive constants, $\mu_i = \mu_i(\varepsilon)$ $(i = \overline{1, N})$, the function $b(\varepsilon) = \max \{ \mu_i(\varepsilon) \colon i = 1, N \}$ is continuous in the domain $\Omega(\varepsilon)$ and $\lim \{ b(\varepsilon) \colon \varepsilon \to 0 \} = 0$.

3. The matrix A_o is constant and $\det A_o \neq 0$.

4. The matrices A_i, $i = \overline{1, N}$, depend on ε ; the function $d(\varepsilon) = \max \{ \| A_i(\varepsilon) \| : i = \overline{1, N} \}$ is continuous in $\Omega(\varepsilon)$ and $\lim \{ d(\varepsilon) \colon \varepsilon \to 0 \} = 0$.

5. The inequality

$$\| (\sum_{i=0}^{N} A_i)^{-1} \| \sum_{i=0}^{N} \mu_i < 1$$

is fulfilled in $\Omega(\varepsilon)$.

6. The limit (3) exists for each $x \in \Omega(x)$. the function $\overline{X}(x)$ is continuous in $\Omega(x)$.

7. For each $\varepsilon \in (0, \bar{\varepsilon}]$ the boundary value problem (1), (2) has an unique continuous solution with values in $\Omega(x)$ for $t \in [0, L\, \varepsilon^{-1}]$ and satisfying the conditions $x(0 + 0) = \varphi(0)$, $\dot{x}(0 + 0) = \dot{\varphi}(0)$.

8. For each $\varepsilon \in (0, \bar{\varepsilon}]$ the boundary value problem (4), (5) has an unique continuous solution whose values belong to the domain $\Omega(x)$ for $t \in [0, L\, \varepsilon^{-1}]$.

Then for each $\eta > 0$ and $L > 0$ there exists a number $\varepsilon_o \in (0, \bar{\varepsilon}]$ ($\varepsilon_o = \varepsilon_o(\eta, L)$) such that for $0 < \varepsilon \leqslant \varepsilon_o$ the inequality

$$\| x(t) - \xi(t) \| < \eta,\ 0 \leqslant t \leqslant L\, \varepsilon^{-1}$$

holds true.

Proof. The solutions of the boundary value problems (1), (2) and (4), (5) can be represented by the relations

$$x(t) = x_o + \varepsilon \int_o^t \Big(X \quad \theta, x(\theta), \max\{x(s): s \in [\theta - h, \theta]\},$$

$$\max\{\dot{x}((s): s \in [\theta - h, \theta]\}\Big)\, d\theta, t > 0, \tag{6}$$

$$x(t) = \varphi(t), \dot{x}(t) = \dot{\varphi}(t), -h \leqslant t \leqslant 0,$$

$$\xi(t) = \xi_o + \varepsilon \int_o^t \overline{X}(\xi(\theta))\, d\theta, \tag{7}$$

$$\sum_{i=0}^{N} A_i (x_o + \varepsilon \beta_i) = \Gamma(x_o + \varepsilon \beta_o, \ldots, x_o + \varepsilon \beta_{N_o} \dot{x}(t_o), \ldots, \dot{x}(t_N), \varepsilon), \tag{8}$$

$$\sum_{i=0}^{N} A_i (\xi_o + \varepsilon \overline{\beta}_i) = \Gamma(\xi_o + \varepsilon \overline{\beta}_o, \ldots, \xi_o + \varepsilon \overline{\beta}_N, \dot{\xi}(t_o), \ldots, \dot{\xi}(t_N), \varepsilon), \tag{9}$$

whre $x_o = x(t_o)$, $\xi_o = \xi(t_o)$,

$$\beta_i = \int_o^{t_i} X(\theta, x(\theta), \max\{x(s): s \in [\theta - h, \theta]\},$$

$$\max\{\dot{x}(s): s \in [\theta - h, \theta]\})\, d\theta, t > 0,$$

$$x(t) = \varphi(t), \dot{x}(t) = \dot{\varphi}(t), -h \leqslant t \leqslant 0,$$

$$\beta_i = \int_{t_o}^{t_i} \overline{X}(\xi, \theta)\, d\theta, i = \overline{0, N}.$$

Substracting for $t > 0$ the representation (7) from (6) one obtains

$$\| x(t) - \xi(t) \| \leqslant \| x_o - \xi_o \| + \varepsilon \| \int_o^t [X(\theta, x(\theta), \max\{x(s): s \in [\theta - h, \theta]\},$$

$$\max\{\dot{x}(s): s \in [\theta - h, \theta]\}) - \overline{X}(\xi(\theta))]\, d\theta \| \leqslant$$

$$\leqslant \| x_o - \xi_o \| + \varepsilon \int_o^t \| X(\theta, x(\theta), \max\{x(s): s \in [\theta - h, \theta]\}, \tag{10}$$

$$\max\{\dot{x}(s): s \in [\theta - h, \theta]\}) - X(\theta, \xi(\theta), \dot{\xi}(\theta), 0) \|\, d\theta +$$

$$+ \varepsilon \| \int_o^t [X(\theta, \xi(\theta), \dot{\xi}(\theta), 0) - \overline{X}(\xi(\theta))]\, d\theta \| \equiv \| x_o - \xi_o \| + I_1 + I_2 .$$

Then terms I_1 and I_2 in the right-hand side of (10) can be estimated for $0 \leqslant t \leqslant L\varepsilon^{-1}$ using the assumptions of the theorem:

$$I_1 = \varepsilon \int_0^t \| X(\theta, x(\sigma), \max\{x(s): s \in [\theta - h, \theta]\},$$

$$\max\{\dot{x}(s): s \in [\theta - h, \theta]\}) - X(\theta, \xi(\theta), \xi(\theta), 0)] \, d\theta \| \leqslant \qquad (11)$$

$$\leqslant 2\varepsilon\lambda \int_0^t \| x(\theta) - \xi(\theta) \| \, d\theta + \varepsilon\lambda \{[2B + 2\varepsilon h M + \max(C, \varepsilon M)] h +$$

$$+ (1 + h)\sqrt{n}\, ML \},$$

$$I_2 = \varepsilon \| \int_0^t [X(\theta, \xi(\theta), \xi(\theta), 0) - X(\xi(\theta))] \, d\theta \| \leqslant \qquad (12)$$

$$\leqslant 2\lambda M L^2/m + F(\varepsilon, m) \equiv a(\varepsilon, m),$$

where

$$B = \sup_{-n \leqslant t \leqslant 0} \| \varphi(t) \|, \qquad C = \sup_{-n \leqslant t \leqslant 0} \| \dot{\varphi}(t) \|,$$

$$F(\varepsilon, m) = L [\sum_{i=0}^{m-1} (\frac{(i+1)L}{\varepsilon m}, \xi_i) + \sum_{i=1}^{m-1} \mathrm{op}(\frac{iL}{\varepsilon m}, \xi_i)] +$$

$$+ \max_{0 \leqslant k \leqslant m-1} \mathrm{op}_0(\varepsilon, \xi_k), \ \xi_k = \xi(\frac{kL}{\varepsilon m}), \ k, m \in \mathbb{N}^+,$$

$$\mathrm{op}(t, \xi) = \| \frac{1}{t} \int_0^t [X(\theta, \xi, \xi, 0) - \overline{X}(\xi)] \, d\theta \|,$$

$$\mathrm{op}_0(\varepsilon, \xi) = \sup_{0 \leqslant \sigma \leqslant L} \tau\,\mathrm{op}(\frac{\tau}{\varepsilon}, \xi).$$

Since for each $\xi \in \Omega(x)$ the function $\mathrm{op}(t, \xi)$ tends to zero as $t \to \infty$ then by an appropriate choice of m sufficiently large and ε sufficiently small, the quantity $a(\varepsilon, m)$ can be made arbitrary small. [3].

Further on it follows from (9) and (8) that for $t \in [0, L\varepsilon^{-1}]$ then inequality

$$\| x_0 - \xi_0 \| \leqslant \varepsilon G(\varepsilon) \sum_{i=1}^{N} \| \beta_i - \overline{\beta}_i \| + \varepsilon D \qquad (13)$$

is valid, where

$$G(\varepsilon)=(b(\varepsilon)+d(\varepsilon))\,\|(\sum_{i=0}^{N}A_i)^{-1}\|(1-\|(\sum_{i=0}^{N}A_i)^{-1}\|\sum_{i=0}^{N}\mu_i)^{-1},$$

$$D=M\,\|(\sum_{i=0}^{N}A_i)^{-1}\|\sum_{i=0}^{N}\nu_i(1-\|(\sum_{i=0}^{N}A_i)^{-1}\|\sum_{i=0}^{N}\mu_i)^{-1}.$$

The relations (10) – (13) yield

$$\|x(t)-\xi(t)\|\leqslant G(\varepsilon)\sum_{i=1}^{N}\{2\varepsilon\lambda\int_0^{t_i}\|x(\theta)-\xi(\theta)\|d\theta+H(\varepsilon,m)\}+$$
$$+2\varepsilon\lambda\int_0^{t}\|x(\theta)-\xi(\theta)\|d\theta+H(\varepsilon,m), \tag{14}$$

$$H(\varepsilon,m)=\varepsilon D+\varepsilon\lambda\{[2B+2\varepsilon hM+\max(C,\varepsilon M)]h+$$
$$+(1+h)\sqrt{n}ML\}+\alpha(\varepsilon,m).$$

Set $H(\varepsilon,m)\,u(t)=x(t)-\xi(t)$ and introduce the notation $\|u\|_T=\sup_{0\leqslant t\leqslant T}\|u(t)\|$. Then it follows from (14) that

$$\|u(t)\|\leqslant 1+G(\varepsilon)(N+2\lambda\alpha L\,\|u\|_T)+2\varepsilon\lambda\int_0^t\|u(\theta)\|d\theta, \tag{15}$$

where

$$\alpha=\sum_{i=1}^{N}\alpha_i.$$

Applying the Gronwall-Bellman inequality to (15) one obtains the estimate

$$\|u(t)\|\leqslant[1+G(\varepsilon)(N+2\lambda\alpha L\,\|u\|_T)]\exp\{2\varepsilon\lambda t\}$$

for the function $\|u(t)\|$. Hence the inequality

$$\|u\|_T\leqslant[1+G(\varepsilon)(N+2\lambda\alpha L\,\|u\|_T)]\exp\{2\lambda L\} \tag{16}$$

is valid for $0\leqslant t\leqslant L\varepsilon^{-1}$.

Since $\lim\{G(\varepsilon):\varepsilon\to 0\}=0$, then there exists a number $\varepsilon_1\in(0,\bar{\varepsilon}]$ such that the inequality

$$2G(\varepsilon)\lambda\alpha L\exp\{2\lambda L\}<1$$

holds true for $\varepsilon \in (0, \varepsilon_1]$. Now it follows from (16) that

$$\| x(t) - \xi(t) \| \leqslant \omega H(\varepsilon, m)$$

for $\varepsilon \in (0, \varepsilon_1]$, where

$$\omega = \frac{(1 + G(\varepsilon)N) \exp\{2\lambda L\}}{1 - 2G(\varepsilon)\lambda'\alpha L \exp\{2\lambda L\}} .$$

Let finally m and $\varepsilon_2 \in (0, \varepsilon]$ be choosen so that $\omega H(\varepsilon, m) < \eta$ for $\varepsilon \in (0, \varepsilon_2]$. The latter inequality for $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$ completes the proof of the theorem.

REFERENCES

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