THE APPROXIMATION PROPERTY OF ORDER p IN LOCALLY CONVEX SPACES(*)

by

J. A. LOPEZ MOLINA

ABSTRACT:

We define the approximation property and the local approximation property of order $p > 1$ of a locally convex space. We prove that, if $E = \varinjlim E_n$ is a regular inductive limit of Frechet spaces E_n with the approximation property of order $p > 1$, then E has this property.

1. INTRODUCTION

In [12], for every real number $p \geqslant 1$, Saphar defines the approximation property of order p (in shortly AP_p) of a Banach space E. The approximation property of order 1 for E is nothing but the classical approximation property (AP) of Grothendieck. Every Banach space has AP_2 and every Banach space with the AP has the AP_p for all $p > 1$. In [11], Reinov notices that there are Banach spaces with the AP_p, $p > 2$, without the AP and gives an example of a reflexive separable Banach space E such that, for every $p \neq 2$, E does not have the AP_p.

It seems that there is no definition of the AP_p, $p > 1$, for locally convex spaces. The purpose of this paper is to introduce the definition of the approximation property of order $p > 1$ of a locally convex space E and to developp a theory similar to the classical one for $p = 1$ as far as possible. This definition is given in section 2. However, after the proof of some general properties, we shall only consider in this paper the problem of the AP_p, $p > 1$, of an inductive limit of a sequence of Frechet spaces. Then, we obtain a theorem similar to a result of Bierstedt and Meise ([2]) for the classical approximation property.

Our notation for separated locally convex spaces (in shortly l.c.s.) over the field $\mathbb{K}$ of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, is standard and we refer the

(*) Supported by the Comisión Asesora de Investigación Científica y Técnica. Proyecto 0258/81.

reader to [6] and [13]. Given a l.c.s. E, we shall denote by $\mathcal{U}(E)$ a basis of absolutely convex closed O-neighbourhoods and by K_U (resp. $\bar{K}_U$) the canonical map from E onto E_U (resp. into $\hat{E}_U$) for each $U \in \mathcal{U}(E)$. If E and F are l.c.s., $\mathcal{L}(E,F)$ will be the space of all continuous linear maps from E into F and $\mathcal{B}_e(E'_\sigma, F'_\sigma)$ will be the espace of all separately continuous bilinear forms on $E'_\sigma \times F'_\sigma$ provided with the topology of the uniform convergence on the sets $U^o \times V^o$, where $U \in \mathcal{U}(E)$ and $V \in \mathcal{U}(F)$.

$\mathbb{N}$ is the set of positive natural numbers. If $p \in \mathbb{R}$, $p \geq 1$, we define its conjugate number $p' \in [1,\infty]$ such that $1/p + 1/p' = 1$. If E is a l.c.s. and $p \geq 1$, in [1] are defined the spaces $\ell^p(E)$ and $\ell^p[E]$ of weakly p-summable and absolutely p-summable sequences (x_i) of E, respectively, We shall consider $\ell^p(E)$ (resp. $\ell^p[E]$) endowed with the topology defined by the system of seminorms $\{ \epsilon_{p,U}, U \in \mathcal{U}(E) \}$ (resp. $\{ \Pi_{p,U}, U \in \mathcal{U}(E) \}$) where

$$\epsilon_{p,U}((x_i)) = \begin{cases} \sup\limits_{x' \in U^o} \left(\sum\limits_{i=1}^{\infty} |\langle x_i, x' \rangle|^p \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sup\limits_{i \in \mathbb{N}} \sup\limits_{x' \in U^o} |\langle x_i, x' \rangle| & \text{if } p = \infty \end{cases}$$

$$\Pi_{p,U}((x_i)) = \begin{cases} \left(\sum\limits_{i=1}^{\infty} (p_U(x_i))^p \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sup\limits_{i \in \mathbb{N}} p_U(x_i) & \text{if } p = \infty \end{cases}$$

We shall consider every finite sequence $(x_1, x_2, \ldots, x_n)$ as a sequence (x_i) with $x_i = 0$ if $i > n$.

If E and F are l.c.s., a map $T \in \mathcal{L}(E,F)$ is called p-absolutely summing $(1 \leq p \leq \infty)$ and we will write $T \in S^p(E,F)$, if for every $(x_i) \in \ell^p(E)$, we have $(Tx_i) \in \ell^p[F]$. If E is bornological, the continuity of T is a consequence of the second condition. The proof is analogous to the proof of [6], pag. 428, in the normed case. A map $T \in S^p(E,F)$ such that the map $\hat{T}: \ell^p(E) \longrightarrow \ell^p[F]$ defined by means of $T((x_i)) = (Tx_i)$, is continuous, will be called totally p-absolutely summing. If E is metrizable or $\ell^p(E)$ is quasibarrelled, every $T \in S^p(E,F)$ is totally p-absolutely summing for all l.c.s. F. (The proof is a slight modification of the proof given in [9], pag. 36 in the case $p = 1$). It is easy to prove that a map $T \in \mathcal{L}(E,F)$ is totally p-absolutely summing if and only if for each $V \in \mathcal{U}(F)$, there is $U \in \mathcal{U}(E)$ such that for every $n \in \mathbb{N}$ and every finite set $\{ x_1, x_2, \ldots, x_n \} \subset E$, we have

$$\Pi_{p,V}((Tx_i)) \leqslant \epsilon_{p,U}((x_i)).$$

With the same method of [6] pag. 433, we can prove the following factorization theorem which we shall use later:

PROPOSITION A: *Let E,F be l.c.s. and $T \in \mathscr{L}(E,F)$ be a totally p-absolutely summing map ($1 < p < \infty$). For each $V \in \mathfrak{U}(F)$, there are $U \in \mathfrak{U}(E)$, a reflexive Banach space M, a totally p-absolutely summing map $J_p \in \mathscr{L}(E,M)$ and a map $B \in \mathscr{L}(M,\hat{F}_V)$, such that*

$$\bar{K}_V T = BJ_p .$$

If E and F are l.c.s. and $p \geqslant 1$, the topology g_p of Saphar in the tensor product $E \otimes F$ is defined by the family of seminorms $\{ g_{p,U,V},\ U \in \mathfrak{U}(E), V \in \mathfrak{U}(F) \}$ where

$$g_{p,U,V}(z) = \inf \left\{ \Pi_{p,U}((x_i)) \cdot \epsilon_{p',V}((y_i)) \ / \ z = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F \right\}$$

taking the inf over all representations of $z \in E \otimes F$. With this topology, $E \otimes F$ is denoted by $E \underset{g_p}{\otimes} F$ and its completion by $E \underset{g_p}{\hat{\otimes}} F$. In [11] it is proved that, if E and F are Banach spaces, $(E \underset{g_p}{\otimes} F)' = S^{p'}(F,E')$ where the isomorphism is defined by

$$<T,z> = \sum_{i=1}^{n} <x_i, Ty_i> \ \forall\, T \in S^{p'}(F,E'),\ \forall\, z = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F.$$

It is easy to prove that these property also holds if F is a Frechet space.

In some cases, we shall identify $E \otimes F$ with a subspace of linear mappings from E′ into F or from F′ into E in the canonical way.

1. THE APPROXIMATION PROPERTY OF ORDER $p > 1$ IN BANACH SPACES

In what follows, p will be always a real number $p > 1$. Let E,F be l.c.s. For each $V \in \mathfrak{U}(F)$, $(x_i) \in \ell^p(E)$ and $T \in S^p(E,F)$, we define

$$P_{(x_i),V}(T) = \Pi_{p,V}((Tx_i)).$$

We consider $S^p(E,F)$ endowed with the topology $\mathcal{T}_p$ defined by the system of seminorms $\{ P_{(x_i),V},\ (x_i) \in \ell^p(E),\ V \in \mathfrak{U}(F) \}$. Clearly $\mathcal{T}_p$ is a separated topology.

PROPOSITION 1: *If E is a Frechet space and F is a Banach space, the topological dual of* $[S^p(E,F), \mathcal{T}_p]$ *is a quotient of* $F' \hat{\otimes}_{g_{p'}} E$.

Proof. Let B be the closed unit ball of F. Every $z \in F' \hat{\otimes}_{g_{p'}} E$ has the form (see [4])

$$z = \sum_{i=1}^{\infty} y_i' \otimes x_i \text{ with } (y_i') \in \ell^{p'}[F'], (x_i) \in \ell^p(E).$$

As $S^p(E,F) \subset (F' \hat{\otimes}_{g_{p'}} E)'$, the linear form on $S^p(E,F)$

$$\varphi_z(T) = <\varphi_z, T> = <z, T> = \sum_{i=1}^{\infty} <Tx_i, y_i'> \qquad \forall\, T \in S^p(E,F),$$

is well defined and, by Hölder's inequality

$$|<\varphi_z, T>| \leqslant \Pi_{p',B^\circ}((y_i')) \cdot P_{(x_i),B}(T).$$

Hence $\varphi_z \in [S^p(E,F), \mathcal{T}_p]'$.

Conversely, let ψ be an element of $[S^p(E,F), \mathcal{T}_p]'$. Then there is $(x_i) \in \ell^p(E)$ such that

$$|\psi(T)| \leqslant 1 \qquad \text{if } T \in S^p(E,F) \text{ and } P_{(x_i),B}(T) < 1$$

where B is the closed unit ball of F. The map G: $[S^p(E,F), \mathcal{T}_p] \longrightarrow \ell^p[F]$ such that

$$G(T) = (Tx_i) \qquad \forall\, T \in S^p(E,F)$$

is continuous. Its dual map G': $\ell^{p'}[F'] \longrightarrow [S^p(E,F), \mathcal{T}_p]'$ is weakly continuous. (See [7], pag. 359). If W is the closed unit ball of $\ell^{p'}[F']$, $G'(W)$ is $\sigma([S^p(E,F), \mathcal{T}_p]', S^p(E,F))$-compact and convex. Let us prove that $\psi \in G'(W)$. If not, by [6] pag. 131, there would be $H \in S^p(E,F)$ such that $|<H, \psi>| > 1$ and $|<H, \eta>| \leqslant 1$ for all $\eta \in G'(W)$. But, by [7], pág. 196, there is $(y_i') \in W$ such that

$$P_{(x_i),B}(H) = <(Hx_i), (y_i')> = |<G(H), (y_i')>| = |<H, G'((y_i'))>| \leqslant 1.$$

Then by hypothesis $|\psi(H)| \leqslant 1$ which is a contradiction. Then there is $(z_i') \in W$ such that $\psi = G'((z_i'))$. Now

$$z = \sum_{i=1}^{\infty} z_i' \otimes x_i \in F' \hat{\otimes}_{g_{p'}} E$$

and it is easy to see that $< \varphi_z, T > = < T, \psi >$ for every $T \in S^p(E,F)$. Then φ is an epimorphism and $[S^p(E,F), \mathcal{T}_p]' = (F' \hat{\otimes}_{g_{p'}} E)/\mathrm{Ker}(\varphi)$.

PROPOSITION 2: *Let E be a Frechet space. The following conditions are equivalent:*

1) For every Banach space F,E' ⊗ F is $\mathcal{T}_{p'}$*-dense in* $S^{p'}(E,F)$.

2) For every Banach space F, the canonical map

$$\chi_F : F \hat{\otimes}_{g_p} E \longrightarrow \mathscr{L}(F',E)$$

is injective.

3) For every Banach space F, the canonical map

$$\psi_F : F' \hat{\otimes}_{g_p} E \longrightarrow \mathscr{L}(F,E)$$

is injective.

Proof: 1 ⇒ 2. Let φ be the map from $F'' \hat{\otimes}_{g_{p'}} E$ onto $[S^{p'}(E,F'), \mathcal{T}_{p'}]'$ of the proposition 1. Let $z \in F \hat{\otimes}_{g_p} E$ be such that $\chi_F(z) = 0$. If

$$z = \sum_{i=1}^{\infty} y_i \otimes x_i$$

with $(y_i) \in \ell^p[F]$ and $(x_i) \in \ell^{p'}(E)$, we can consider z as an element of $F'' \hat{\otimes}_{g_p} E$. Then, for every $y' \in F'$ and every $x' \in E'$ we have

$$0 = < \chi_F(z)(y'), x' > = \sum_{i=1}^{\infty} <y_i,y'><x_i,x'> = <\varphi(z), x' \otimes y'>.$$

Then, since $E' \otimes F'$ is $\mathcal{T}_{p'}$-dense on $S^{p'}(E,F')$, $\varphi(z) = 0$ on $S^{p'}(E,F') = (F \hat{\otimes}_{g_p} E)'$. Now, for every $T \in S^{p'}(E,F')$

$$< z, T > = \sum_{i=1}^{\infty} < Tx_i, y_i > = < \varphi(z), T > = 0$$

and hence $z = 0$ and χ_F is injective.

2) ⇒ 3). Given

$$z = \sum_{i=1}^{\infty} y_i' \otimes x_i \in F' \hat{\otimes}_{g_p} E \qquad ,(y_i') \in \ell^p[F'], \qquad (x_i) \in \ell^{p'}(E),$$

the map $S = \psi_F(z) \in \mathscr{L}(F,E)$ is the restriction to F of the map $\chi_{F'}(z) \in \mathscr{L}(F'',E)$. Moreover, for each $x' \in E'$

$$g(x') = z' = \sum_{i=1}^{\infty} \langle x_i, x' \rangle y'_i$$

is a convergent series in F'. Given $y'' \in F''$, there is a net $\{y_a, a \in A\}$ in F $\sigma(F'', F')$-convergent to y''. Then for every $x' \in E'$

$$|\langle \chi_{F'}(z)(y'') - \chi_{F'}(z)(y_a), x' \rangle| = |\langle y'' - y_a, g(x') \rangle|$$

and

$$\chi_{F'}(z)(y'') = \lim_{a \in A} \chi_{F'}(z)(y_a) = \lim_{a \in A} \psi_F(z)(y_a) \quad \text{in} \quad \sigma(E,E').$$

Then, if $z \in F' \hat{\otimes}_{g_p} E$ is such that $\psi_F(z) = 0$, we have $\chi_{F'}(z) = 0$ in $\mathscr{L}(F'',E)$. By hypothesis $z = 0$ and ψ_F is injective.

3) $\Rightarrow$ 1). Let $G \in [S^{p'}(E,F), \mathcal{T}_{p'}]'$ be such that $G(z) = 0$ for every $z \in E' \otimes F$. By proposition 1, there is

$$z = \sum_{i=1}^{\infty} y'_i \otimes x_i \in F' \hat{\otimes}_{g_p} E$$

such that

$$\langle G,T \rangle = \sum_{i=1}^{\infty} \langle Tx_i, y'_i \rangle \qquad \forall T \in S^{p'}(E,F).$$

Then, for every $x' \in E'$ and every $y \in F$

$$0 = \langle G, x' \otimes y \rangle = \sum_{i=1}^{\infty} \langle x', x_i \rangle \langle y, y'_i \rangle = \langle \psi_F(z)(y), x' \rangle.$$

Hence $\psi_F(z) = 0$ in $\mathscr{L}(F,E)$ and by 3), $z = 0$. Then $G = 0$ and $E' \otimes F$ is $\mathcal{T}_{p'}$-dense in $S^{p'}(E,F)$.

PROPOSITION 3: *If E is a l.c.s., the following conditions are equivalent:*

1) For every Banach space F, $E' \otimes F$ is $\mathcal{T}_{p'}$-dense in $S^{p'}(E,F)$.

2) For every l.c.s. F, $E' \otimes F$ is $\mathcal{T}_{p'}$-dense in $S^{p'}(E,F)$.

Proof: 1) $\Rightarrow$ 2). Let us suppose that $T_o \in S^{p'}(E,F)$, $(x_i) \in \ell^{p'}(E)$ and $V \in \mathcal{U}(F)$. Given $\epsilon > 0$ we consider the $\mathcal{T}_{p'}$-neighbourhood of T_o

$$W = \left\{ T \in S^{p'}(E,F) \,/\, P_{(x_i),V}(T-T_o) < \epsilon \right\}.$$

As $\bar{K}_V T_o \in S^{p'}(E,\hat{F}_V)$, by 1) there is

$$T = \sum_{j=1}^{h} x_j' \otimes w_j \in E' \otimes \hat{F}_V \subset S^{p'}(E,\hat{F}_V)$$

such that $P_{(x_i),V}(\bar{K}_V T_o - T) < \epsilon/2$. For each $j = 1,2,\ldots,h$ we have

$$N_j = \left(\sum_{i=1}^{\infty} |<x_j', x_i>|^{p'} \right)^{1/p'} < \infty.$$

Let us define $M_j = N_j$ if $N_j \neq 0$ and $M_j = 1$ if $N_j = 0$. Now, we choose $y_j \in F$ such that

$$p_V(w_j - \bar{K}_V(y_j)) < \epsilon/(2hM_j) \qquad \forall j = 1,2,\ldots,h$$

and we define

$$S = \sum_{j=1}^{h} x_j' \otimes y_j \in E' \otimes F.$$

Then,

$$P_{(x_i),V}(S-T_o) = \left(\sum_{i=1}^{\infty} \left(p_V\left(\sum_{j=1}^{h} <x_j', x_i> \bar{K}_V(y_j) - (\bar{K}_V T_o)(x_i)\right)\right)^{p'} \right)^{1/p'} \leq$$

$$\leq \left(\sum_{i=1}^{\infty} \left(p_V\left(\sum_{j=1}^{h} <x_j', x_i> (\bar{K}_V(y_j) - w_j)\right)\right)^{p'} \right)^{1/p'} +$$

$$+ \left(\sum_{i=1}^{\infty} \left(p_V\left(\sum_{j=1}^{h} <x_j', x_i> w_j - (\bar{K}_V T_o)(x_i)\right)\right)^{p'} \right)^{1/p'} \leq$$

$$\leq \sum_{j=1}^{h} \left(\sum_{i=1}^{\infty} \left(|<x_j', x_i>| \, p_V(\bar{K}_V(y_j) - w_j)\right)^{p'} \right)^{1/p'} +$$

$$+ P_{(x_i),V}(\bar{K}_V T_o - T) \leq \epsilon.$$

Hence $S \in W$.

2) $\Rightarrow$ 1). Trivial.

It is known (see [3], [6], [8]) that the AP of a Banach space E is equivalent to the fact that, for every Banach space F, the canonical map from $F \hat{\otimes}_\pi E$ into

$\mathscr{L}$ (F',E) in injective. As the projective tensor topology π coincides with the tensor topology g_1 of Saphar, Reinov ([11]) (and Saphar ([12]) with a slightly different formulation) gave the following definition:

DEFINITION A: *A Banach space E has the AP_p $(p \geqslant 1)$ if, for every Banach space F the canonical map χ_F from* $F \hat{\otimes}_{g_p} E$ *into $\mathscr{L}$ (F', E) is injective.*

Then, the proposition 2 is a new characterization of the $AP_p, p > 1$ of a Banach space E.

2. THE APPROXIMATION PROPERTY OF ORDER $p > 1$ IN LOCALLY CONVEX SPACES

Motivated by propositions 2 and 3, we shall give the following definition: (always $p > 1$)

DEFINITION 1: *A l.c.s. E is said to satisfy the AP_p, if for every l.c.s. F, $E' \otimes F$ is $\mathcal{C}_{p'}$-dense in $S^{p'}(E,F)$.*

By propositions 3 and 2, this definition is consistent with the definition A of Reinov in the case of Banach spaces E.

PROPOSITION 4: *Let E be a l.c.s. with the AP_p. If H is a dense subspace of E,H has the AP_p.*

Proof: Since $\ell^{p'}(H) \subset \ell^{p'}(E)$, the proof is inmediate.

Consequently, if the completion $\hat{E}$ of a l.c.s. E has the AP_p, E has also the AP_p. Now, we introduce the concept of local approximation property of order p (local AP_p):

DEFINITION 2: *A l.c.s. E is said to satisfy the local AP_p if there is a basis of 0-neighbourhoods $\mathcal{U}(E)$ such that the Banach space $\hat{E}_U$ has the AP_p for each $U \in \mathcal{U}(E)$.*

In this case, according to proposition 4, each E_U has the AP_p.

PROPOSITION 5: *Let E be a l.c.s. with the local AP_p and such that, for every l.c.s. F, every $T \in S^{p'}(E,F)$ is totally p'-absolutely summing. Then E has the AP_p.*

Proof: Given a l.c.s. F, $(x_i) \in \ell^{p'}(E)$, $V \in \mathcal{U}(F)$ and $T \in S^{p'}(E,F)$, there is $U \in \mathcal{U}(E)$ such taht

$$\Pi_{p',V}((Tt_i)) \leqslant \epsilon_{p',U}((t_i)) \qquad \forall\, t_1, t_2, \ldots, t_n \in E, \qquad \forall n \in \mathbb{N}$$

Then, the map $\overline{T}$: $E_U \longrightarrow \hat{F}_V$ defined by $\overline{T}(K_U(x)) = (\overline{K}_V T)(x)$ for all $x \in E$, is well defined and $\overline{T} \in S^{p'}(E_U, F_V)$. As $(K_U(x_i)) \in \ell^{p'}(E_U)$, by proposition 4, given $\epsilon > 0$, there is

$$\overline{z} = \sum_{j=1}^{h} x_j' \otimes K_V(y_j) \in E'_{U^\circ} \otimes F_V$$

such that $P_{(K_U(x_i)),V}(\overline{T} - \overline{z}) \leqslant \epsilon$. Then, for

$$z = \sum_{j=1}^{h} x_j' \otimes y_j \in E' \otimes F,$$

$P_{(x_i),V}(T - z)$ holds and E has the AP_p.

In [9], Nelimarkka has shown that in each Frechet space F with a Schauder basis, there is a system $\mathcal{U}(F)$ such that for every $U \in \mathcal{U}(F)$, $\hat{F}_U$ has the AP. Hence, every Frechet space with a Schauder basis has the AP_p for every $p > 1$.

THEOREM 1. *Let E,F be l.c.s. such that F has the local AP_p or F is a Frechet space with the AP_p. Then $E' \otimes F'$ is $\sigma((E \hat{\otimes}_{g_p} F)', E \hat{\otimes}_{g_p} F)$-dense in $(E \hat{\otimes}_{g_p} F)'$.*

Proof: Let us suppose that F is a l.c.s. with the local AP_p and $z_i, i = 1,2, \ldots, n$ are in $E \hat{\otimes}_{g_p} F$. Given $T \in (E \hat{\otimes}_{g_p} F)'$, there are $U \in \mathcal{U}(E)$ and $V \in \mathcal{U}(F)$ such that the linear form $\overline{T}$ on $E_U \otimes_{g_p} F_V$ defined by

$$<(K_U \otimes K_V)(z), \overline{T}> = <z, T> \qquad \forall z \in E \otimes F,$$

is well defined an $\overline{T} \in (E_U \otimes_{g_p} F_V)' = (\hat{E}_U \hat{\otimes}_{g_p} \hat{F}_V)'$. Let $\varphi \in (\hat{E}_U \hat{\otimes}_{g_p} \hat{F}_V)$ be such that $\varphi(E'_{U^\circ} \otimes F'_{V^\circ}) = 0$ but $\varphi \neq 0$. By proposition 2, the canonical map $\chi : \hat{E}_U \hat{\otimes}_{g_p} \hat{F}_V \longrightarrow \mathscr{L}(E'_{U^\circ}, \hat{F}_V)$ is injective. Then, there are $x' \in E'_{U^\circ}$ and $y' \in F'_{V^\circ}$ such that $0 \neq <\chi(\varphi)(x'), y'> = <\varphi, x' \otimes y'>$, which is a contradiction. Hence $E'_{U^\circ} \otimes F'_{V^\circ}$ is $\sigma((\hat{E}_U \hat{\otimes}_{g_p} \hat{F}_V)', (\hat{E}_U \hat{\otimes}_{g_p} \hat{F}_V))$-dense in $(\hat{E}_U \hat{\otimes}_{g_p} \hat{F}_V)'$.

Now, let $\overline{K}_U \hat{\otimes} \overline{K}_V$ be the canonical map from $E \hat{\otimes}_{g_p} F$ into $E_U \hat{\otimes}_{g_p} F_V = \hat{E}_U \hat{\otimes}_{g_p} \hat{F}_V$. Given $\epsilon > 0$, there is $w \in E'_{U^\circ} \otimes F'_{V^\circ} \subset E' \otimes F'$ such that

$$|<\overline{T} - w, (\overline{K}_U \hat{\otimes} \overline{K}_V)(z_i)>| = |<T - w, z_i>| \leqslant \epsilon \qquad i = 1,2, \ldots, n$$

and the proof is complete. If F is a Frechet space with the AP_p, the proof is similar replacing F_V by F and using the propositions 3 and 2.

COROLLARY 1: *Let E,F be l.c.s. such that F has the local* AP_p *or F is a Frechet space with the* AP_p. *Then* $< E \hat{\otimes}_{g_p} F, E' \otimes F' >$ *is a dual pair.*

Proof: Inmediate, by theorem 1.

COROLLARY 2. *Let E,F be l.c.s. such that F has the local* AP_p *or F is a Frechet space with the* AP_p. *Then the canonical map*

$$\hat{\Delta} : E \hat{\otimes}_{g_p} F \longrightarrow \hat{\mathcal{B}}_e (E'_\sigma, F'_\sigma)$$

is injective.

Proof: It is easy to see that every $\varphi \in \hat{\mathcal{B}}_e (E'_\sigma, F'_\sigma)$ can be identified with a bilinear form on $E' \times F'$. Let $z \in E \hat{\otimes}_{g_p} F$ be such that $\hat{\Delta}(z) = 0$. There is a net $\{ z_a, a \in A \}$ in $E \otimes_{g_p} F$ convergent to z in the completion. Then, for every $x' \in E'$ and $y' \in F'$.

$$< z, x' \otimes y' > = \lim_{a \in A} < z_a, x' \otimes y' > = \lim_{a \in A} \ddot{\Delta}(z_a)(x', y') = \hat{\Delta}(z)(x', y') = 0.$$

By theorem 1, $z = 0$ and $\hat{\Delta}$ is injective.

COROLLARY 3: *Let F,G be complete l.c.s. such that G has the local* AP_p *or G is a Frechet space with the* AP_p. *Let H be a l.c.s. and T a continuous injective linear map from G into H. If I is the identity map on F, the continuous linear map*

$$I \hat{\otimes} T : F \hat{\otimes}_{g_p} G \longrightarrow F \hat{\otimes}_{g_p} H$$

is injective.

Proof: The space $\mathcal{B}_e (F'_\sigma, G'_\sigma)$ is complete (see [8] pág. 167). We consider the canonical continuous linear maps

$$\hat{\Delta}_1 : F \hat{\otimes}_{g_p} G \longrightarrow \mathcal{B}_e (F'_\sigma, G'_\sigma) \text{ and } \hat{\Delta}_2 : F \hat{\otimes}_{g_p} H \longrightarrow \hat{\mathcal{B}}_e (F'_\sigma, H'_\sigma)$$

as in corollary 2. If $z \in F \hat{\otimes}_{g_p} G$ is such that $(I \hat{\otimes} T)(z) = 0$, we take a net

$$\left\{ z_a = \sum_{i=1}^{n_a} x_i^a \otimes y_i^a \qquad , a \in A \right\}$$

in $F \otimes G$ convergent to z in $F \hat{\otimes}_{g_p} G$. Since $\hat{\Delta}_2 (I \hat{\otimes} T)(z) = 0$, given $(x',h') \in F' \times H'$,

$$0 = \lim_{a \in A} \left| \sum_{i=1}^{n_a} < x_i^a, x' > < Ty_i^a, h' > \right| = \lim_{a \in A} |\hat{\Delta}_1 (z_a)(x', T'h')| =$$

$$= |\hat{\Delta}_1 (z)(x', T'h')|.$$

Since T is injective, $T'(H')$ is $\sigma(G', G)$-dense in G'. As $\hat{\Delta}_1 (z) \in \mathcal{B}_e (F'_\sigma, G'_\sigma)$, we have $\hat{\Delta}_1 (z) = 0$ on $F' \times G'$. By corollary 2, $z = 0$ and $I \hat{\otimes} T$ is injective.

3. THE APPROXIMATION PROPERTY OF ORDER $p > 1$ IN INDUCTIVE LIMITS

We begin with a previous result which seems to be interesting in itself.

PROPOSITION 6: *Let E be a l.c.s. and* $(x_i) \in \ell^p(E)$. *Then there is a bounded set B in E such that B is contained in the closed linear span of* $\{x_i, i \in \mathbb{N}\}$ *and* $(x_i) \in \ell^p(E_B)$.

Proof: Set

$$F = \left\{ \sum_{i=1}^{n} b_i x_i \ / \ b_i \in \mathbb{K},\ i = 1,2,\ldots,n\ ;\ n \in \mathbb{N} \text{ and } \left(\sum_{i=1}^{n} |b_i|^{p'} \right)^{1/p'} \leq 1 \right\}$$

and

$$\lambda(U) = \sup_{x' \in U^o} \left(\sum_{i=1}^{\infty} |< x_i, x' >|^p \right)^{1/p} < \infty \qquad \forall\, U \in \mathcal{U}(E).$$

By Hölder's inequality, every $z \in F$ lies in $\lambda(U)U^{oo} = \lambda(U)\, U$ for each $U \in \mathcal{U}(E)$. Then F is bounded and its closed convex hull B is also bounded and is contained in the closed linear span of $\{x_i, i \in \mathbb{N}\}$.

Let us see that $(x_i) \in \ell^p(E_B)$. Let z' be in $(E_B)'$ such that $\|z'\| \leq 1$ and let V be the closed unit ball of $\ell^{p'}$. Given $(b_i) \in V$, there is $(c_i) \in \mathbb{K}^{\mathbb{N}}$ such that $|c_i| = 1$ and $c_i b_i < x_i, z' > = |b_i < x_i, z' >|$ for all $i \in \mathbb{N}$. Then for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} b_i c_i x_i \in F \subset B$$

and

$$\sum_{i=1}^{n} |< x_i, z' >| \, |b_i| = \sum_{i=1}^{n} b_i c_i < x_i, z' > = < \sum_{i=1}^{n} b_i c_i x_i, z' > \leq \|z'\|$$

Consequently

$$\left(\sum_{i=1}^{\infty} |<x_i, z'>|^p\right)^{1/p} = \sup \left\{ \left| \sum_{i=1}^{\infty} b_i <x_i, z'> \right|, (b_i) \in V \right\} \leqslant \| z' \|$$

and $(x_i) \in \ell^p (E_B)$.

In the case $p = 1$, this result has been obtained by Hollstein in [5].

For our study of the AP_p on inductive limits, we shall need the following lemmas:

LEMMA 1: *Let E be a Frechet space and* $(x_i) \in \ell^{p'} (E)$. *The closed absolutely convex cover of the set*

$$K = \left\{ \sum_{i=1}^{t} b_i x_i \ / \ b_i \in \mathbb{K} \ i = 1,2, \ldots, t; \ t \in \mathbb{N} \text{ and } \left(\sum_{i=1}^{t} |b_i|^p \right)^{1/p} \leqslant 1 \right\},$$

is $\sigma(E,E')$*-compact.*

Proof: By the theorems of Krein an Eberlein ([7], pág. 325 and 313), it is enough to see that each sequence

$$z_j = \sum_{i=1}^{\infty} b_i^j x_i, \qquad b_i^j = 0 \text{ if } i > t_j, \quad j \in \mathbb{N},$$

in K has a $\sigma(E,E')$-convergent subsequence.

For every $j \in \mathbb{N}$, $(b_i^j)_{i=1}^{\infty}$ belongs to the closed unit ball B of ℓ^p. Then there is a subsequence (again denoted by (b_i^j)) weakly convergent in ℓ^p to a sequence $(b_i) \in B$. Since $(x_i) \in \ell^{p'} (E)$, it is easy to see that

$$z = \sum_{i=1}^{\infty} b_i x_i \in E.$$

Then, given $x' \in E'$, the sequence $(<x_i, x'>)_{i=1}^{\infty} \in \ell^{p'}$ and hence, $z = \lim_{j \to \infty} z_j$ in $\sigma(E,E')$. This completes the proof.

LEMMA 2: *Let E be a reflexive Banach space, F a l.c.s. and* $(x_i) \in \ell^{p'} (F)$. *Let B be the closed unid ball of* ℓ^p. *Then, the set*

$$H = \left\{ \sum_{i=1}^{\infty} a_i \ f_i \otimes x_i \ / \ \| f_i \| \leqslant 1 \quad \forall i \in \mathbb{N} \quad \text{and} \quad (a_i) \in B \right\},$$

is contained in $E \hat{\otimes}_{g_p} F$ *and is* $\sigma(E \hat{\otimes}_{g_p} F, (E \hat{\otimes}_{g_p} F)')$*-relatively compact.*

Proof: It is easy to see that

$$\sum_{i=1}^{\infty} a_i\ f_i \otimes x_i\,, \qquad \|f_i\| \leqslant 1 \quad \forall\, i \in \mathbb{N},\ (a_i) \in B$$

is convergent in $E \hat{\underset{g_p}{\otimes}} F$. As this space is complete, it will be also complete for the finer topology $\tau(E \hat{\underset{g_p}{\otimes}} F, (E \hat{\underset{g_p}{\otimes}} F)')$. By Eberlein's theorem ([7], pag. 313) the lemma will be proved if we show that each sequence

$$z_n = \sum_{i=1}^{\infty} a_i^n f_i^n \otimes x_i \qquad n \in \mathbb{N},$$

in H has a weakly convergent subnet. Let U be the closed unit ball of E endowed with the induced topology by $\sigma(E,E')$. We consider on B the induced topology by $\sigma(\ell^p, \ell^{p'})$. Then the topological space

$$X = B \times U \times U \times U \times U \ \ldots$$

endowed with the product topology, is compact. Given the sequence on X

$$w_n = ((a_i^n), f_1^n, f_2^n, \ldots, f_j^n, \ldots\ ), \quad n \in \mathbb{N},$$

there is a subnet $\{ w_{n(d)}, d \in D \}$ such that

$$\lim_{d \in D} (a_i^{n(d)}) = (a_i) \in B \ \text{ in } \ \sigma(\ell^p, \ell^{p'}) \tag{1}$$

and

$$\lim_{d \in D} f_j^{n(d)} = f_j \in U \ \text{ in } \ \sigma(E,E') \qquad \forall\, j \in \mathbb{N}. \tag{2}$$

Let us see that

$$\sum_{i=1}^{\infty} a_i\ f_i \otimes x_i = \lim_{d \in D} z_{n(d)}$$

in $\sigma(E \hat{\underset{g_p}{\otimes}} F, (E \hat{\underset{g_p}{\otimes}} F)')$. Let $\varphi \in (E \hat{\underset{g_p}{\otimes}} F)'$. There is $V \in \mathcal{U}(F)$ such that the linear form

$$<\overline{\varphi}, \sum_{i=1}^{n} x_i \otimes K_V(y_i)> = <\varphi, \sum_{i=1}^{n} x_i \otimes y_i>, \ \forall \sum_{i=1}^{n} x_i \otimes K_V(y_i) \in E \otimes F_V$$

is well defined and $\bar{\varphi} \in S^{p'}(\hat{F}_V, E')$. As $(\bar{K}_V(x_i)) \in \ell^{p'}(\hat{F}_V)$, we have $(\| \bar{\varphi}(\bar{K}_V(x_i)\|) \in \ell^{p'}$. Given $\epsilon > 0$, by (1), (2) and the inequalities of Hölder and Minskowski, there are $r \in \mathbb{N}$, $t \in \mathbb{N}$ and $d_o \in D$ such that if $d \geq d_o$

$$|<\sum_{i=i}^{\infty} (a_i^{n(d)} f_i^{n(d)} - a_i f_i) \otimes x_i, \varphi>| \leq$$

$$\leq |\sum_{i=1}^{r} <(a^{n(d)} - a_i) f^{n(d)}, \bar{\varphi}(\bar{K}_V(x_i))> | +$$

$$+ |\sum_{i=r+1}^{\infty} <(a_i^{n(d)} - a_i) f_i^{n(d)}, \bar{\varphi}(\bar{K}_V(x_i))> | +$$

$$+ |\sum_{i=1}^{t} <a_i (f_i^{n(d)} - f_i), \bar{\varphi}(\bar{K}_V(x_i))> | +$$

$$+ |\sum_{i=t+1}^{\infty} <a_i (f_i^{n(d)} - f_i), \bar{\varphi}(\bar{K}_V(x_i))> | \leq \epsilon$$

and the proof is complete.

LEMMA 3: *Let M be a reflexive Banach space and let* $E = \varinjlim E_n$ *be an inductive limit of Frechet spaces* E_n *such that each* E_n *has the* AP_p. *Then, for every* $n \in \mathbb{N}$, $E' \otimes M$ *is* $\mathcal{C}_{p'}$*-dense in* $E_n' \otimes M$.

Proof: We fix $n \in \mathbb{N}$. If I is the identity map on M,I' is its dual identity map on M' and I_n is the inclusion of E_n into E, by corollary 3, the canonical map

$$I' \hat{\otimes} I_n : M' \hat{\otimes}_{g_p} E_n \longrightarrow M' \hat{\otimes}_{g_p} E$$

is injective. We define

$$H = \left\{ z \in M' \hat{\otimes}_{g_p} E \ / \ <z, M \hat{\otimes} E'> = 0 \right\} = (M \otimes E')^{\perp} \text{ in } M' \hat{\otimes}_{g_p} E,$$

and we consider the canonical quotient map K_H from $M' \hat{\otimes}_{g_p} E$ onto the quotient space $N = (M' \hat{\otimes}_{g_p} E)/H$. Each $K_H(z) \in N$ defines an element φ_z of the algebraic dual $(M \otimes E')^*$ by means of $<\varphi_z, u> = <z, u>$ for all $u \in M \otimes E'$. By the definition of H, if $K_H(z) = K_H(w)$, we have $\varphi_z = \varphi_w$. Moreover, the map $D: K_H(z) \longrightarrow \varphi_z$ is injective because $\varphi_z = 0$ implies $z \in H$, that is, $K_H(z) = 0$.

Let us see that $J = DK_H(I' \hat{\otimes} I_n)$ is also injective. Let us suppose that $J(z) = 0$. Then $(I' \hat{\otimes} I_n)(z) \in H$ and for every $m \in M$ and every $x' \in E'$ we have

$$0 = <(I' \hat{\otimes} I_n)(z), m \otimes x'> = <z, (I' \hat{\otimes} I_n)'(m \otimes x')> = <z, m \otimes I_n'(x')> (1)$$

Now, given $m \in M, y' \in E'_n$ and

$$w = \sum_{j=1}^{\infty} m'_j \otimes e_j \in M' \hat{\otimes}_{g_p} E_n \text{ with } (m'_j) \in \ell^p [M'] \text{ and } (e_j) \in \ell^{p'}(E_n) \qquad (2)$$

(see [4]), since $I'_n(E')$ is $\sigma(E'_n, E_n)$-dense in E'_n (and hence $\tau(E'_n, E_n)$-dense), by lemma 1, there is a net $\{ x'_a, a \in A \}$ in E' such that, given $\epsilon > 0$,

$$\sup \left\{ |< \sum_{j=1}^{t} b_j e_j, y' - I'_n(x'_a) >| \,/\, b_j \in \mathbb{K}\ j = 1,2,\ldots,t; t \in \mathbb{N}; \right.$$
$$\left. (\sum_{j=1}^{t} |b_j|^p)^{1/p} \leqslant 1 \right\} \leqslant \epsilon$$

for every $a \in A$ such that $a \geqslant a_o$ for some $a_o \in A$. In this case, for every $h \in \mathbb{N}$, by (2), we have

$$|\sum_{j=1}^{h} < m'_j, m >< e_j, y' - I'_n(x'_a) > | = |< \sum_{j=1}^{h} < m'_j, m > e_j, y' - I'_n(x'_a) >| \leqslant$$
$$\leqslant \epsilon (1 + (\sum_{j=1}^{\infty} |< m'_j, m >|^p)^{1/p}).$$

This proves that $m \otimes y' = \lim_{a \in A} m \otimes I'_n(x'_a)$ in $\sigma((M' \hat{\otimes}_{g_p} E_n)', M' \hat{\otimes}_{g_p} E_n)$. Then, by (1), $< z, M \otimes E'_n > = 0$. By theorem 1, $z = 0$ and J is injective.

It is easy to see that J is continuous from $M' \hat{\otimes}_{g_p} E_n$ into $(M \otimes E')^*$ when this space is endowed with the topology $\sigma((M \otimes E')^*, M \otimes E')$. Then, $J'(M \otimes E')$ is $\sigma((M' \hat{\otimes}_{g_p} E_n)', M' \hat{\otimes}_{g_p} E_n)$-dense in $(M' \hat{\otimes}_{g_p} E_n)'$ and also is $\tau((M' \hat{\otimes}_{g_p} E_n)', M' \hat{\otimes}_{g_p} E_n)$-dense.

We consider now

$$z = \sum_{r=1}^{k} x'_r \otimes m_r \in E'_n \otimes M, \qquad z^t = \sum_{r=1}^{k} m_r \otimes x'_r \in M \otimes E'_n,$$

$(x_i) \in \ell^{p'}(E_n)$ and $\epsilon > 0$. By lemma 2, the set

$$P = \left\{ \sum_{i=1}^{\infty} a_i f'_i \otimes x_i \,/\, a_i \in \mathbb{K}\ \ \forall i \in \mathbb{N}; (\sum_{i=1}^{\infty} |a_i|^p)^{1/p} \leqslant 1; \|f'_i\| \leqslant 1 \quad \forall i \in \mathbb{N} \right\}$$

is $\sigma(M' \hat{\otimes}_{g_p} E_n), (M' \hat{\otimes}_{g_p} E_n)')$-relatively compact. Then, there are

$$w = \sum_{h=1}^{s} u'_h \otimes y_h \in E' \otimes M \quad \text{and} \quad w^t = \sum_{h=1}^{s} y_h \otimes u'_h \in M \otimes E'$$

such that

$$\sup_{v \in P} |< z^t - J'(w^t), v >| = \sup_{v \in P} |<z^t, v> - <J(v), w^t>| =$$

$$= \sup_{v \in P} |< z^t - w^t, v >| \leqslant \epsilon \qquad (3)$$

Now, we choose, for every $i \in \mathbb{N}$, an element $\overline{f'_i} \in M'$ such that $\| \overline{f'_i} \| \leqslant 1$ and

$$\| (z^t - w^t)(x_i) \| = < (z^t - w^t)(x_i), \overline{f'_i} > = < z^t - w^t, \overline{f'_i} \otimes x_i > .$$

Then, by (3)

$$\left(\sum_{i=1}^{\infty} \| (z^t - w^t)(x_i) \|^{p'} \right)^{1/p'} = \sup \left\{ \left| \sum_{i=1}^{\infty} a_i \| (z^t - w^t)(x_i) \| \right| \Big/ \right.$$

$$\left. \Big/ \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \leqslant 1 \right\} = \sup \left\{ \left| < z^t - w^t, \sum_{i=1}^{\infty} a_i \overline{f'_i} \otimes x_i > \right| \Big/ \right.$$

$$\left. \Big/ \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \leqslant 1 \right\} \leqslant \epsilon$$

and the lemma is proved with help of w.

THEOREM 2: *Let* $E = \lim\limits_{\rightarrow} E_n$ *be a regular inductive limit of Frechet spaces* E_n *such that every* E_n *has the* AP_p. *Then E has the* AP_p.

Proof: Given a l.c.s. F, $\varphi \in S^{p'}(E,F)$, $(x_i) \in \ell^{p'}(E)$, $V \in \mathcal{U}(F)$ and $\epsilon > 0$, we must show that there is $w \in E' \otimes F$ such that $P_{(x_i),V}(\varphi - w) \leqslant \epsilon$.

By proposition 6, there is a bounded set B in E such that $(x_i) \in \ell^{p'}(E_B)$. Since E is regular, there is $n \in \mathbb{N}$ such that $B \subset E_B \subset E_n$ and B is bounded in E_n. Then $(x_i) \in \ell^{p'}(E_n)$ and the restriction φ_n of φ to E_n belongs to $S^{p'}(E_n,F)$. By proposition A, there are a reflexive Banach space M, a map $A \in S^{p'}(E_n,M)$ and a map $B_o \in \mathscr{L}(M,\hat{F}_V)$ such that $\overline{K}_V \varphi_n = B_o A$. Moreover, we can suppose that $\overline{A(E_n)} = M$ restricting B_o to the reflexive Banach space $\overline{A(E_n)}$ if necessary.

Let $\| B_o \|$ be the norm of the map B_o. Since E_n has the AP_p, there is $z \in E'_n \otimes M$ such that

$$\left(\sum_{i=1}^{\infty} \| A(x_i) - z(x_i) \|^{p'} \right)^{1/p'} \leqslant \epsilon / 3 (1 + \| B_o \|).$$

By lemma 3, there is

$$t = \sum_{j=1}^{k} x_j' \otimes m_j \in E' \otimes M$$

such that

$$\left(\sum_{i=1}^{\infty} \| (z - t)(x_i) \|^{p'} \right)^{1/p'} \leqslant \epsilon / 3\,(1 + \| B_o \|). \tag{1}$$

If we define

$$\eta = 1 + \sum_{j=1}^{k} \left(\sum_{i=1}^{\infty} | < x_j', x_i > |^{p'} \right)^{1/p'}, \tag{2}$$

since $A(E_n)$ is dense in M, we choose $e_j \in E_n$, $j = 1,2, \ldots k$, such that

$$\| A(e_j) - m_j \| \leqslant \epsilon / 3\, \eta\, (1 + \| B_o \|) \qquad \forall\, j = 1,2, \ldots ,k. \tag{3}$$

Then, if

$$w = \sum_{j=1}^{k} x_j' \otimes \varphi_n (e_j) \in E' \otimes F,$$

we have

$$P_{(x_i),V}(\varphi - w) = P_{(x_i),V}(\overline{K}_V\, \varphi_n - \overline{K}_V w) =$$

$$= \left(\sum_{i=1}^{\infty} \left(p_V\, (B_o\, A)(x_i) - \sum_{j=1}^{k} < x_j', x_i > B_o\, A(e_j)\right)\right)^{p'})^{1/p'} \leqslant$$

$$\leqslant \| B_o \| \left(\sum_{i=1}^{\infty} \| A(x_i) - \sum_{j=1}^{k} < x_j', x_i > A(e_j) \|^{p'} \right)^{1/p'} \leqslant$$

$$\leqslant \| B_o \| \left(\sum_{i=1}^{\infty} \| A(x_i) - z(x_i) \|^{p'} \right)^{1/p'} + \| B_o \| \left(\sum_{i=1}^{\infty} \| z(x_i) - t(x_i) \|^{p'} \right)^{1/p'} +$$

$$+ \| B_o \| \left(\sum_{i=1}^{\infty} \| \sum_{j=1}^{k} < x_j', x_j > (m_j - A(e_j)) \|^{p'} \right)^{1/p'} \leqslant \epsilon$$

by Minkowski's inequality and (1), (2) and (3). Then, the proof is complete.

REFERENCES

[1] Apiola, H.: Duality between spaces of p-summable sequences, (p,q)-summing operators and characterizations of nuclearity. Math. Ann. 219, 53-64 (1976).

[2] Bierstedt, K. and Meise, R.: Bemerkungen über die Approximationeigenschaft lokalkonvexer Funktionenräume. Math. Ann. 209, 99-107 (1974).

[3] Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires. Memoirs of the A.M.S. (1955).

[4] Harksen, J.: Tensornormtopologien. Dissertation. Kiel (1979).

[5] Hollstein, R.: Extension and lifting of continuous linear mappings in locally convex spaces. Math. Nach. 108, 275-297 (1982).

[6] Jarchow, H.: Locally convex spaces. Teubner. Stuttgart (1981).

[7] Köthe, G.: Topological vector spaces I. Springer. Berlin, Heidelberg, New York, (1969).

[8] Köthe, G.: Topological vector spaces II. Springer. Berlin. Heidelberg. New York (1979).

[9] Nelimarkka, E.: The approximation property and locally convex spaces defined by the ideal of approximable operators. Math. Nach. 107, 349-356 (1982).

[10] Pietsch, A.: Nuclear locally convex spaces. Springer. Berlin. Heildelberg, New York. (1972).

[11] Reinov, O.: Approximation properties of order p and the existence of non p-nuclear operators with p-nuclear second adjoints. Math. Nach. 109, 125-134 (1982).

[12] Saphar, P.: Produits tensoriels d'espaces de Banach et classes d'applications linéaires. Studia Math. 38, 71-100 (1970).

[13] Schaefer, H.: Topological vector spaces. Springer. Berlin. Heidelberg. New York (1970).

J.A. López Molina
E.T.S. Ingenieros Agrónomos
Cátedra Ampliación de Matemáticas
Camino de Vera
46022 Valencia. Spain.