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On Grothendieck space ideals

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Abstract

We develop some techniques to prove that some well known classes of Schwartz spaces cannot be generated by a pre-ideal of operators. We also construct new proofs for Montel spaces and spaces of maximal diametral dimension.

Introduction

Since the appearance of nuclear spaces due to the work of Grothendieck [12] and the formalization of the term operator ideal by Pietsch [21], many new ideals of operators have come to light and consequently many new classes of locally convex spaces generated by an operator ideal, or Grothendieck space ideals (see below for definitions), have emerged.

In fact, almost all the classes of locally convex spaces which appear in the literature are Grothendieck space ideals, and only three exceptions seem to be known: the class of Montel spaces [16, 7.9.3; 21, 29.6.11], the class of $\Lambda_N(\alpha)$-nuclear spaces [20], and Ω, the class of Schwarz spaces of maximal diametral dimension [10].

Probably the most remarkable pathology in this topic is that the three corresponding proofs appear to be unrelated.

In part 3 of this paper we shall construct new classes not generated by an ideal of operators. This is done by exploiting the results of part 2 (are they a partial answer to the problem posed by Pietsch in [21, p. 402]: characterize Grothendieck space ideals by means of certain "inner" properties?).

0. Notation

For terms not explained below, see [15,17,18]. Other relevant definitions will be stated at the apropriate place in the text.

When E is a locally convex space (in short lcs), $\mathcal{U}(E)$ will denote a fundamental system of absolutely convex closed neighborhoods of 0. If $U \in \mathcal{U}(E)$ with gauge p_U, then E_U is the space $E/\mathrm{Ker}\, p_U$ endowed with the norm $\|\phi_U x\| = p_U(x)$ where ϕ_U is the quotient map; $\hat{E}_U$ denotes its completion. If $V \in \mathcal{U}(E)$, $V \subset U$, then $\hat{T}_{UV}$ is the extension to the completions of the map $T_{UV} : E_V \to E_U$ defined by the equation $T_{UV}\phi_V x = \phi_U x$.

φ will denote the countable-dimensional space endowed with the strongest locally convex topology: $\varphi = \oplus_{\mathbf{N}}\mathbf{K}$. If (E, F) is a dual pair, $\mu(E, F)$ denotes the Mackey topology and $\sigma(E, F)$ the weak toplogy in E.

A Köthe set is a set P of real valued sequences satisfying:

1. $\forall a \in P$, $a \geq 0$.
2. $\forall n \in \mathbf{N}$, $\exists a \in P : a_n > 0$.
3. $\forall a, b \in P$, $\exists c \in P : \max\{a, b\} \leq c$.

(Inequalities and operations between sequences are understood component-wise).

The space

$$\Lambda(P) = \left\{z \in \mathbf{K}^{\mathbf{N}} : \forall a \in P \ za \in \ell_1\right\}$$

endowed with the topology given by the seminorms $p_a(z) = \|za\|_1$, $a \in P$, is called a Köthe sequence space. When moreover the sequences a of P are monotone increasing, $a \geq 1$ and P satisfies

4. $\forall a \in P$, $\exists b \in P : a^2 \leq kb$ for some scalar $k > 0$,

then P is called a power set, and the corresponding $\Lambda(P)$, a G_∞ space.

When E is a Hausdorff lcs, the diametral dimension of E is defined to be the set

$$\Delta(E) = \left\{z \in \mathbf{K}^{\mathbf{N}} : \forall U \in \mathcal{U}(E)\, \exists V \in \mathcal{U}(E) \ : z_n\delta_n(V, U) \xrightarrow{n} 0\right\}$$

where $\delta_n(V, U)$ stands for the nth Kolmogorov diameter of V with respect to U

$$\delta_n(V, U) = \inf\{\sigma > 0 : V \subset L + \sigma U,\ dim L < n\}.$$

1. Operator ideals and Grothendieck ideals

L denotes the class of all operators acting between Banach spaces.

An operator ideal $\mathfrak{A}$ is a subclass of $\mathfrak{L}$ satisfying the following:

1. The finite-dimensional operators belong to $\mathfrak{A}$.
2. $\mathfrak{A} + \mathfrak{A} \subset \mathfrak{A}$.
3. $\mathfrak{L}\mathfrak{A}\mathfrak{L} \subset \mathfrak{A}$.

When a subclass only satisfies 1 and 3 it is called [20] a pre-ideal; we will see that $3 \Longrightarrow 1$ (further of trivial case $\mathfrak{A} = 0$); 3 is referred to as the ideal-property. Thus a class of operators with the ideal property is a pre-ideal.

If X and Y are Banach (normed) spaces then $A(X,Y)$ denotes all the operators of A acting between X and Y.

A Hausdorff lcs E is termed an $\mathfrak{A}$–space when for each $U \in \mathcal{U}(E)$ there exists a $V \in \mathcal{U}(E)$, $V \subset U$, such that $\hat{T}_{VU} \in \mathfrak{A}(\hat{E}_V, \hat{E}_U)$. The class formed by all the $\mathfrak{A}$-spaces is denoted $\mathrm{Groth}(\mathfrak{A})$.

A class $\mathcal{B}$ of lcs is called a Grothendieck space ideal (pre-ideal) if $\mathcal{B} = \mathrm{Groth}(\mathfrak{A})$ for some ideal (pre-ideal) $\mathfrak{A}$ of operators.

Well-known ideals of operators are: the ideal $\mathfrak{F}$ of finite rank operators ($T \in \mathfrak{F}$ if and only if the range of T has finite dimension), the ideal $\mathfrak{N}$ of nuclear operators ($T \in \mathfrak{N}(X,Y)$ if and only if T is representable as

$$Tx = \sum_{n=1}^{\infty} \langle a_n, x\rangle\, y_n \qquad \forall x \in X,$$

with $(a_n) \subset X'$, $(y_n) \subset Y$ and

$$\sum_{n=1}^{\infty} \|a_n\|\,\|y_n\| < +\infty),$$

the ideal $\mathfrak{K}$ of compact operators ($T \in \mathfrak{K}(X,Y)$ if and only if the image of the unit ball of X is relatively compact in Y). If we require the image to be relatively $\sigma(Y,Y')$ compact we obtain the ideal $\mathfrak{W}$ of weakly compact operators.

These operator ideals generate the classes $\mathrm{Groth}(\mathfrak{F})$ = lcs endowed with the weak topology, $\mathrm{Groth}(\mathfrak{N})$ = nuclear spaces, $\mathrm{Groth}(\mathfrak{K})$ = Schwartz spaces, and $\mathrm{Groth}(\mathfrak{W})$ = lcs having a fundamental system of zero neighborhoods such that the associated Banach spaces are reflexive [13,15].

2. Necessary conditions

We are now interested in the question: how is the presence of an operator ideal felt in the topological structure of a locally convex space?

It is clear that if F is a dense subspace of E then F is an $\mathfrak{A}$ space if and only if E is an $\mathfrak{A}$–space. We will refer to this property as stability under the formation of dense subspaces and completions.

It is also well-known that if $\mathfrak{A}$ is an operator ideal then Groth($\mathfrak{A}$) is stable under the formation of arbitrary products. This property is not necessarily true for pre-ideals, and we will see counterexamples later.

Complemented subspaces of $\mathfrak{A}$ spaces are $\mathfrak{A}$–spaces; a direct proof runs as follows: it is easy to see that $E \in$ Groth($\mathfrak{A}$) if and only if each operator $T \in \mathfrak{L}(E, G)$ belongs to $\mathfrak{A}$ when it is interpreted as an operator $\mathfrak{L}(\hat{E}_V, \hat{G}_U)$ for all $U \in \mathcal{U}(G)$ and adequate $V \in \mathcal{U}(E)$. Let $T \in \mathfrak{L}(E, G)$ and $P : F \to E$ be a continuous projection. Then $TP \in \mathfrak{L}(F, G)$ and $TP \in \mathfrak{A}(\hat{F}_V, \hat{G}_U)$. It may be checked that TP acting from $\hat{E}_{V\cap E} \to \hat{G}_U$ coincides with T interpreted through $\hat{E}_{V\cap E}$.

Note that this proof only uses the ideal property of $\mathfrak{A}$. If Groth($\mathfrak{A}$), $\mathfrak{A}$ a pre-ideal, contains an infinite dimensional space then, since finite dimensional spaces are complemented in each lcs, it follows that the operator $\mathrm{Id}_{\mathbf{K}^n}$ belongs to $\mathfrak{A}$ for all n, and therefore any class $\mathfrak{A}$ with the ideal property contains $\mathfrak{F}$.

The class Groth($\mathfrak{F}_n$) $\mathfrak{F}_n$ being the pre-ideal of operators of rank $\leq n$, shows that nothing more can be said in general.

In [25, p. 24] there is a different proof of the above for ideals. Another proof of the stability of Groth($\mathfrak{F}_n A$) by complemented subspaces can be seen in [21].

In [6] the author introduced the notion of local complementation:

DEFINITION. A subspace E is said to be locally complemented in a Hausdorff lcs F when a fundamental system of zero neighborhoods $\mathcal{U}(F)$ can be found in F such that, for all $U \in \mathcal{U}(F)$, the Banach space $\hat{E}_{U\cap E}$ is complemented in the Banach space $\hat{F}_U$.

Proposition 1

Let $\mathfrak{A}$ be a pre-ideal of operators. If E is a locally complemented subspace of some $F \in$ Groth($\mathfrak{A}$), then $E \in$ Groth($\mathfrak{A}$).

Proof. Given $U \in \mathcal{U}(F)$ find $W \in \mathcal{U}(F)$, $W \subset U$, such that $\hat{E}_{W\cap E}$ is complemented in $\hat{F}_U$, and then a $V \subset W$ such that $\hat{T}_{VW} : \hat{F}_V \to \hat{F}_W$ belongs to $\mathfrak{A}$. That $E \in \mathrm{Groth}(\mathfrak{A})$ follows from the commutative diagram

$$\begin{array}{ccccc} \hat{E}_{V\cap E} & \longrightarrow & \hat{E}_{W\cap E} & \longrightarrow & \hat{E}_{U\cap E} \\ \downarrow & & \uparrow & & \\ \hat{F}_V & \longrightarrow & \hat{F}_W & \longrightarrow & \hat{F}_U \end{array}$$

□

Definition. We will say that a class of lcs M possesses enough metrizables when for each $E \in M$ there exists a collection F_i, $i \in I$, of metrizable spaces in M such that E is locally complemented in $\prod_I F_i$.

Proposition 2

If $\mathfrak{A}$ is an operator pre-ideal, then $\mathrm{Groth}(\mathfrak{A})$ *possesses enough metrizables.*

Proof. Let $E \in \mathrm{Groth}(\mathfrak{A})$. Let $\mathcal{U}(E)$ be a fundamental system of zero neighborhoods in E. Take $U \in \mathcal{U}(E)$ and find a $U_1 \in \mathcal{U}(E)$ such that the linking map $\hat{T}_{U_1U}$ belongs to $\mathfrak{A}$, then a $U_2 \in \mathcal{U}(E)$ with $\hat{T}_{U_2U_1}$ also in $\mathfrak{A}$, and so on. The projective spectrum

$$\cdots \longrightarrow \hat{E}_{U_3} \longrightarrow \hat{E}_{U_2} \longrightarrow \hat{E}_{U_1} \longrightarrow \hat{E}_U$$

defines a metrizable $\mathfrak{A}$ space that we call E^U.

Start again with another $V \in \mathcal{U}(E)$ and construct E^V. Proceed until the system $\mathcal{U}(E)$ is exhausted. Following [17, 18.3(7)] it is not difficult to see that E is a subspace (locally complemented by construction) of the product $\prod_{U\in\mathcal{U}(E)} E^U$.
□

Thus we have:

Theorem 1

Let $\mathcal{M}$ be a class of Hausdorff lcs. Necessary conditions for $\mathcal{M}$ to be generated by an ideal of operators are:

1. *To contain the product spaces K^I, I any set.*
2. *To be stable under the formation of dense subspaces and completions.*
3. *To be stable under the formation of arbitrary products.*
4. *To be stable by local complementation.*
5. *To have enough metrizables.*

Remarks.

1. To be generated by a pre-ideal condition (3) is not necessary.

2. The idea of local complementation was inspired by [19], where it is proved that complementation implies local complementation. It is plain that subspaces of Hilbertizable spaces (i.e. spaces having a fundamental system of zero neighborhoods whose associated Banach spaces are Hilbert spaces) are locally complemented.

3. The idea of "enough metrizables" was inspired by [2].

In [4] the author gave an elementary proof of the fact that the class of Schwartz spaces of maximal diametral dimension ($\Delta(E) = \mathbf{K}^{\mathbf{N}}$), Ω, cannot be ideal generated. The proof was based upon proving that Ω does not possess enough metrizables: the only metrizable spaces in Ω are the subspaces of $\mathbf{K}^{\mathbf{N}}$, and φ is not a subspace of any product of copies of $\mathbf{K}$.

Concerning Montel spaces the classical proof runs through proving that quotients of Hilbertizable $\mathfrak{A}$-spaces are $\mathfrak{A}$ spaces and recalling that there are Fréchet-Montel-Hilbertizable sequence spaces such that some quotient is isomorphic to ℓ_2 [16]. This proof is constructed as a violation of a, in some sense, dual of (4). Let us turn it to the right: we only need to find a non-Montel subspace of a Montel Hilbertizable space; choose $\Lambda(P)$ a Fréchet nuclear G_∞–space satisfying

$$\forall a \in P \ \exists b \in P : a_{n^2} \leq k b_n$$

for some $k > 0$.

Following [27,4.2] $\Lambda(P) \otimes \Lambda(P) = \Lambda(P)$. By Saxon's theorem [24, 1.4] φ is a subspace of $\Lambda(P)^I$, $\operatorname{card} I \geq 2^{\aleph_0}$. Therefore $\varphi \otimes \Lambda(P)$ is a subspace of $\Lambda(P)^I \otimes \Lambda(P)$, in turn isomorphic to $\Lambda(P)^I$, which is both Montel and Hilbertizable. But $\varphi \otimes \Lambda(P)$, not being barrelled [3], cannot be Montel.

3. New classes not ideal-generated

If a class Groth($\mathfrak{A}$) contains all Banach spaces then clearly it contains all locally convex spaces. This is the reason why we will restrict ourselves in what follows to Schwartz spaces. Without a restriction excluding the Banach spaces, the results become more obvious.

The preceding argumentation actually proves:

Proposition 3

The class of barrelled Schwartz spaces is not a Grothendieck space pre-ideal.

We extend proposition 3 to other kind of barrelledness. Recall from [15, Chapter 12] that a Hausdorff lcs E is said to be ℓ_∞-barrelled (resp. c_0 barrelled) if every bounded sequence (resp. null sequence) in $(E', \sigma(E', E))$ is equicontinuous. It is said to be $\aleph_0$-barrelled if every $\sigma(E', E)$ bounded set in E' which is the countable union of equicontinuous sets is itself equicontinuous. It is clear that $\aleph_0$ barrelledness implies ℓ_∞-barrelledness and this implies c_0-barrelledness.

We have:

Proposition 4

The class of $\aleph_0$-barrelled—resp. ℓ_∞-barrelled, c_0 barrelled—Schwartz spaces is not a Grothendieck space pre-ideal.

Proof. First note that, for a pre-ideal generated class, the necessity of having enough metrizables implies that when a class $\mathcal{B}$ of lcs contains all Fréchet spaces, then $\mathcal{B} \cap \mathrm{Groth}(\mathfrak{A})$ cannot be pre-ideal generated unless $\mathrm{Groth}(\mathfrak{A}) \subset \mathcal{B}$. Therefore in these cases we only need to find a Schwartz space not $\aleph_0$-, ℓ_∞- or c_0-barrelled.

For $\aleph_0$- and ℓ_∞ barrelledness we only need to see that the universal Schwartz space $(\ell_\infty, \mu(\ell_\infty, \ell_1))$ [15, 10.5] is not ℓ_∞ barrelled [11, Example 3.5].

Difficulties increase for c_0 barrelledness in accordance with the close interplay existent between this property and the Schwartz character of a space [14]. We shall denote by τ_0 the Schwartz topology in ℓ_1 given by (ℓ_1, c_0) [11, 10.4], and by $n(\ell_1, c_0)$ the topology of the uniform convergence on all the $\sigma(c_0, \ell_1)$ null sequences of c_0. Since

$$n(\ell_1, c_0) \leq \tau_0(\ell_1, c_0) \leq \mu(\ell_1, c_0),$$

it follows from [11, Corollary 1.2] that $(\ell_1, \tau_0(\ell_1, c_0))$ is c_0-barrelled.

On the other hand this space does not have a weakly sequentially complete dual [11, p. 71]. Then applying [11, 3.4], $(\ell_1, \tau_0(\ell_1, c_0))$ contains a dense hyperplane not c_0 barrelled. Recalling property (2) of Grothedieck space ideals, the proof ends. □

A Hausdorff lcs E is said to possess the Approximation Property (AP) when for each precompact set K and each zero neighborhood U there exists a finite rank operator $T \in \mathfrak{F}(E, E)$ such that $(I - T)(K) \subset U$, where I is the identity map of E. E is said to have the Bounded Approximation Property (BAP) when the set of operators defining the AP is equicontinuous.

Proposition 5

The class $\mathcal{S} + BAP$ of Schwartz spaces with the Bounded Approximation Property is not a Grothendieck space pre-ideal.

Proof. Should not be the case, any subspace of a Hilbertizable space with BAP would have BAP. But Dubinsky [8] constructed a nuclear Fréchet space without the BAP which, by the Komura-Komura's theorem, is a subspace of $(s)^{\mathbb{N}}$, which has the BAP. □

Remarks.

1. The above proof shows that local complementation does not imply complementation since the BAP is stable under complementation.

2. The class $\mathcal{S} + BAP$ has properties (1), (2) and (3) of theorem 1. That it also has enough metrizables follows from the main result of [6].

3. The class $\mathcal{S} + BAP$ is stable under local complementation, but we do not know wether it has enough metrizables. It is an open question posed by Ramanujan and treated in [19] if $\mathcal{S} + BAP = \mathrm{Groth}(\mathfrak{S})$. See also [6] for additional information.

In [5] the class of Schwartz Δ–stable spaces is introduced; that is, Schwartz spaces such that $\Delta(E \times E) = \Delta(E)$. We shall prove that this class cannot be ideal generated:

Proposition 6

The class of all Schwartz Δ–stable spaces is not a Grothendieck space pre-ideal.

Proof. Let us call this class Δ_s, and assume $\Delta_s = \mathrm{Groth}(\mathfrak{A})$. In [5, Proposition and Remark] it is proven that a G_∞-Schwartz space $\Lambda(P)$ is Λ–stable if and only if

$$\forall a \in P\, \exists b \in P : (a_{2n}/b_n) \in c_0 \tag{$*$}$$

(for convenience of the reader we sketch the "only if": Ramanujan and Terzioglu proved [23] that $(*)$ characterizes stable G_∞-Schwartz spaces $\Lambda(P)$. We show that stable and Δ–stable are equivalent for $\Lambda(P)$: $\Lambda(P) \times \Lambda(P)$ is isomorphic to the G_∞ Schwartz space $\Lambda(P * P)$, being

$$P * P = \{a * b = (a_1, b_1, a_2, b_2, \ldots) : a, b \in P\}.$$

The proof is a modification of [23]. Next

$$\Delta(\Lambda(P)) = \Delta(\Lambda(P)) \times \Lambda(P) = \Delta(\Lambda(P * P)),$$

and it is well known that two G_∞–Schwartz spaces are isomorphic if tey have equal diametral dimension, thus

$$\Lambda(P) = \Lambda(P * P) = \Lambda(P) \times \Lambda(P).$$

Recall that a monotone increasing sequence (k_n) is said to be stable when

$$\sup_n \frac{k_{2n}}{k_n} < +\infty.$$

Let then $x \in c_0$ be a monotone decreasing sequence, $x_n \neq 0$ for each $n \in \mathbb{N}$, and be such that x^{-1} is stable, the G_∞ Schwartz space $\Lambda(P_x)$ with $P_x = \{x^{-k} : k \in \mathbb{N}\}$ belongs to Δ_s; thus a power x^n of x makes the diagonal map $D_{x^n} \in \mathfrak{A}(\ell_1, \ell_1)$.

From this it follows that any diagonal operator $D_z : \ell_1 \to \ell_1$, with $z \in c_0$ such that $z_n \neq 0$ for each $n \in \mathbb{N}$, must belong to $\mathfrak{A}$; given such a z, find $y \in c_0$ such that $z \leq y^k$ for all k and large n, and then a stable sequence x^{-1} monotone increasing such that $y \leq x$.

From the factorization:

$$\begin{tikzcd} \ell_1 \arrow[rr, "D_z"] \arrow[dr, "D_{x^k}"'] & & \ell_1 \\ & \ell_1 \arrow[ur, "D_{z/x^k}"'] & \end{tikzcd}$$

taking into account that $z/x^k \in \ell_\infty$ for each $k \in \mathbb{N}$ and that $D_{x^k} \in \mathfrak{A}(\ell_1, \ell_1)$ for some k, it follows that $D_z \in \mathfrak{A}(\ell_1, \ell_1)$.

Therefore, any Köthe Schwartz space with continuous norm should be in Δ_s, which is false by the above mentioned result for G_∞ Schwartz spaces. □

Remarks. It is obvious that Δ–stable spaces satisfy (1) and (2). They do not satisfy (4): we need a

Lemma

If E is locally complemented in F then $\Delta(F) \subset \Delta(E)$.

Proof. Let $U, V \in \mathcal{U}(E)$ be such that $V \subset L + rU$, $\dim L \leq n$. Looking at the diagram

$$\begin{tikzcd} \hat{E}_{V \cap E} \arrow[r] \arrow[d] & \hat{E}_{U \cap E} \\ \hat{F}_V \arrow[r, dashed] & \hat{F}_U \arrow[u, "P_U"'] \end{tikzcd}$$

we see that

$$V \cap E = P_U(V \cap E) \subset P_U(V) \subset P_U + L + r(U \cap E),$$

and thus

$$\delta_n(V \cap E, U \cap E) \leq \delta_n(V, U).$$

which gives

$$\Delta(F) \leq \Delta(E). \quad \square$$

Now let $(\ell_2)_o$ be the universal Schwartz-Hilbert space [1, Th. 4.2]. Regarding this lemma and the fact that there are Fréchet-Schwartz-Hilbert spaces with arbitrarily "slow" diametral dimension

$$\Delta\big((\ell_2)_o^{\mathbb{N}}\big) = \ell_\infty$$

and thus it is Δ stable. But not all the Fréchet-Schwartz-Hilbert spaces are Δ stable. □

Corollary

The class of Fréchet-Schwartz-Hilbert Δ-stable spaces is not a Grothendieck space pre-ideal.

Analogously to Ω we introduce the class Ω_0 of Schwartz spaces of minimal diametral dimension ($\Delta(E) = \ell_\infty$). In [5] we gave a characterization of those peculiar Schwartz spaces.

It is clear that Ω_0 cannot be ideal generated since it excludes finite dimensional spaces. We see that its "complement" cannot be ideal generated either:

Proposition 7

The class of Schwartz spaces E such that $\Delta(E) \neq \ell_\infty$ is not a Grothendieck space pre-ideal.

Proof. Let us call this class $\tilde{\Omega}_0$. In [5, Th. 1] it was proved that all G_∞ Schwartz spaces are in Ω_0. Thus following the proof of Proposition 5, $\Omega_0 = \mathrm{Groth}(\mathfrak{A})$ would imply that all Köthe Schwartz spaces would be in $\tilde{\Omega}_0$, which is not the case since $\Delta(\Lambda[c_0]) = \ell_\infty$ [5, Remark 1]. □

Obviously Ω_0 satisfies (1) and (2). It also satisfies (4) due to the lemma. In [5, Th. 2] it is proved that if F is a metrizable space then $\Delta(F) = \ell_\infty$; $\mathrm{Groth}(\mathfrak{K})$ has enough metrizables and hence Ω_0 has enough metrizables and satisfies (5). However it fails in (3): let $x \in c_0$ and $F_x = \Lambda(P_x)$. We know that $\Delta(F_x) \neq \ell_\infty$. It is not hard to check that

$$\Delta\left(\prod_{x\in c_0} F_x\right) = \ell_\infty.$$

Corollary

The class of Schwartz-Hilbert spaces E such that $\Delta(E) \neq \ell_\infty$ is not a Grothendieck space pre-ideal.

Proof. Analogous to proposition 6 but using [9, p. 21]

$$\Delta\big(\Lambda^2(|c_0|)\big) = \ell_\infty. \qquad \square$$

Remark. We have just encountered a curious phenomenon: a Grothendieck space ideal split into two non ideal-generated classes

$$\mathrm{Groth}(\mathfrak{K}) = \Omega_0 \cup \check{\Omega}_0.$$

There are more ways of obtaining such decompositions: let us call $\mathcal{S}_\varphi$ the class of Schwartz spaces containing φ, and $\check{\mathcal{S}}_\varphi$ the class of those which do not. Clearly $\mathcal{S}_\varphi$ is not ideal generated (it fails (1)). Recalling Saxon's theorem $\check{\mathcal{S}}_\varphi$ is not ideal generated since it fails (3) (it plainly satisfies (1), (2) and (4); and (5) because all metrizable Schwartz spaces are in $\check{\mathcal{S}}_\varphi$). A proof valid for pre-ideals needs then a reasoning we used before: if $\check{\mathcal{S}}_\varphi = \mathrm{Groth}(\mathfrak{A})$, since all Fréchet spaces are in $\mathcal{S}_\varphi$, all Schwartz spaces should also be. This is an absurd since φ is a Schwartz space.

Is a decomposition of a Grothendieck space ideal into two Grothendieck space ideals possible then? The answer is no:

Proposition 8

$\mathrm{Groth}(\mathfrak{A}) \cup \mathrm{Groth}(\mathfrak{B}) = \mathrm{Groth}(\mathfrak{C})$ *if and only if* $\mathrm{Groth}(\mathfrak{A}) \subset \mathrm{Groth}(\mathfrak{B})$ *or* $\mathrm{Groth}(\mathfrak{B}) \subset \mathrm{Groth}(\mathfrak{A})$.

Proof. The if part is clear. We prove the only if part. Suppose not and find $A \in \mathrm{Groth}(\mathfrak{A})$, $A \in \mathrm{Groth}(\mathfrak{B})$ and $B \in \mathrm{Groth}(\mathfrak{B})$, $B \notin \mathrm{Groth}(\mathfrak{A})$. Both A and B belong to $\mathrm{Groth}(\mathfrak{C})$, and so does $A \times B$. If $A \times B \in \mathrm{Groth}(\mathfrak{A})$ then $B \in \mathrm{Groth}(\mathfrak{A})$ which is a contradiction. Analogously if $A \times B \in \mathrm{Groth}(\mathfrak{B})$. $\square$

4. Counterexamples and further results

The five conditions of theorem 1 are not sufficient to ensure that a class of lcs is ideal generated; to see this, note that if $(\mathfrak{A}_k)$ is a sequence of operator ideals then the class

$$\bigcap_{k\in\mathbb{N}} \mathrm{Groth}(\mathfrak{A}_k)$$

satisfies the conditions of theorem 1. The only not evident one is (5): if $E \in \bigcap_{\mathbb{N}} \mathrm{Groth}(\mathfrak{A}_k)$ then given $U \in \mathcal{U}(E)$ find $U_1 \in \mathcal{U}(E)$ with $\hat{T}_1 \in \mathfrak{A}_1$, then $U_2 \in \mathcal{U}(E)$ with $\hat{T}_2 \in \mathfrak{A}_1 \cap \mathfrak{A}_2$, and in general $\hat{T}_{k+1} \in \mathfrak{A}_{k+1} \cap \ldots \cap \mathfrak{A}_1$. The limit space

$$E^U = \varprojlim \hat{T}_{k+1}(\hat{E}_k)$$

belongs to $\bigcap_{\mathbf{N}} \mathrm{Groth}\mathfrak{A}_k)$, and with the standard argument we see that this class possesses enough metrizables. However a class of this kind might not be pre-ideal generated. We quote the main result of [20]: let α be a nuclear exponent; i.e.

$$\sum_n k^{-\alpha_n} < +\infty \qquad \forall k \geq 1.$$

Let k be a natural number. According to [22] we define the power series space $\Lambda_k(\alpha)$ of finite type as

$$\Lambda_k(\alpha) = \left\{ x \in \mathbf{K}^{\mathbf{N}} : \sum_n |x_n| R^{\alpha_n} < +\infty \quad \forall R < k \right\}.$$

When $k = \infty$ we obtain the power-series space $\Lambda(\alpha)$ of infinite type, which is in fact the G_∞- space constructed over the Köthe set $\{e^{r\alpha} : r \in \mathbf{N}\}$.

Recall from [22] that an operator $T \in L(X,Y)$ is said to be $\Lambda_k(\alpha)$ nuclear, $k \in \mathbf{N}$ and α a monotone increasing nuclear exponent, when $Tx = \sum_n \langle a_n, x \rangle\, y_n$ for every $x \in E$ with $(a_n) \subset X'$, $(y_n) \subset Y$ and $(\|a_n\|\,\|y_n\|) \in \Lambda_k(\alpha)$.

A Hausdorff lcs E is said to be $\Lambda_k(\alpha)$ nuclear when it is a $\Lambda_k(\alpha)$ space, and $\Lambda_{\mathbf{N}}(\alpha)$ nuclear when it is $\Lambda_k(\alpha)$–nuclear for all $k \in \mathbf{N}$.

Therefore:

$$\Lambda_k(\alpha)\text{–nuclear spaces} = \mathrm{Groth}(\Lambda_k(\alpha) \text{ nuclear operators})$$

and

$$\Lambda_{\mathbf{N}}(\alpha) \text{ nuclear spaces} = \bigcap_{k \in \mathbf{N}} \mathrm{Groth}(\Lambda_k(\alpha) \text{ nuclear operators}).$$

Theorem [20]

Let α be as before with $\liminf \alpha_{n+1}/\alpha_n > 1$. Then the class of $\Lambda_{\mathbf{N}}(\alpha)$ nuclear spaces cannot be pre-ideal generated.

Let us try a different approach to this result. Given a monotone increasing sequence ϕ converging to infinity, define the sequence space:

$$\lambda_\phi = \left\{ x \in \mathbf{R}^{\mathbf{N}} : x_n \leq k\phi_n^{-1} \text{ for some } k > 0 \right\}$$

and then the pre-ideal of operators:

$$\mathfrak{T}_\phi = \left\{ T \in \mathfrak{L}(X,Y) : (\delta_n(T))_n \in \lambda_\phi \right\}$$

where $\delta_n(T)$ means $\delta_n(T(U_X), U_Y)$.

When ϕ is stable we have:

Proposition 9

$\mathrm{Groth}(\mathfrak{T}_\phi) = \{E \text{ lcs} : \phi \in \Delta(E)\}$.

Proof. We only prove the inclusion $\subseteq$. From [5] recall that when ϕ is stable then $\phi \in \Delta(E)$ if and only if $\phi^p \in \Delta(E)$ for all (some) $p > 0$. From $E \in \mathrm{Groth}(\mathfrak{T}_\phi)$, $\phi^{1/2} \in \Delta(E)$ follows and also $\phi \in \Delta(E)$. □

Appealing again to the aforementioned result of [5]:

Corollary

$E \in \mathrm{Groth}(\mathfrak{T}_\phi)$ if and only if $\phi, \phi^2, \phi^3, \ldots \in \Delta(E)$.

On the other hand by putting $\phi = e^\alpha$ it is easy to see that:

E is $\Lambda_{\mathbf{N}}(\alpha)$-nuclear if and only if $\phi, \phi^2, \phi^3, \ldots \in \Delta(E)$

and the same proof provides

$$\Lambda_{\mathbf{N}}(\alpha)\text{-nuclear spaces} = \bigcap_{\mathbf{N}} \mathrm{Groth}(\mathfrak{T}_{\phi^k})$$

and thus:

Proposition 10

Let α be a nuclear exponent such that $\liminf \alpha_{n+1}/\alpha_n > 1$ and $\phi = e^\alpha$. Then the class

$$\bigcap_{\mathbf{N}} \mathrm{Groth}(\mathfrak{T}_{\phi^k})$$

is not a Grothendieck space pre-ideal.

Therefore, when ϕ is not stable the situation is entirely different to that of the stable case, and the class of those lcs for which $\phi, \phi^2, \phi^3, \ldots \in \Lambda(E)$ may not be pre-ideal generated.

If we turn to the relations between $\mathfrak{T}_{\phi^k}$ and $\mathfrak{T}_\phi^k$ we see that $\mathfrak{T}_{\phi^k} \subseteq \mathfrak{T}_\phi^k$, but the other inclusion does not hold: since $\mathrm{Groth}(\mathfrak{A}) = \mathrm{Groth}(\mathfrak{A}^k)$, $\mathfrak{T}_\phi^k \subseteq \mathfrak{T}_{\phi^k}$ would imply

$$\bigcap_{\mathbf{N}} \mathrm{Groth}(\mathfrak{T}_{\phi^k}) = \bigcap_{\mathbf{N}} \mathrm{Groth}(\mathfrak{T}_\phi^k) = \mathrm{Groth}(\mathfrak{T}_\phi).$$

But this is not the case, and then, when ϕ is not stable a $k \in \mathbf{N}$ must exist (and from the proof in [20] we know that if $c = \liminf \alpha_{n+1}/\alpha_n > 1$, then $k > c/(c-1)$ serves) such that for all $n \geq k$, $\mathfrak{T}_\phi^n$ is not contained in $\mathfrak{T}_{\phi^{k+1}}$. Therefore a Hausdorff lcs, constructed (in [20]) as a countable projective limit of diagonal maps acting on ℓ_2, exists such that $\phi \in \Delta(E)$ but $\phi^{k+1} \notin \Delta(E)$.

We get

Proposition 11

There exists a Fréchet space E such that, for some sequence $a \in \Delta(E)$, $a^2 \notin \Delta(E)$ (clearly a is a power of $\phi = e^{\alpha}$ and E is the just mentioned space).

This is apparently the first counterexample to [26]: "Characterize the locally convex spaces E with the following property: for each $\delta \in \bar{\Delta}(E)$, $\gamma \in \Delta(E)$, there exists $\beta \in \Delta(E)$ with $\beta_n \leq \delta_n \gamma_n$. All spaces with regular bases (Dragilev) have this property".

There we used the same notation as in [5]:

$$\dot{\Delta}(E) = \left\{x \in \mathbb{R}^{\mathbb{N}} : \forall U \in \mathcal{U}(E)\ \exists V \in \mathcal{U}(E) : x_n^{-1}\delta_n(V,U) \xrightarrow{n} 0\right\}.$$

The equivalence of the different definitions of diametral dimension appearing in the literature was treated in [5]. The equivalence between $\Delta(E)$ and $\Delta(E)$ is, roughly, inversion.

Therefore the above question is to characterize those lcs such that

$$\forall \delta, \gamma \in \Delta(E)\ \exists \beta \in \Delta(E) : \ \delta\gamma \leq \beta.$$

We see that this is equivalent to the property

$$a \in \Delta(E) \Longrightarrow a^2 \in \Delta(E).$$

One implication is clear. Take $a = \max\{\delta, \gamma\}$ for the other.

In [5] we treated this problem from the positive side and obtained that it has an affirmative answer when E is Δ-stable or a G_∞ space.

Since $\Lambda_k(\alpha)$ nuclear operators are contained in $\mathfrak{T}_{\varphi^k}$ we have

Proposition 12

If E is Δ-stable or a G_∞ space then E is $\Lambda_{\mathbb{N}}(\alpha)$-nuclear if and only if E is $\Lambda_k(\alpha)$ nuclear (for some $k \in \mathbb{N}$), and if and only if $E \in$ Groth$(\mathfrak{T}_\phi)$, $\phi = e^{\alpha}$.

By combining proposition 11 and the fact [23] that $\Delta(E \times E) = \Delta(E) * \Delta(E)$, that is

$$(x_n) \in \Delta(E \times E) \qquad \text{iff} \qquad (x_{2n}) \in \Delta(E) \text{ and } (x_{2n-1}) \in \Delta(E),$$

we see that the same proof of proposition 10 (preferably in terms of diametral dimension) serves to show that when $\phi = e^{\alpha}$ and $\liminf \alpha_{n+1}/\alpha_n > 1$ then the class Groth$(\mathfrak{T}_\phi)$ is not stable under Cartesian products: indeed, if E is the constructed counterexample with, suppose for te sake of simplicity, $\phi \in \Delta(E)$ but $\phi^2 \notin \Delta(E)$, then $\phi \notin \Delta(E \times E)$.

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