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A NOTE ON ISOMORPHISMS BETWEEN POWERS OF BANACH SPACES

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ABSTRACT. We are concerned with the following problem: "Let E and F be Banach spaces such that E^I is isomorphic to F^I for some infinite set I. Then, when does it follow that E is isomorphic to F?" Here, we provide a partial answer to this problem and characterize the Banach spaces E which are isomorphic to any F whenever $F^{\mathbb{N}}$ is isomorphic to $E^{\mathbb{N}}$.

e use standard terminology of the theory of locally convex spaces (l.c.s from now).

Given E and F l.c.s, $F < E$ means that F is a complemented subspace of E. l.c.s. E is called prime (resp. primary) if, whenever $F < E$, then $\dim F < \infty$ or $\simeq E$ (resp. whenever $E \simeq F \oplus G$ then either $E \simeq F$ or $E \simeq G$).

For a Banach sequence space $(\mu, \|\cdot\|_\mu)$ and a Banach space E we shall let $\mu(E)$ the Banach space of all sequences $(x_n)_n$ such that $x_n \in E$, for all $n \in \mathbb{N}$, and $x_n\|) \in \mu$, with the norm

$$\|(x_n)_n\| = \big\|(\|x_n\|)\big\|_\mu .$$

The following lemma is the key of this paper.

mma 1. *Let E be an l.c.s. and let us fix $\mathcal{U}(E)$ a base of absolutely convex neighurhoods of 0 in E. If B is a (complemented) normable subspace of E, then there ists a neighbourhood $V_0 \in \mathcal{U}(E)$ such that B is isomorphic to a (complemented) bspace of the local Banach space E_V, for each $V \in \mathcal{U}(E)$, $V \subset V_0$.*

oof. Let us denote by Q the norm defining the topology of B, and by P_V the nkowski functional of V, for each $V \in \mathcal{U}(E)$. Then, it is straightforward that there is $V_1 \in \mathcal{U}(E)$, and, for each $V \in \mathcal{U}(E)$, there is $M_V \in \mathbb{R}$ such that

$$Q(x) \leq P_{V_1}(x), \qquad P_V(x) \leq M_V\, Q(x), \qquad \forall x \in B.$$

So. if $V \subset V_1$ we get

$$P_{V_1}(x) \leq P_V(x) \leq M_V\, P_{V_1}(x), \qquad \forall x \in B.$$

Hence, for each $V \subset V_1$, the canonical mapping $I_V : E \to E_V$, restricted to B, is isomorphism onto $I_V(B)$.

Let us now assume that B is complemented in E and denote by H the project from E onto B. There exists $V_2 \in \mathcal{U}(E)$ such that

$$P_{V_1}(H(x)) \leq P_{V_2}(x), \qquad \forall x \in E.$$

Then, for each $V \in \mathcal{U}(E)$, $V \subset V_2$, we can define a continuous mapping $H_V : E_V -$ so that $H_V \circ I_V = H$. It follows that $I_V(B)$ is isomorphic to B and complemente E_V, for each $V \in \mathcal{U}(E)$ with $V \subset V_1 \cap V_2$. ■

Remark. Let us recall the following fact which we shall need below: Let E an be l.c.s. isomorphic to their Cartesian square, and such that $E < F < E$. Then isomorphic to F.

We are now in conditions to prove our main result.

Theorem 2. *Let E, F be Banach spaces and let I be an infinite set. The follow are equivalent:*

1) *E^I is isomorphic to F^I.*
2) *There is an index $n \in \mathbb{N}$ such that $E < F^n$ and $F < E^n$.*
3) *$E^{\mathbb{N}}$ is isomorphic to $F^{\mathbb{N}}$.*

Proof. 1) $\to$ 2) Let us assume that E^I is isomorphic to F^I. Then E is isomor to a complemented subspace of F^I, hence, from Lemma 1, E is isomorphic complemented subspace of F^J, for some finite set $J \subset I$. In the same way, there finite set $J' \subset I$ such that F is isomorphic to a complemented subspace of $E^{J'}$. result follows by taking

$$n = \max\{\operatorname{card}(J), \operatorname{card}(J')\}.$$

2) $\to$ 3) Under the assumption, it is easy to see that $E^{\mathbb{N}} < F^{\mathbb{N}} < E^{\mathbb{N}}$. M over, $E^{\mathbb{N}}$ and $F^{\mathbb{N}}$ are isomorphic to their Cartesian square. The assertion is th consequence of the remark.

3) $\to$ 1) It is immediate. ■

From 2) of Theorem 2 we get the following.

›rollary 3. *Let E and F be Banach spaces isomorphic to their Cartesian square d let I be an infinite set. If E^I is isomorphic to F^I then E is isomorphic to F.*

Notice that the hypothesis on E and F in Corollary 3 can't be dropped unless ne additional assumption is added. Indeed, take a Banach space E non isomorphic its Cartesian square, (e.g. see [1]) and put $F := E \oplus E$. Then $F^{\mathbf{N}}$ is isomorphic $E^{\mathbf{N}}$ but E is not isomorphic to F. These results lead us to introduce and study : class $\mathcal{R}$ consisting on those Banach spaces E such that E is isomorphic to F, for y Banach space F such that $E^{\mathbf{N}}$ is isomorphic to $F^{\mathbf{N}}$. We are now concerned with ding a characterization of $\mathcal{R}$.

·oposition 4. *A Banach space E is in $\mathcal{R}$ if and only if the following three conditions ld:*

i. $E \simeq E \oplus E$.

ii. *For every Banach space such that $E < F < E$, one has $E \simeq F$.*

iii. *For every Banach space such that $E \simeq F^n$ for some $n \in \mathbf{N}$, one has $E \simeq F$.*

oof. In a first place, let us assume that $E \in \mathcal{R}$. Condition i. hold as we have eady noted. To show condition ii. (resp. condition iii.) let F be a Banach space :h that $E < F < E$ (resp. $E \simeq F^n$ for some $n \in \mathbf{N}$), then $E^{\mathbf{N}} < F^{\mathbf{N}} < E^{\mathbf{N}}$, hence $^{\mathbf{N}} \simeq F^{\mathbf{N}}$ (resp. it is clear that $E^{\mathbf{N}} \simeq F^{\mathbf{N}}$). Thus, in any case, E is isomorphic to F.

To show the converse implication let us take a Banach space F such that $^{\mathbf{N}} \simeq F^{\mathbf{N}}$. From 2) of Theorem 2, and by i., we have $F^n < E < F^n$, for some $\in \mathbf{N}$. By applying conditions ii. and iii. we get $E \simeq F$, so $E \in \mathcal{R}$. ■

By using Proposition 4 it is easy to see that $E \in \mathcal{R}$ whenever E is a prime Banach ace isomorphic to its Cartesian square. So, the familiar Banach spaces c_0 and ℓ_p, $\leq p \leq \infty$, are in $\mathcal{R}$ (let us note, however, that there are no prime Banach spaces own other than c_0 and ℓ_p, $1 \leq p \leq \infty$). The spaces $\ell^{p_1} \oplus \ell^{p_2} \oplus \cdots \oplus \ell^{p_n}$, $1 \leq p_i < \infty$, o are in $\mathcal{R}$ (cf. [6]). Furthermore, we shall obtain from Proposition 4 the following ficient condition for a Banach space to be in $\mathcal{R}$.

›rollary 5. *Let E be a Banach space such that:*

a) *E is isomorphic to $\mu(E)$, where μ is one of the spaces c_0 or ℓ^p, $1 \leq p \leq \infty$.*

b) *E is primary.*

Then $E \in \mathcal{R}$.

Proof. It is left to the reader to check that a) implies conditions i. and ii. of Prc sition 3 (to see ii. use the Pelczyński's decomposition method), and b) clearly imp condition iii. ■

Now, a large class of examples arises. Indeed, the following Banach spaces in $\mathcal{R}$.

α) $L^p([0,1])$, $1 \leq p < \infty$.

β) $C([0,1])$ (cf. [5]).

γ) $\ell^p(X)$ where X is a Banach space with a symmetric basis, not isomor to ℓ^1, and $1 < p < \infty$, $\ell^p(\ell^\infty)$; $c_0(\ell^\infty)$; the Pelczyński's complementa universal space U for all Banach spaces with unconditional bases (cf. and [4]).

δ) $\ell^p(L^r)$, $1 \leq p, r < \infty$ ([2] and its references).

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