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A KOROVKIN-TYPE THEOREM IN THE SPACE OF RIEMANN INTEGRABLE FUNCTIONS *

Michele Campiti

ABSTRACT. We give a characterization of the Korovkin subspaces of the space $\mathcal{R}(X)$ of Riemann integrable functions over a Hausdorff compact topological space X, equipped with the $\mathcal{R}$–sequential convergence. Some applications are presented in the context of the space of 2π periodic real functions which are Riemann integrable on the compact real interval $[0, 2\pi]$ and of the space of Riemann integrable functions on the standard simplex and the hypercube of $\mathbb{R}^p$ $(p \geq 1)$.

Introduction

Our starting point is the concept of $\mathcal{R}$ sequential convergence introduced in [6] in the space $\mathcal{R}([a,b])$ of Riemann integrable functions over a compact real interval $[a,b]$.

Among the most important consequences of this type of convergence, we point out the property that polynomials are not only dense in the space $\mathcal{C}([a,b])$ of real continuous functions on $[a,b]$, but also in $\mathcal{R}([a,b])$ (cf. [6], Th. 2.6 or [8], Th. 1), giving fundamental results in approximation theory.

Moreover, if we introduce in $\mathcal{R}([a,b])$ the concept of $\mathcal{R}$–Cauchy sequence (cf. [6] or [8]), we find the space $\mathcal{R}([a,b])$ to be also $\mathcal{R}$-(sequentially) complete (cf. [6], Th. 2.5 or [8], Th. 1).

In this paper, we are interested in a consequence of this type of convergence which states that the classical Korovkin theorem also holds in the space $\mathcal{R}([a,b])$; in fact.

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if $(T_n)_{n\in\mathbf{N}}$ is a sequence of positive linear operators on the space $\mathcal{R}([a,b])$ into itself, then the sequence $(T_n(f))_{n\in\mathbf{N}}$ converges $\mathcal{R}$-sequentially to f for each $f \in \mathcal{R}([a,b])$ if and only if the $\mathcal{R}$-sequential convergence is established for the three test functions p_0, p_1, p_2 (where for each $i = 0,1,2$ and $x \in [a,b]$, $p_i(x) = x^i$) (cf. [8], Th. 2).

The first question which arises from the above theorem is concerned with the existence of other types of test functions in $\mathcal{R}([a,b])$.

Later on, the concept of $\mathcal{R}$-sequential convergence was generalized in [10] to the space of Riemann integrable functions on arbitrary sets; therefore it seems also interesting to see if, under appropriate hypotheses, other Korovkin-type theorems remain true in this space.

In the present paper, we consider this last problem and the main result that we obtain (Th. 2.4) also answers to the first question.

More precisely, we consider a compact Hausdorff topological space X, an algebra Ω of subsets of X which is a base of X and a bounded regular and coregular (cf. Def. 2.2) content on Ω and characterize the $\mathcal{R}$-Korovkin subspaces in $\mathcal{R}(X)$ for sequences of positive linear operators, that is those subspaces H of $\mathcal{R}(X)$ such that for every sequence $(T_n)_{n\in\mathbf{N}}$ of positive linear operators of $\mathcal{R}(X)$ into $\mathcal{R}(X)$, the $\mathcal{R}$-sequential convergence to h of the sequence $(T_n(h))_{n\in\mathbf{N}}$ for each $h \in H$, implies the $\mathcal{R}$-sequential convergence to f of the sequence $(T_n(f))_{n\in\mathbf{N}}$ for each $f \in \mathcal{R}(X)$.

In particular, we show that if H is a Korovkin subspace in $\mathcal{C}(X)$, then H is an $\mathcal{R}$-Korovkin subspace in $\mathcal{R}(X)$ (for the $\mathcal{R}$-sequential convergence).

We also consider the space $\mathcal{R}_{2\pi}$ of Riemann integrable functions on $[0,2\pi]$ taking the same values at the points 0 and 2π and establish that the subspace generated by the constant functions and the functions sin and cos is an $\mathcal{R}$-Korovkin subspace in $\mathcal{R}_{2\pi}$.

Some applications are given showing that on the standard simplex of $\mathbf{R}^p$ and on the hypercube of $\mathbf{R}^p$ the polynomials are dense in the space of Riemann integrable functions with respect to the $\mathcal{R}$-sequential convergence.

1. Preliminary results

We first recall some preliminary definitions and properties.

Let X be an arbitrary set and Ω an algebra of subsets of X (i.e. Ω is closed under finite unions and relative complement and the whole space X belongs to Ω); the elements of Ω are called elementary sets.

Let $\mu : \Omega \longrightarrow \mathbb{R}$ be a bounded content on Ω, that is μ satisfies the following conditions:

$$\mu(X) < +\infty;$$

$$\mu(\emptyset) = 0;$$

$$\mu(A) \geq 0 \qquad \text{for each } A \in \Omega;$$

$$\mu(A \cup B) = \mu(A) + \mu(B) \qquad \text{for each } A, B \in \Omega \text{ such that } A \cap B = \emptyset.$$

Let $\mathcal{B}(X)$ be the space of all real bounded functions on X endowed with the natural order and the sup-norm: for each $f \in \mathcal{B}(X)$,

$$\|f\| = \sup_{x \in X} |f(x)|.$$

For each subset B of X, we put

$$M(f, B) = \sup_{x \in X} f(x), \qquad m(f, B) = \inf_{x \in X} f(x), \tag{1}$$

$$\omega(f, B) = \sup_{x, y \in B} |f(x) - f(y)|; \tag{2}$$

we easily have

$$\omega(f, B) = M(f, B) - m(f, B). \tag{3}$$

Now, we denote by $\mathfrak{S}$ the set of all partitions of X, whose elements are non empty elementary sets. Thus, we have $P \in \mathfrak{S}$ if there exist $A_0, \ldots, A_n \in \Omega \setminus \{\emptyset\}$ $(n \in \mathbb{N})$ such that $P = \{A_0, \ldots, A_n\}$ and

$$X = \bigcup_{i=1}^{n} A_i.$$

$$A_i \cap A_j = \emptyset \qquad \text{for each } i = 0, \ldots, n,\ j = 0, \ldots, n\ i \neq j.$$

For each $f \in \mathcal{B}(X)$ and $P = \{A_0, \ldots, A_n\} \in \mathfrak{S}$, we put

$$(4) \qquad S(f, P, \mu) = \sum_{i=0}^{n} M(f, A_i)\, \mu(A_i), \qquad s(f, P, \mu) = \sum_{i=0}^{n} m(f, A_i)\, \mu(A_i)$$

$$(5) \qquad \overline{\int} f(x)\, d\mu(x) = \inf_{P \in \mathfrak{S}} S(f, P, \mu), \qquad \underline{\int} f(x)\, d\mu(x) = \sup_{P \in \mathfrak{S}} s(f, P, \mu)$$

(if no confusion arises, we simply write $\overline{\int} f$ and $\underline{\int} f$ instead of $\overline{\int} f(x)\, d\mu(x)$ and respectively $\underline{\int} f(x)\, d\mu(x)$),

$$(6) \qquad O(f, P, \mu) = \sum_{i=0}^{n} \omega(f, A_i)\, \mu(A_i) \qquad (= S(f, P, \mu) - s(f, P, \mu))$$

(cf. [3]),

$$(7) \qquad I(f, \mu) = \inf_{P \in \mathfrak{S}} O(f, P, \mu) \qquad \left(= \overline{\int} f(x)\, d\mu(x) - \underline{\int} f(x)\, d\mu(x) \right).$$

We refer to [10] for the following properties which are true for every $f, g \in \mathcal{B}(X)$ and $\alpha \in \mathbb{R}_+^*$:

$$(8) \qquad 0 \leq I(f, \mu) = I(-f, \mu) \leq 2 \overline{\int} |f(x)|\, d\mu(x) \leq 2\, \mu(X)\, \|f\|;$$

$$(9) \qquad \overline{\int} (f + g) \leq \overline{\int} f + \overline{\int} g, \qquad \overline{\int} \alpha f = \alpha \overline{\int} f, \qquad \overline{\int} (-\alpha f) = -\alpha \underline{\int} f,$$

$$(10) \qquad \text{if } f \leq g, \text{ then } \underline{\int} f \leq \underline{\int} g,\ \overline{\int} f \leq \overline{\int} g.$$

Finally, if we put

$$\overline{\int_B} f(x)\,d\mu(x) = \overline{\int} f(x)\,1_B(x)\,d\mu(x)$$

for each $B \in \Omega$ (where 1_B denotes the characteristic function of B), we have

$$\overline{\int_B} |f(x)|\,d\mu(x) \le \mu(B)\,\|f\,1_B\| \tag{11}$$

for each $B \subset \Omega$.

Now the set of Riemann integrable functions is defined by:

$$\mathcal{R}(X) = \{f \in \mathcal{B}(X) : I(f,\mu) = 0\} \qquad \left(= \left\{ f \in \mathcal{B}(X) : \overline{\int} f = \underline{\int} f \right\}\right) \tag{12}$$

(sometimes we write $\mathcal{R}(X,\mu)$ instead of $\mathcal{R}(X)$).

If $f \in \mathcal{R}(X)$, we put

$$\int f(x)\,d\mu(x) = \overline{\int} f(x)\,d\mu(x) \qquad \left(= \underline{\int} f(x)\,d\mu(x)\right).$$

If $B \in \Omega$, we recall that a function $f \in \mathcal{B}(X)$ is Riemann integrable if and only if the functions $f\,1_B$ and $f\,1_{X\setminus B}$ are Riemann integrable and, in this case

$$\int f = \int_B f + \int_{X\setminus B} f.$$

In what follows, we suppose that X is a compact Hausdorff topological space and that the algebra Ω of subsets of X is a base of X, that is

(13) *for each $x \in X$ and for each neighbourhood U of x, there exists a neighbourhood A of x with the properties*

$$A \in \Omega, \qquad A \subset U.$$

Further, we denote by $\mathcal{C}(X)$ the space of all real continuous functions on X; since X is compact we have $\mathcal{C}(X) \subset \mathcal{B}(X)$ and since Ω is a base of X, we have $\mathcal{C}(X) \subset \mathcal{R}(X)$ (cf. [10], Lemma 2).

Proposition 1.1. *Let X be a compact Hausdorff topological space, Ω an algebra of subsets of X which is a base of X and $\mu : \Omega \to \mathbb{R}$ a bounded content.*

If $f \in \mathcal{B}(X)$ and if for each $\epsilon \in \mathbb{R}_+^*$ there exist $\phi \in \mathcal{C}(X)$ and $\psi \in \mathcal{C}(X)$ such that

$$\phi \leq f \leq \psi, \qquad \int(\psi - \phi) < \epsilon,$$

then $f \in \mathcal{R}(X)$.

Proof. Let $\epsilon \in \mathbb{R}_+^*$ and $\phi \in \mathcal{C}(X)$, $\psi \in \mathcal{C}(X)$ be such that

$$\phi \leq f \leq \psi, \qquad \int(\psi - \phi) < \epsilon,$$

(we have rightly considered $\int(\psi - \phi)$ since $\mathcal{C}(X) \subset \mathcal{R}(X)$ (cf. [10], Lemma 2). We have (cf. (10))

$$\underline{\int} \phi \leq \underline{\int} f \leq \underline{\int} \psi \leq \overline{\int} \psi$$

and therefore

$$\begin{aligned} I(f,\mu) &= \overline{\int} f - \underline{\int} f \\ &\leq \overline{\int} \psi - \underline{\int} \phi \\ &= \int \psi - \int \phi \\ &= \int (\psi - \phi) < \epsilon. \end{aligned}$$

Since $\epsilon \in \mathbb{R}_+^*$ is arbitrary, it follows $I(f) = 0$, that is $f \in \mathcal{R}(X)$ (cf. (12)). ■

At this point, we recall that (cf. [3], 7.3.1, p. 208) a content $\mu : \Omega \to \mathbb{R}$ is said to be regular if for $A \in \Omega$ and $\epsilon \in \mathbb{R}_+^*$, there exist a compact set $F \in \Omega$ and an open set $G \in \Omega$ such that

$$F \subset A \subset G, \qquad \mu(G \setminus F) < \epsilon.$$

In order to obtain the converse of Prop. 1.1, we give the following definition:

Definition 1.2. Let X be a topological space and Ω an algebra of subsets of X.

A content $\mu : \Omega \to \mathbf{R}$ is said to be *coregular* if, for each elementary set $A \in \Omega$ and for each $\epsilon \in \mathbf{R}_+^*$, there exist an open elementary set $G \in \Omega$ and a closed elementary set $F \in \Omega$ such that

$$G \subset A \subset F, \qquad \mu(F \setminus G) < \epsilon.$$

We give here some general properties of coregular contents.

Proposition 1.3. *Let X be a topological space, Ω an algebra of subsets of X and $\mu : \Omega \to \mathbf{R}$ a regular content.*

Then the following statements are equivalent:

a) *μ is coregular.*

b) 1. *For each closed elementary set $F \in \Omega$ and for each $\epsilon \in \mathbf{R}_+^*$, there exists an open elementary set $G \in \Omega$ such that*

$$G \subset F, \qquad \mu(F \setminus G) < \epsilon;$$

2. *for each open elementary set $G \in \Omega$ and for each $\epsilon \in \mathbf{R}_+^*$, there exists a closed elementary set $F \in \Omega$ such that*

$$G \subset F, \qquad \mu(F \setminus G) < \epsilon.$$

Proof. The condition a) trivially implies the condition b).

Conversely, let $A \in \Omega$ and $\epsilon \in \mathbf{R}_+^*$; since μ is regular there exist a closed (compact) elementary set $F_1 \in \Omega$ and an open elementary set $G_1 \in \Omega$ such that

$$F_1 \subset A \subset G_1, \qquad \mu(G_1 \setminus F_1) < \frac{\epsilon}{3}.$$

The condition 1 of b) implies the existence of an open elementary set $G \in \Omega$ such that

$$G \subset F_1, \qquad \mu(F_1 \setminus G) < \frac{\epsilon}{3},$$

while the condition 2 of of b) implies the existence of a closed elementary set $F \in \Omega$ such that

$$G_1 \subset F, \qquad \mu(F \setminus G_1) < \frac{\epsilon}{3}.$$

It follows $G \subset A \subset F$ and

$$\begin{aligned}\mu(F \setminus G) &= \mu(F \setminus G_1) + \mu(G_1 \setminus F_1) + \mu(F_1 \setminus G) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \blacksquare\end{aligned}$$

Proposition 1.4. *Let X be a topological space, Ω an algebra of subsets of X containing the open subsets of X and $\mu : \Omega \to \mathbb{R}$ a bounded content. Then, the following statements are equivalent:*

a) *μ is coregular.*

b) *For each $A \in \Omega$ one also has*

$$\bar{A} \in \Omega \qquad \text{and} \qquad \mu(A) = \mu(\bar{A}).$$

c) *For each $A \in \Omega$ one also has*

$$A^\circ \in \Omega \qquad \text{and} \qquad \mu(A) = \mu(A^\circ).$$

d) *For each $A \in \Omega$ such that $A^\circ = \emptyset$, it results $\mu(A) = 0$.*

e) *For each $A \in \Omega$ such that $\bar{A} = X$, it results $\mu(A) = \mu(X)$.*

Proof. a)$\Rightarrow$b) Let $A \in \Omega$; since $A \subset \bar{A}$, we have $\mu(A) \le \mu(\bar{A})$. Suppose $\mu(A) < \mu(\bar{A})$ and put $\epsilon = \mu(\bar{A}) - \mu(A)$ $(\in \mathbb{R}_+^*)$; by a), there exist a closed set $F \in \Omega$ and an open set $G \in \Omega$ such that

$$G \subset A \subset F, \qquad \mu(F \setminus G) < \epsilon.$$

The set F is closed and therefore $\bar{A} \subset F$ and $\mu(\bar{A}) \le \mu(F)$; it follows

$$\mu(\bar{A}) - \mu(A) = \mu(\bar{A} \setminus A) \le \mu(F \setminus G) < \epsilon,$$

that is a contradiction; then $\mu(\bar{A}) = \mu(A)$.

b)$\Rightarrow$c) Let $A \in \Omega$; by the formula $X \setminus A^\circ = \overline{X \setminus A}$, we obtain $A^\circ = X \setminus (\overline{X \setminus A})$ and therefore, by b),

$$\begin{aligned} \mu(A^\circ) &= \mu(X) - \mu(\overline{X \setminus A}) \\ &= \mu(X) - \mu(X \setminus A) \\ &= \mu(X) - \big(\mu(X) - \mu(A)\big) \\ &= \mu(A). \end{aligned}$$

c)$\Rightarrow$d) It is obvious since $\mu(\emptyset) = 0$.

d)$\Rightarrow$e) Let $A \in \Omega$ such that $\bar{A} = X$; then $X \setminus A$ has empty interior and, by d), $\mu(X \setminus A) = 0$; hence

$$\mu(A) = \mu(X) - \mu(X \setminus A) = \mu(X).$$

e)⇒a) Let $A \in \Omega$ and consider the open set $G = A^\circ \in \Omega$ and the closed set $F = \bar{A} \in \Omega$; since $G \subset A \subset F$, it is enough to prove that $\mu(F \setminus G) = 0$. In fact the set $F \setminus G$ has empty interior so its complement $X \setminus (F \setminus G)$ is dense in X; by e), we obtain

$$\mu(X) - \mu(F \setminus G) = \mu(X \setminus (F \setminus G)) = \mu(X),$$

and this implies $\mu(F \setminus G) = 0$, completing the proof. ■

Remark 1.5. 1) We observe that if we do not suppose that Ω contains the open subsets of X, the implications a)⇒d) and a)⇒e) of Proposition 1.4 remain true, and further the conditions b) and c) together imply a).

2) Let $p \in \mathbb{N}$ and consider a Peano-Jordan measurable compact subset X of $\mathbb{R}^p$; further, denote by μ the Peano-Jordan content on the algebra of all Peano-Jordan measurable subsets of $\mathbb{R}^p$ and by μ_X the Peano-Jordan content on the algebra Ω of all Peano-Jordan measurable subsets of X (that is, $\mu_X = \mu_{|\Omega}$). Then μ_X is clearly regular since, if $A \in \Omega$ and $\epsilon \in \mathbb{R}_+^*$, ther exist a closed p-dimensional figure P_1 (i.e. P_1 is the union of a finite number of intervals in $\mathbb{R}^p$) and an open p-dimensional figure P_2 such that

$$X \cap P_1 \subset A \subset X \cap P_2 \qquad \text{and} \qquad \mu(P_2 \setminus P_1) < \epsilon,$$

from which

$$\mu_X\big((X \cap P_2) \setminus (X \cap P_1)\big) = \mu_X\big(X \cap (P_2 \setminus P_1)\big) < \epsilon;$$

further, the interior P_1' of P_1 and the closure P_2' of P_2 are again p dimensional figures and have the same measure of P_1 and respectively P_2; since

$$X \cap P_1' \subset A \subset X \cap P_2' \qquad \text{and} \qquad \mu(P_2' \setminus P_1') = \mu(P_2 \setminus P_1) < \epsilon,$$

it follows that μ_X is also coregular.

Moreover, the Lebesgue measure λ on X is regular (cf. [3], 7.3.3, p. 210), but it is not coregular (if $\lambda(X) > 0$); for example, the set $\mathbb{Q}^p \cap X$ is dense in X but $\lambda(\mathbb{Q}^p \cap X) = 0$ and therefore the condition e) of Prop. 1.4 is not fulfilled.

Proposition 1.6. *Let X be a compact Hausdorff topological space, Ω an algebra of subsets of X which is also a base of X (cf. (13)), $\mu : \Omega \to \mathbb{R}$ a bounded content both regular and coregular and $f \in \mathcal{B}(X)$.*

Then $f \in \mathcal{R}(X)$ if and only if, for each $\epsilon \in \mathbb{R}_+^*$, there exist $\phi \in C(X)$ and $\psi \in C(X)$ such that

$$(14) \qquad \varphi \le f \le \psi, \qquad \int (\psi - \varphi) < \epsilon.$$

Proof. By virtue of Prop. 1.1, we have only to show the necessity of the condition (14). We proceed by different steps:

1) Suppose that $f = 1_A$ with $A \in \Omega$. Let $\epsilon \in \mathbf{R}_+^*$; since μ is coregular there exist an open set $G \in \Omega$ and a closed set $F \in \Omega$ such that

$$G \subset A \subset F, \qquad \mu(F \setminus G) < \frac{\epsilon}{3}\,. \tag{15}$$

Since μ is regular and $G \in \Omega$, there exists a closed set $F_1 \in \Omega$ such that

$$F_1 \subset G, \qquad \mu(G \setminus F_1) < \frac{\epsilon}{3} \tag{16}$$

and since $F \in \Omega$ there exists an open set $G_1 \in \Omega$ such that

$$F \subset G_1, \qquad \mu(G_1 \setminus F) < \frac{\epsilon}{3}\,. \tag{17}$$

The sets F_1 and $X \setminus G$ are closed and disjoint; by the normality of X, there exists a continuous function $\phi \in \mathcal{C}(X)$ such that

$$0 \le \phi \le 1, \qquad \phi = 1 \text{ on } F_1, \qquad \phi = 0 \text{ on } X \setminus G. \tag{18}$$

Analogously, the sets F and $X \setminus G_1$ are closed and disjoint and therefore there exists a continuous function $\psi \in \mathcal{C}(X)$ such that

$$0 \le \psi \le 1, \qquad \psi = 1 \text{ on } F, \qquad \psi = 0 \text{ on } X \setminus G_1. \tag{19}$$

We have $\phi \le 1$ and $\phi = 0$ on $X \setminus A$ $(G \subset A)$ and further $0 \le \psi$ and $\psi = 1$ on A $(A \subset F)$; it follows $\phi \le f \le \psi$.

Finally, by (15)-(19),

$$\begin{aligned}
\int(\psi - \phi) &= \int_{F_1}(\psi - \phi) + \int_{G_1 \setminus F_1}(\psi - \phi) + \int_{X \setminus G_1}(\psi - \phi)\\
&= \int_{G_1 \setminus F_1}(\psi - \phi)\\
&\le \int_{G_1 \setminus F_1} 1_X\\
&= \mu(G_1 \setminus F_1)\\
&= \mu(G_1 \setminus F) + \mu(G \setminus F) + \mu(G \setminus F_1)\\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\\
&= \epsilon,
\end{aligned}$$

and this completes the proof in the first case.

2) Suppose that

$$f = \sum_{i=0}^{n} \alpha_i 1_{A_i}$$

with $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ and $P = \{A_0, \dots, A_n\} \in \mathfrak{S}$.

In this case the result follows from step 1), since the set of all $f \in \mathcal{R}(X)$ satisfying (14) is clearly a (real) vector space.

3) We examine now the general case $f \in \mathcal{R}(X)$.

Let $\epsilon \in \mathbb{R}_+^*$; since $I(f, \mu) = 0$ (cf. (12)), there exists $P = \{A_0, \dots, A_n\} \in \Omega$ such that (cf. (6) and (7))

$$S(f, P, \mu) - s(f, P, \mu) < \frac{\epsilon}{3}. \tag{20}$$

Consider the functions

$$g = \sum_{i=0}^{n} m(f, A_i)\, 1_{A_i}, \qquad h = \sum_{i=0}^{n} M(f, A_i)\, 1_{A_i};$$

obviously (cf. (20))

$$g \le f \le h, \qquad \int (h - g) < \frac{\epsilon}{3}; \tag{21}$$

moreover, by step 2), there exist $\phi \in \mathcal{C}(X)$ and $\psi \in \mathcal{C}(X)$ such that

$$\phi \le g, \qquad \int (g - \phi) < \frac{\epsilon}{3}$$

and

$$h \le \psi, \qquad \int (\psi - h) < \epsilon.$$

Then $\phi \le f \le \psi$ and (cf. (21))

$$\begin{aligned} \int (\psi - \phi) &= \int (\psi - h) + \int (h - g) + \int (g - \phi) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon, \end{aligned}$$

completing the proof. ■

Remark 1.7. Suppose that Ω contains the open subsets of X. If μ is not coregular, the Proposition 1.4 ensures the existence of an elementary set $A \in \Omega$ with empty interior and $\mu(A) > 0$; it follows that the characteristic function 1_A is Riemann integrable but, for each continuous functions $\phi, \psi \in \mathcal{C}(X)$ satisfying $\phi \leq 1_A \leq \psi$, it results

$$\underline{\int}(\psi - \phi) \geq \mu(A).$$

Therefore, if Ω contains the open subsets of X, the validity of Prop. 1.6 requires μ to be coregular.

2. The main theorem

We fix a compact Hausdorff topological space X, an algebra Ω of subsets of X which is also a base of X (cf. (13)) and a bounded content $\mu : \Omega \to \mathbf{R}$.

Denote by $\mathcal{P}(X)$ the set of all subsets of X; we need to consider the outer content $\bar{\mu} : \mathcal{P}(X) \to \mathbf{R}$ defined by putting, for each subset A of X,

$$\bar{\mu}(A) = \inf_{\substack{B \in \Omega \\ A \subset B}} \mu(B). \tag{22}$$

Further, we say that a subset A of X is a $\bar{\mu}$–null set if $\bar{\mu}(A) = 0$.

If μ is regular (resp. coregular), for each subset A of X, it results

$$\bar{\mu}(A) = \inf\{\mu(G) : G \in \Omega,\ A \subset G,\ G \text{ open}\} \tag{23}$$

$$(\text{resp. } \bar{\mu}(A) = \inf\{\mu(K) : K \in \Omega,\ A \subset K,\ K \text{ compact}\}).$$

In fact, if $\epsilon \in \mathbf{R}_+^*$, by (22) there exists $B \in \Omega$ such that $A \subset B$ and

$$\mu(B) < \bar{\mu}(A) + \frac{\epsilon}{2};$$

then $A \subset G$ and

$$\mu(G) = \mu(B) + \mu(G \setminus B) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

From now on, we consider the natural order on the spaces $\mathcal{C}(X)$ and $\mathcal{R}(X)$.

For each $x \in X$, we denote by $\delta_x : \mathcal{C}(X) \to \mathbf{R}$ the valuation functional defined by putting, for each $g \in \mathcal{C}(X)$, $\delta_x(g) = g(x)$.

A subspace H of $\mathcal{C}(X)$ is said to be cofinal if for each $f \in \mathcal{C}(X)$ there exists $h \in H$ such that $f \le h$ (clearly, in this case for each $f \in \mathcal{C}(X)$ we also have $k \le f$ for some $k \in H$).

We observe that if H is a cofinal subspace of $\mathcal{C}(X)$, $f \in \mathcal{C}(X)$ and $v : \mathcal{C}(X) \to \mathbf{R}$ is a positive linear form on $\mathcal{C}(X)$, then a straightforward application of the Hahn-Banach theorem yields a positive linear form $\tilde{v} : \mathcal{R}(X) \to \mathbf{R}$ such that $\tilde{v} = v$ on H and $\tilde{v}(f) = v(f)$.

At this point we can introduce the concept of $\mathcal{R}$-sequential convergence on the space $\mathcal{R}(X)$.

Definition 2.1. A sequence $(f_n)_{n\in\mathbf{N}}$ of elements of $\mathcal{R}(X)$ is called $\mathcal{R}$ *convergent to* $f \in \mathcal{R}(X)$ (in notation, $f = \mathcal{R} - \lim_{n\to\infty} f_n$) if

$$(\mathcal{R}1) \qquad \sup_{n\in\mathbf{N}} \|f_n\| < +\infty,$$

$$(\mathcal{R}2) \qquad \lim_{n\to\infty} \overline{\int} \sup_{k\geq n} |f_k - f| = 0.$$

Uniform convergence clearly implies $\mathcal{R}$-convergence.

Further, $\mathcal{R}$ convergence implies the convergence in L^1 norm and pointwise convergence Lebesgue almost everywhere (cf. [6] and [10]).

Observe also that the integrand in $(\mathcal{R}2)$ is monotone with respect to n and this, together with the equiboundedness property in $(\mathcal{R}1)$, is useful in many proofs as a substitute of the σ-additivity of the algebra Ω.

In what follows, we consider the space $\mathcal{R}(X)$ equipped with the $\mathcal{R}$-sequential convergence and the natural order.

We recall the following:

Definition 2.2. A subspace H of the vector space $\mathcal{R}(X)$ is an $\mathcal{R}$*-Korovkin subspace* in $\mathcal{R}(X)$ for sequences of positive linear operators if for every sequence $(T_n)_{n\in\mathbf{N}}$ of positive linear operators of $\mathcal{R}(X)$ into $\mathcal{R}(X)$, the condition

$$h = \mathcal{R} - \lim_{n\to\infty} T_n(h) \qquad \text{for each } h \in H,$$

also ensures

$$f = \mathcal{R} - \lim_{n\to\infty} T_n(f) \qquad \text{for each } f \in \mathcal{R}(X).$$

In order to give a characterization of the $\mathcal{R}$ Korovkin subspaces in $\mathcal{R}(X)$ for sequences of positive linear operators, we need the following lemma:

Lemma 2.3. *Let X be a compact metrizable topological space, Ω an algebra of subsets of X which is a base of X and $\mu : \Omega \to \Re$ a coregular bounded content.*

Then, for each non empty subset A of X there exists a sequence $(x_n)_{n\in\mathbf{N}}$ of elements of A such that, for each $n \in \mathbf{N}$,

$$\mu\left(\bigcup_{k\geq n}\{x_k\}\right) = \mu(A).$$

Proof. Since X is a compact metrizable space, there exists a countably base for the neighbourhood system $(A_n)_{n\in\mathbf{N}}$ of X and since Ω is a base of X, we may also assume $A_n \in \Omega$ for each $n \in \mathbf{N}$.

Let A be a non empty subset of X and put

$$M = \{m \in \mathbf{N} : A \cap A_m \neq \emptyset\};$$

further, for each $m \in \mathbf{N}$, choose $y \in A \cap A_m$.

At first, we show that

$$\bar{\mu}\left(\bigcup_{m\in M} \{y_m\}\right) = \bar{\mu}(A).$$

Obviously

$$\bar{\mu}\left(\bigcup_{m\in M} \{y_m\}\right) \leq \bar{\mu}(A);$$

conversely let K be a compact subset of X such that $K \in \Omega$ and

$$\bigcup_{m\in M} \{y_m\} \subset K.$$

If $x \in A \setminus K$ ($\subset X \setminus K$), there exists $m \in \mathbf{N}$ such that $x \in A_m \subset X \setminus K$ and therefore $x \in A \cap A_m$; it follows $m \in M$ and $y_m \in A \cap A_m \subset X \setminus K$, contradicting $y_m \in K$. Hence $A \subset K$ and therefore by (23)

$$\mu(A) \leq \bar{\mu}\left(\bigcup_{m\in M} \{y_m\}\right).$$

Now, we define the map $\phi : \mathbf{N} \to M$ by putting $\varphi(0) = \min M$ and, for each $n \in \mathbf{N}$,

$$\phi(n+1) = \begin{cases} \min\{m \in M : \phi(n) < m\} & \text{if } \{m \in M : \varphi(n) < m\} \neq \emptyset \\ \min M & \text{if } \{m \in M : \varphi(n) < m\} = \emptyset \end{cases}$$

if M is finite, or

$$\varphi(n+1) = \begin{cases} \min\{m \in M : \phi(n) < m\} & \text{if } n+1 \neq k! \text{ for each } k \in \mathbf{N} \\ \min M & \text{if } n+1 = k! \text{ for some } k \in \mathbf{N} \end{cases}$$

if M is not finite.

Finally, we consider the sequence $(x_n)_{n\in\mathbf{N}}$ defined by putting, for each $n \in \mathbf{N}$, $x_n = y_{\phi(n)}$. For each $n \in \mathbf{N}$, we have

$$\bigcup_{k\geq n}\{x_k\} = \bigcup_{m\in M}\{y_m\}$$

and this completes the proof. ■

If H is a subspace of $\mathcal{C}(X)$ and if $f \in \mathcal{C}(X)$, it is useful to consider the set $U_H(f)$ of all the $x \in X$ such that $v(f) = f(x)$ for each positive linear form v on $\mathcal{C}(X)$ satisfying the condition $v = \delta_x$ on H. It is well known that (cf. [2])

$$U_H(f) = \left\{ x \in X : \sup_{\substack{h\in H\\ h\leq f}} h(x) = \inf_{\substack{h\in H\\ f\leq h}} h(x) \right\}. \tag{24}$$

Now, we can state our main theorem.

Theorem 2.4. *Let X be a compact Hausdorff topological space, Ω an algebra of subsets of X which is a base of X (cf. (13)) and $\mu : \Omega \to \mathbf{R}$ a bounded content both regular and coregular (cf. Def. 1.3).*

Let H be a cofinal vector subspace of $\mathcal{C}(X)$. If H fulfills the following condition

a) *For each $f \in \mathcal{C}(X)$ and for each compact subset K of X disjoint from $U_H(f)$, it results $\bar{\mu}(K) = 0$,*

then H is an $\mathcal{R}$ Korovkin subspace in $\mathcal{R}(X)$ for sequences of positive linear operators (cf. Def. 2.2).

Moreover, if X is metrizable and, for each $x \in X$, $\{x\} \in \Omega$, then also the converse holds.

Proof. Let $(T_n)_{n\in\mathbf{N}}$ be a sequence of positive linear operators of $\mathcal{R}(X)$ into itself and suppose that, for each $h \in H$,

$$h = \mathcal{R} - \lim_{n\to\infty} T_n(h). \tag{25}$$

We show primarily that, for each $f \in \mathcal{R}(X)$, the sequence $(T_n(f))_{n\in\mathbf{N}}$ is equibounded.

Denote by $p_0 : X \to \mathbf{R}$ the constant function of constant value 1. Since H is cofinal, there exist $h_1 \in H$ and $h_2 \in H$ such that $h_1 \leq p_0 \leq h_2$; the sequences $(T_n(h_1))_{n\in\mathbf{N}}$ and $(T_n(h_2))_{n\in\mathbf{N}}$ are equibounded (cf. (25) and $(\mathcal{R}_1)$) and further, for each $n \in \mathbf{N}$,

$$T_h(h_1) \leq T_n(p_0) \leq T_n(h_2);$$

it follows that the sequence $(T_n(p_0))_{n\in\mathbf{N}}$ is equibounded. Now let $f \in \mathcal{R}(X)$; for each $n \in \mathbf{N}$,

$$|T_n(f)| = \left|T_n(f\,p_0)\right| \le \left|T_n(\|f\|\,p_0)\right| = \|f\|\,T_n(p_0)$$

and therefore the sequence $(T_n(f))_{n\in\mathbf{N}}$ is equibounded.

Now we show that, for each $f \in \mathcal{R}(X)$,

$$\lim_{n\to\infty} \overline{\int} \sup_{k\ge n} |T_k(f) - f| = 0.$$

At first, we suppose $f \in \mathcal{C}(X)$. Let $\epsilon \in \mathbf{R}_+^*$ and $x_0 \in U_H(f)$; by (24), it follows the existence of $\phi \in H$ and $\psi \in H$ such that

$$\phi \le f \le \psi, \qquad \psi(x_0) - \frac{\epsilon}{8\,\mu(X)} < f(x_0) < \phi(x_0) + \frac{\epsilon}{8\,\mu(X)}\,. \tag{26}$$

Since ϕ, ψ and f are continuous, there exists an open neighbourhood $U(x_0)$ of x_0 such that, for each $x \in U(x_0)$,

$$\psi(x) - \frac{\epsilon}{8\,\mu(X)} < f(x) < \phi(x) + \frac{\epsilon}{8\,\mu(X)}\,. \tag{27}$$

Let $k \in \mathbf{N}$; by (26), it follows

$$T_k(\phi) \le T_k(f) \le T_k(\psi)$$

and therefore, for each $x \in X$,

$$T_k(f)(x) - f(x) \le T_k(\psi)(x) - \phi(x) \le |T_k(\psi)(x) - \phi(x)| + |T_k(\phi)(x) - \psi(x)|,$$

$$f(x) - T_k(f)(x) \le \psi(x) - T_k(\phi)(x) \le |T_k(\psi)(x) - \phi(x)| + |T_k(\phi)(x) - \psi(x)|;$$

hence, for each $x \in X$,

$$|T_k(f)(x) - f(x)| \le |T_k(\psi)(x) - \psi(x)| + |T_k(\phi)(x) - \phi(x)| + 2\,|\phi(x) - \psi(x)|$$

and by (27), for each $x \in U(x_0)$,

$$|T_k(f)(x) - f(x)| \le |T_k(\psi)(x) - \psi(x)| + |T_k(\phi)(x) - \phi(x)| + \frac{\epsilon}{2\,\mu(X)}\,. \tag{28}$$

Put $\Omega(x_0) = \{A \in \Omega : A \subset U(x_0)\}$. If $A \in \Omega(x_0)$ (28) remains true for all $x \in A$ and therefore, for every $n \in \mathbf{N}$,

$$\overline{\int_A} \sup_{k\ge n} |T_k(f) - f| \le \overline{\int} \sup_{k\ge n} |T_k(\psi) - \psi| + \overline{\int} \sup_{k\ge n} |T_k(\phi) - \phi| + \frac{\epsilon\,\mu(A)}{2\,\mu(X)}$$

from which (cf. (25) and $(\mathcal{R}2)$) we deduce

$$\lim_{n\to\infty} \overline{\int_A} \sup_{k\geq n} |T_k(f) - f| \leq \frac{\epsilon\,\mu(A)}{2\,\mu(X)}\,. \tag{29}$$

Now let x_0 vary in $U_H(f)$ and consider the sets:

$$\Gamma = \bigcup_{x_0\in U_H(f)} \Omega(x_0), \qquad U = \bigcup_{a\in\Gamma} A; \tag{30}$$

we may assume that (29) holds for each element $A \in \Gamma$; further, since Ω is a base of X, the family $(A)_{A\in\Omega(x_0)}$ is a covering of $U(x_0)$ for each $x_0 \in U_H(f)$; consequently

$$U = \bigcup_{x_0\in U_H(f)} U(x_0)$$

is an open subset of X and the family $(A)_{A\in\Gamma}$ is a covering of $U_H(f)$ such that every $x_0 \in U_H(f)$ is in the interior of some $A \in \Gamma$ (cf. (13)).

Put $K = X \setminus U$; K is a compact subset of X disjoint from $U_H(f)$ and then, by a), $\mu(K) = 0$. Since μ is regular, there exists an open subset G of X such that $G \in \Omega$, $K \subset G$ and

$$\mu(G) < \frac{\epsilon}{2M+1} \tag{31}$$

where $M = \sup_{n\in\mathbf{N}} \|T_n(f) - f\|$, (cf. (23)).

The compact set $X \setminus G$ is contained in U and then the family $(A)_{A\in\Gamma}$ is a cover of $X \setminus G$ (cf. (30)) and each $x_0 \in X \setminus G$ is in the interior of some $A \in \Gamma$; it follows the existence of a finite cover $A_0, \ldots, A_p$ ($p \in \mathbf{N}$) of elements of Γ.

Put $B_0 = A_0 \setminus G$ and, for each $k = 1, \ldots, p$,

$$B_k = A_k \setminus \left(\bigcup_{i=0}^{k-1} A_i \cup G\right);$$

then $P = \{B_0, \ldots, B_p, G\} \in \mathfrak{S}$ (we have supposed that each element of P is not empty; otherwise we may consider $P \setminus \{\emptyset\}$) and, for each $i = 0, \ldots, p$ (cf. (29))

$$\lim_{n\to\infty} \overline{\int_{B_i}} \sup_{k\geq n} |T_k(f) - f| \leq \frac{\epsilon}{2\,\mu(X)}\,\mu(B_i). \tag{32}$$

Hence (cf. (32) and (31))

$$\begin{aligned}\lim_{n\to\infty}\overline{\int}\sup_{k\geq n}|T_k(f)-f| &\leq \lim_{n\to\infty}\left(\sum_{i=0}^{p}\left(\overline{\int_{B_i}}\sup_{k\geq n}|T_k(f)-f|\right)\right.\\ &\qquad\left.+\overline{\int_{G}}\sup_{k\geq n}|T_k(f)-f|\right)\\ &=\sum_{i=0}^{p}\lim_{n\to\infty}\overline{\int_{B_i}}\sup_{k\geq n}|T_k(f)-f|\\ &\qquad+\lim_{n\to\infty}\overline{\int_{G}}\sup_{k\geq n}|T_k(f)-f|\\ &\leq\sum_{i=0}^{p}\frac{\epsilon}{2\,\mu(X)}\,\mu(B_i)+\sup_{n\in\mathbf{N}}\|T_n(f)-f\|\,\mu(G)\\ &<\frac{\epsilon}{2}+\frac{\epsilon}{2}\\ &=\epsilon.\end{aligned}$$

Since $\epsilon\in\mathbf{R}_+^*$ is arbitrary, the result is true for $f\in\mathcal{C}(X)$.

Now, suppose $f\in\mathcal{R}(X)$, and let $\epsilon\in\mathbf{R}_+^*$; by virtue of Prop. 1.6, there exist $\phi\in\mathcal{C}(X)$ and $\psi\in\mathcal{C}(X)$ such that

$$\phi\leq f\leq\psi,\qquad\int(\psi-\phi)<\frac{\epsilon}{2}\,.$$

For each $k\in\mathbf{N}$, we have

$$T_k(\phi)\leq T_k(f)\leq T_k(\psi)$$

and therefore, for each $n\in\mathbf{N}$,

$$\sup_{k\geq n}|T_k(f)-f|\leq\sup_{k\geq n}|T_k(\psi)-\psi|+\sup_{k\geq n}|T_k(\phi)-\phi|+2\,(\psi-\phi).$$

Since $\phi,\psi\in\mathcal{C}(X)$, it results

$$\lim_{n\to\infty}\overline{\int}\sup_{k\geq n}|T_k(\psi)-\psi|=0,$$

$$\lim_{n\to\infty} \overline{\int} \sup_{k\geq n} |T_k(\phi) - \phi| = 0$$

and consequently

$$\lim_{n\to\infty} \overline{\int} \sup_{k\geq n} |T_k(f) - f| \leq 2 \int(\psi - \phi) < \epsilon.$$

Since $\epsilon \in \mathbf{R}_+^*$ is arbitrary, it follows

$$\lim_{n\to\infty} \overline{\int} \sup_{k\geq n} |T_k(f) - f| = 0;$$

thus the first part of the proof is complete.

Conversely, suppose that X is metrizable and, for each $x \in X$, $\{x\} \in \Omega$. Let H be an $\mathcal{R}$ Korovkin subspace in $\mathcal{R}(X)$ for sequences of positive linear operators; we argue by contradiction and suppose that there exist $f \in \mathcal{C}(X)$ and a compact subset K of X disjoint from $U_H(f)$ such that $\bar{\mu}(K) > 0$.

For each $x \in X \setminus U_H(f)$, there exists a positive linear form $v_x : \mathcal{C}(X) \to \mathbf{R}$ such that $v_x = \delta_x$ on H and $v_x(f) \neq f(x)$ (cf. (24)) and then there exists also a positive linear form $\tilde{v}_x : \mathcal{R}(X) \to \mathbf{R}$ such that $\tilde{v}_x = \delta_x$ on H and $\tilde{v}_x(f) \neq f(x)$. For each $n \in \mathbf{N}$, put

$$A_n = \left\{ x \in X \setminus U_H(f) : |\tilde{v}_x(f) - f(x)| \geq \frac{1}{n+1} \right\}. \tag{33}$$

If $\mu(A_n) = 0$ for each $n \in \mathbf{N}$, then for each $\epsilon \in \mathbf{R}_+^*$ and $n \in \mathbf{N}$, we can find an open subset $G_n \in \Omega$ such that $A_n \subset G_n$ and $\mu(G_n) < \epsilon/2^{n+1}$ (cf. (23)); since K is compact, there exists a finite subset M of $\mathbf{N}$ such that

$$K \subset \bigcup_{m\in M} G_m;$$

then

$$\bigcup_{m\in M} G_m \in \Omega$$

and

$$\mu\left(\bigcup_{m\in M} G_m\right) \leq \sum_{m\in M} \mu(G_m) < \epsilon$$

(cf. [3], (1.3.9), p. 11); since $\epsilon \in \mathbf{R}_+^*$ is arbitrary, it follows $\bar{\mu}(K) = 0$ (cf. (22)), that is a contradiction.

Hence, there exists $p \in \mathbf{N}$ such that $\mu(A_p) = \delta > 0$. By virtue of Lemma 2.3, there exists a sequence $(x_n)_{n\in\mathbf{N}}$ of elements of A_p such that, for each $n \in \mathbf{N}$,

$$\bar{\mu}\left(\bigcup_{k\geq n}\{x_k\}\right) = \delta. \tag{34}$$

Fix $n \in \mathbf{N}$ and let $f \in \mathcal{R}(X)$. Consider the map $\tilde{f}_n : X \to \mathbf{R}$ which agrees with f on $X \setminus \{x_n\}$ and assume the value $\tilde{v}_{x_n}(f)$ at x_n; since $\{x_n\} \in \Omega$, we have $\tilde{f}\, 1_{X\setminus\{x_n\}} \in \mathcal{R}(X)$ and $\tilde{f}\, 1_{\{x_n\}} \in \mathcal{R}(X)$; hence $\tilde{f}_n \in \mathcal{R}(X)$.

Now, we consider the map $T_n : \mathcal{R}(X) \to \mathcal{R}(X)$ defined by putting, for each $f \in \mathcal{R}(X)$, $T_n(f) = \tilde{f}_n$; since $\tilde{v}_{x_n}$ is positive and linear, T_n is a positive linear operator of $\mathcal{R}(X)$ into itself.

Next, consider the sequence $(T_n)_{n\in\mathbf{N}}$; for each $n \in \mathbf{N}$ and $h \in H$, we have $T_n(h) = h$ (since $\tilde{v}_{x_n} = \delta_{x_n}$ on H) and therefore

$$h = \mathcal{R} - \lim_{n\to\infty} T_n(h).$$

We conclude the proof by showing that f cannot be the $\mathcal{R}$ limit of the sequence $(T_n(f))_{n\in\mathbf{N}}$, that is a contradiction.

If the sequence $(T_n(f))_{n\in\mathbf{N}}$ is not equibounded, it cannot be $\mathcal{R}$-convergent. Suppose that $(T_n(f))_{n\in\mathbf{N}}$ is equibounded and let $n \in \mathbf{N}$ and $P = \{B_0, \dots, B_q\} \in \mathfrak{S}$ $(q \in \mathbf{N})$; denote by $C_0, \dots, C_r$ $(r \in \mathbf{N})$ all the elements of P containing some x_k with $k \geq n$.

Then

$$\bigcup_{k\geq n}\{x_k\} \subset \bigcup_{i=0}^{r} C_i \qquad (\in \Omega)$$

and (cf. (34) and (22))

$$\delta \leq \mu\left(\bigcup_{i=0}^{r} C_i\right) = \sum_{i=0}^{r}\mu(C_i). \tag{35}$$

For simplicity, put $f_n = \sup_{k\geq n}|T_k(f) - f|$; for each $i = 0, \dots, r$, there exists $k \geq n$ such that $x_k \in C_i$ and therefore (cf. (1) and (33))

$$\begin{aligned} M(f_n, C_i) &\geq f_n(x_k) \\ &\geq |T_k(f)(x_k) - f(x_k)| \\ &= |\tilde{v}_{x_k}(f) - f(x_k)| \\ &\geq \frac{1}{p+1}\,. \end{aligned} \tag{36}$$

It follows (cf. (4), (36) and (35))

$$\begin{aligned} S(f_n, P, \mu) &= \sum_{i=0}^{q} M(f_n, B_i)\,\mu(B_i) \\ &\geq \sum_{i=0}^{r} M(f_n, C_i)\,\mu(C_i) \\ &\geq \frac{1}{p+1} \sum_{i=0}^{r} \mu(C_i) \\ &\geq \frac{\delta}{p+1}\,, \end{aligned}$$

and finally, by (5)

$$\overline{\int} \sup_{k\geq n} |T_k(f) - f| = \overline{\int} f_n \geq \frac{\delta}{p+1}\,.$$

Since $n \in \mathbf{N}$ is arbitrary, the sequence $(T_n(f))_{n\in\mathbf{N}}$ cannot be $\mathcal{R}$ convergent to f. ■

Let H be a subspace of $\mathcal{C}(X)$; if we introduce the Choquet boundary $\partial_H X$ of X with respect to H (cf. [5]. Def. 29.1, p. 176) as the set of all the $x \in X$ such that $v = \delta_x$ for each positive linear form $v : \mathcal{C}(X) \to \mathbf{R}$ satisfying the condition $v = \delta_x$ on H, we have, by the definition of $U_H(f)$

(37) $$\partial_H X = \bigcap_{f\in\mathcal{C}(X)} U_H(f).$$

Corollary 2.5. *Let X be a compact Hausdorff topological space, Ω an algebra of subsets of X which is a base of X and $\mu : \Omega \to \mathbf{R}$ a bounded content both regular and coregular.*

If a subspace H of $\mathcal{C}(X)$ satisfies the following condition:

(38) *for each compact subset K of X disjoint from $\partial_H X$, it results $\mu(K) = 0$,*

then H is an $\mathcal{R}$ Korovkin subspace in $\mathcal{R}(X)$ for sequences of positive linear operators.

In particular the condition (38) is true in the case in which $\bar{\mu}(X \setminus \partial_H X) = 0$ or $\partial_H X = X$ (i.e. H is a Korovkin space in $\mathcal{C}(X)$).

Proof. The condition (38) clearly implies the condition a) of Th. 2.4. ■

Now we apply Cor. 2.5 to the following corollaries.

Corollary 2.6. *Let X be a compact Hausdorff topological space, Ω an algebra of subsets of X which is a base of X, $\mu : \Omega \to \mathbf{R}$ a bounded content both regular and coregular and A a closed $\bar{\mu}$-null set.*

Then, the subspace H of $\mathcal{C}(X)$ formed by all the $h \in \mathcal{C}(X)$ such that the restriction $h_{|A}$ of h to A is constant is an $\mathcal{R}$–Korovkin subspace in $\mathcal{R}(X)$ for sequences of positive linear operators.

Proof. Since $\mu(A) = 0$, by (38) it is enough to show that $X \setminus A \subset \partial_H X$. Let $x \in X \setminus A$ and fix $\epsilon \in \mathbf{R}_+^*$; further, let $f \in \mathcal{C}(X)$ and suppose $f \geq 0$. Since f is continuous and A is closed, there exists a neighbourhood U of x such that $U \subset X \setminus A$ and, for each $y \in U$, $|f(x) - f(y)| < \epsilon$; further, there exists a continuous function $g : X \to \mathbf{R}$ such that $0 \leq g \leq 1$, $g = 0$ on $X \setminus U$ and $g(x) = 1$.

Now, put

$$m = \inf_{y \in X} f(y), \qquad M = \sup_{y \in X} f(y)$$

and consider the maps $h : X \to \mathbf{R}$ and $k : X \to \mathbf{R}$ defined by putting, for each $y \in X$,

$$h(y) = m\,(1 - g(y)) + (f(x) - \epsilon)\, g(y),$$
$$k(y) = M\,(1 - g(y)) + (f(x) + \epsilon)\, g(y).$$

If $y \in X \setminus U$, we have $g(y) = 0$ and therefore

$$h(y) = m \leq f(y) \leq M = k(y)$$

and if $y \in U$

$$\begin{aligned} h(y) &\leq f(y)\,(1 - g(y)) + f(y)\, g(y) \\ &= f(y) \\ &= f(y)\,(1 - g(y)) + f(y)\, g(y) \\ &\leq k(y); \end{aligned}$$

hence $h \leq f \leq k$.

Further, $h \in H$, $k \in H$ and $h(x) = f(x) - \epsilon$, $k(x) = f(x) + \epsilon$.

Since $\epsilon \in \mathbf{R}_+^*$ is arbitrary, by virtue of (24), it follows $x \in U_H(f)$.

If f is not positive, we obtain again $x \in U_H(f)$ by applying the preceding arguments to the positive and negative part of f.

By (37), we conclude $x \in \partial_H X$. ■

Corollary 2.7. *Let X be a compact Hausdorff topological space, Ω an algebra of subsets of X which is a base of X, $\mu : \Omega \to \mathbf{R}$ a bounded content both regular and coregular and S a subset of $\mathcal{C}(X)$ which separates the points of X.*

Then, the subspace H of $C(X)$ generated by

$$p_0 \cup S \cup S^2$$

is an $\mathcal{R}$-Korovkin subspace in $\mathcal{R}(X)$ for sequences of positive linear operators, where $S^2 = \{h^2 : h \in S\}$.

Proof. It follows by Cor. 2.6 since it results $\partial_H X = X$ (cf. [2] and [9]). ■

Corollary 2.8. *Let X be a compact Hausdorff topological space, Ω an algebra of subsets of X which is a base of X, $\mu : \Omega \to \mathbf{R}$ a bounded content both regular and coregular.*

If a subspace H of $C(X)$ contains the constant functions and, for each $x_0 \in X$, there exists $h \in H$ such that

$$h(x_0) = 0 \quad \text{and} \quad h(x) > 0 \quad \text{for each} \quad x \in X \setminus \{x_0\},$$

then H is an $\mathcal{R}$-Korovkin subspace in $\mathcal{R}(X)$ for sequences of positive linear operators.

Moreover, if $h_1, \ldots, h_n$ are continuous functions on X which separate the points of X, then the subspace H of $C(X)$ generated by

$$\left\{ p_0, h_1, \ldots, h_n, \sum_{i=1}^{n} h_i^2 \right\}$$

is an $\mathcal{R}$-Korovkin subspace in $\mathcal{R}(X)$ for sequences of positive linear operators.

Proof. It follows by Cor. 2.6 since it results $\partial_H X = X$ (cf. [4]). ■

Finally, we consider the case of a Peano-Jordan measurable compact subset X of $\mathbb{R}^p$ and the Peano-Jordan content (cf. Remark 1.5, 2)).

Corollary 2.9. *Let X be a Peano-Jordan measurable compact subset of $\mathbb{R}^p$ $(p \geq 1)$ and consider the Peano-Jordan content μ_X on the algebra Ω of all Peano-Jordan measurable subsets of X.*

For each $i = 1, \ldots, p$ denote by $\mathrm{pr}_i : X \to \mathbb{R}$ the restriction to X of the i-th projection of $\mathbb{R}^p$ onto $\mathbb{R}$.

Then, the subspace H of $C(X)$ generated by the constant functions and

$$\mathrm{pr}_1, \ldots, \mathrm{pr}_p, \mathrm{pr}_1^2 + \cdots + \mathrm{pr}_p^2$$

is an $\mathcal{R}$-Korovkin subspace in $\mathcal{R}(X)$ for sequences of positive linear operators.

Proof. It follows by the preceding Cor. 2.8. ■

In particular, if $p = 1$, we obtain the Korovkin type theorem stated in [8], Th. 2.

3. Applications

In this section we give some other applications of the preceding corollaries in the case of particular compact subsets X of $\mathbb{R}^p$ $(p \geq 1)$ and the Peano-Jordan content μ_X on X.

1. Let us consider the compact interval $[0, 2\pi]$ of $\mathbb{R}$ and assume the following notations:

$$\mathcal{R}_{2\pi} = \{f \in \mathcal{R}([0, 2\pi]) : f(0) = f(2\pi)\}$$

($\mathcal{R}_{2\pi}$ may be identified with the space of all 2π-periodic real functions on $\mathbb{R}$ which are Riemann integrable on $[0, 2\pi]$),

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Consider the projections pr_1 and pr_2 of S on $\mathbb{R}$: since $\mathrm{pr}_1^2 + \mathrm{pr}_2^2 = p_0$ (as usual, p_0 is the constant function on S of constant value 1), it follows from Cor. 2.9 that *the subspace of $\mathcal{R}(S)$ generated by the functions*

$$p_0, \mathrm{pr}_1, \mathrm{pr}_2,$$

is an $\mathcal{R}$ Korovkin subspace in $\mathcal{R}(S)$ for sequences of positive linear operators.

Further, the map

$$\varphi : \mathcal{R}(S) \longrightarrow \mathcal{R}_{2\pi}$$

defined by setting

$$\varphi(f)(t) = f(\cos t, \sin t)$$

for each $f \in \mathcal{R}(S)$ and $t \in [0, 2\pi]$ and $t \in [0, 2\pi]$, is a linear isomorphism of $\mathcal{R}(S)$ into $\mathcal{R}_{2\pi}$ (both equipped with the $\mathcal{R}$ sequential convergence) and therefore since $p_0 = \varphi \circ p_0$, $\cos = \varphi \circ \mathrm{pr}_1$, $\sin = \varphi \circ \mathrm{pr}_2$, *the subspace of $\mathcal{R}_{2\pi}$ generated by*

$$p_0, \cos, \sin$$

is an $\mathcal{R}$-Korovkin subspace in $\mathcal{R}_{2\pi}$ for sequences of positive linear operators. ∎

Now, we give an application of this last result; as observed above, each $f \in \mathcal{R}_{2\pi}$ can be extended in a natural way to a 2π-periodic function on $\mathbb{R}$, so it makes sense to consider, for each fixed positive $g \in \mathcal{R}([0, 2\pi])$ and $\alpha \in \mathbb{R}^*$, the map $\xi_f : \mathbb{R} \to \mathbb{R}$ defined by setting, for each $x \in \mathbb{R}$,

$$\xi_f(x) = \int_0^{2\pi} f(x - \alpha t)\, g(t)\, dt;$$

we have $\xi_f \in \mathcal{R}_{2\pi}$; in fact, it is clear in the case in which f and g are continuous while in the general case we fix $\epsilon \in \mathbb{R}_+^*$, and by Prop. 1.6 we consider $\phi, \psi \in \mathcal{C}(X)$ such that

$$\phi \le f \le \psi, \qquad \int_0^{2\pi} (\phi - \psi)(t)\, dt < \epsilon;$$

it results $\xi_\phi \le \xi_f \le \xi_\psi$ and, for each $x \in \mathbb{R}$, the inequalities

$$\begin{aligned}\int_0^{2\pi} (\psi - \phi)(x - \alpha\, t)\, g(t)\, dt &\le \|g\| \int_0^{2\pi} (\psi - \phi)(x - \alpha\, t)\, g(t)\, dt \\ &= -\|g\|\, \frac{1}{\alpha} \int_0^{x - 2\pi\alpha} (\psi - \phi)(s)\, ds \\ &= \|g\|\, \frac{1}{\alpha} \int_0^{2\pi\alpha} (\psi - \phi)(s)\, ds \\ &\le M \int_0^{2\pi} (\psi - \phi)(s)\, ds \\ &\le M\, \epsilon,\end{aligned}$$

(for a suitable $M \in \mathbb{R}_+$ which does not depend on ϵ) imply

$$\int_0^{2\pi} (\xi_\psi - \xi_\phi)(x)\, dx = \int_0^{2\pi} dx \int_0^{2\pi} (\psi - \phi)(x - \alpha\, t)\, g(t)\, dt \le 2\,\pi\, M\, \epsilon;$$

since $\epsilon \in \mathbb{R}_+^*$ is arbitrary, by Prop. 1.6 it follows $\xi_f \in \mathcal{R}_{2\pi}$.

Now, we can state the following result; the sequence of positive linear operators that we define is similar to a particular one considered in [1], Prop. 2.5 (for complex-valued Lebesgue integrable functions).

Proposition 3.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of positive elements of $\mathcal{R}_{2\pi}$ such that:*

i)
$$\lim_{n\to\infty} \int_0^{2\pi} f_n(t)\, dt = 1;$$

ii) *there exists $\alpha \in \mathbb{R}^*$ such that*

$$\lim_{n\to\infty} \hat{f}_n(\alpha) = \frac{1}{2\,\pi}\ ,$$

where

$$\hat{f}_n(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-\mathrm{i}\, \alpha\, t)\, f_n(t)\, dt$$

for each $n \in \mathbb{N}$.

Further, for each $n \in \mathbb{N}$, consider the positive linear operator $L_n : \mathcal{R}_{2\pi} \to \mathcal{R}_{2\pi}$ defined by setting

$$L_n(f)(x) = \int_0^{2\pi} f(x - \alpha t) f_n(t)\, dt$$

for each $f \in \mathcal{R}_{2\pi}$ and $x \in \mathbb{R}$.

Then, for each $f \in \mathcal{R}_{2\pi}$,

$$f = \mathcal{R} - \lim_{n \to \infty} L_n(f) \tag{39}$$

on the compact interval $[0, 2\pi]$.

Proof. It is enough to prove that (39) holds for p_0 and the functions sin and cos.

For each $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$L_n(p_0)(x) = \int_0^{2\pi} f_n(t)\, dt;$$

by i), the (39) follows with $f = p_0$.

Further, we observe that the condition ii) implies

$$\lim_{n \to \infty} \int_0^{2\pi} \sin(x - \alpha t) f_n(t)\, dt = \sin x,$$

$$\lim_{n \to \infty} \int_0^{2\pi} \cos(x - \alpha t) f_n(t)\, dt = \cos x$$

uniformly for $x \in \mathbb{R}$; hence

$$\lim_{n \to \infty} L_n(\sin) = \sin$$

and

$$\lim_{n \to \infty} L_n(\cos) = \cos$$

for the uniform convergence of $\mathcal{R}_{2\pi}$, which implies the $\mathcal{R}$–sequential convergence. ■

Remark 3.2. For example, the conditions i) and ii) of Prop. 3.1 are satisfied by

a) the sequence of the Féjer's kernels $((1/2\pi) K_n)_{n \in \mathbb{N}}$, where

$$K_n(t) = \begin{cases} \dfrac{1}{n+1} \left(\dfrac{\sin(n+1)t/2}{\sin t/2} \right)^2 & \text{if } t \in]0, 2\pi[\\ \dfrac{1}{n+1} & \text{if } t \in \{0, 2\pi\} \end{cases}$$

(cf. [11], Ch. III, §3, (A));

b) each sequence of Poisson's kernels $((1/2\,\pi)\,P_n)_{n\in\mathbb{N}}$, where

$$P_n(t) = \frac{1-r_n^2}{1+r_n^2-2\,r_n\,\cos t} \qquad \text{for each } t\in[0,2\,\pi],$$

$r_n\in]0,1[$ and the sequence $(r_n)_{n\in\mathbb{N}}$ converges to 1 (cf. [6], p. 302, (V), (VI)). ■

The next examples are concerned with the $\mathcal{R}$-sequential convergence of the Bernstein polynomials on the standard simplex and on the hypercube of $\mathbb{R}^p$ $(p\geq 1)$.

2. Let us consider the standard p simplex

$$X = \left\{(x_1,\ldots,x_p)\in\mathbb{R}^p \;\middle|\; \forall i=1,\ldots,n : x_i\geq 0,\ \sum_{i=1}^{p}\leq 1\right\}$$

of $\mathbb{R}^p$ $(p\geq 1)$ and, for each $n\in\mathbb{N}$, $n\geq 1$, the n-th Bernstein operator $B_n:\mathcal{R}(X)\to\mathcal{R}(X)$ defined by setting

$$B_n(f)(x_1,\ldots,x_p) = \sum_{\substack{h_1,\ldots,h_p\in\mathbb{N}\\ h_1+\cdots+h_p\leq n}} \frac{n!}{h_1!\cdots h_p!\,(1-h_1-\cdots-h_p)!}\, f(h_1/n,\ldots,h_p/n)$$
$$x_1^{h_1}\cdots x_p^{h_p}\left(1-\sum_{i=1}^{p}x_i\right)^{n-h_1-\cdots-h_p}.$$

for each $f\subset\mathcal{R}(X)$ and $(x_1,\ldots,x_p)\in X$.

Then, for each $f\in\mathcal{R}(X)$, it results

$$f = \mathcal{R}-\lim_{n\to\infty} B_n(f). \tag{40}$$

In fact, consider the subspace H of $C(X)$ generated by p_0 and $\mathrm{pr}_1,\ldots,\mathrm{pr}_p$, $\mathrm{pr}_1^2,\ldots,\mathrm{pr}_p^2$ (cf. Cor. 2.9): we have $B_n(p_0)=p_0$ and for each $n\in\mathbb{N}$, $n\geq 1$ and $i=1,\ldots,p$, $B_n(\mathrm{pr}_i)=\mathrm{pr}_i$ and

$$B_n(\mathrm{pr}_i^2) = \frac{1}{n}\,B_1(\mathrm{pr}_i^2) + \left(1-\frac{1}{n}\right)\mathrm{pr}_i^2 = \frac{1}{n}\,\mathrm{pr}_i + \left(1-\frac{1}{n}\right)\mathrm{pr}_i^2.$$

It follows

$$p_0 = \mathcal{R}-\lim_{n\to\infty} B_n(p_0)$$

and, for each $i=1,\ldots,p$,

$$\mathrm{pr}_i = \mathcal{R}-\lim_{n\to\infty} B_n(\mathrm{pr}_i):$$

since for each $n \geq 1$ and $i = 1, \ldots, p$

$$\sup_{k \geq n} \left| B_k(\mathrm{pr}_i^2) - \mathrm{pr}_i^2 \right| = \sup_{k \geq n} \frac{1}{k} \left| \mathrm{pr}_i - \mathrm{pr}_i^2 \right| \leq \frac{1}{n} ,$$

it results also

$$\mathrm{pr}_i^2 = \mathcal{R} - \lim_{n \to \infty} B_n(\mathrm{pr}_i^2).$$

By virtue of the Cor. 2.9 we obtain the (40). ■

3. Let us consider the hypercube $X = [0,1]^p$ of $\mathbb{R}^p$ ($p \geq 1$) and, for each $n \in \mathbf{N}$, $n \geq 1$, the n-th Bernstein operator $B_n : \mathcal{R}(X) \to \mathcal{R}(X)$ defined by setting

$$B_n(f)(x_1, \ldots, x_p) = \sum_{h_1, \ldots, h_p = 0}^{n} \binom{n}{h_1} \cdots \binom{n}{h_p} f(h_1/n, \ldots, h_p/n)\, x_1^{h_1}(1 - x_1)^{h_1} \cdots x_p^{h_p}(1 - x_p)^{h_p}$$

for each $f \in \mathcal{R}(X)$ and $(x_1, \ldots, x_p) \in X$.

Then, for each $f \in \mathcal{R}(X)$, it results

$$f = \mathcal{R} - \lim_{n \to \infty} B_n(f).$$

Also in this case we have $B_n(p_0) = p_0$ and for each $n \in \mathbf{N}$, $n \geq 1$ and $i = 1, \ldots, p$, $B_n(\mathrm{pr}_i) = \mathrm{pr}_i$ and

$$B_n(\mathrm{pr}_i^2) = \frac{1}{n}\, \mathrm{pr}_i + \left(1 - \frac{1}{n}\right) \mathrm{pr}_i^2;$$

as before, we obtain the proof by Cor. 2.9. ■

The applications 2. and 3. show that on the standard simplex of $\mathbb{R}^p$ and on the hypercube of $\mathbb{R}^p$ the Bernstein polynomials are dense in the space of Riemann integrable functions with respect to the $\mathcal{R}$-sequential convergence.

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Michele Campiti
Università degli Studi Basilicata
Istituto di Matematica
Via Nazario Sauro, 85
85100 Potenza
ITALY