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# On the geometric means of an entire function of several complex variables represented by multiple Dirichlet series

S. N. Srivastava and Vinita Vijai

*Department of Mathematics, Lucknow University, Lucknow, India*

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Abstract

Let $f(s_1, s_2)$ be an entire function of two complex variables represented by double Dirichlet series. In the present paper, we have studied certain growth properties of the geometric means of $|f(s_1, s_2)|$ and obtained some inequalities which are the best possible.

## 1. Introduction

Consider the double Dirichlet series

$$(1.1)\qquad f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(s_1\lambda_m + s_2\mu_n) \qquad (s_j = \sigma_j + it_j,\ j = 1, 2)$$

where $a_{m,n} \in \mathbb{C}$, the field of complex numbers, $\lambda_m$, and $\mu_n$ are real, and

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m \to \infty, \qquad 0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty.$$

A.I. Janusauskas [1] has shown that if

$$(1.2)\qquad \lim_{m\to\infty} \frac{\log m}{\lambda_m} = 0, \qquad \lim_{n\to\infty} \frac{\log n}{\mu_n} = 0$$

then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence. Also, Sarkar [2, pp. 99] has shown that the necessary and sufficient condition for the series (1.1) satisfying (1.2) to be entire is that

$$\lim_{(m,n)\to\infty} \frac{\log|a_{m,n}|}{\lambda_m + \mu_n} = -\infty. \tag{1.3}$$

Let the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3) be denoted by $\mathcal{F}$. Then $f \in \mathcal{F}$ denotes an entire function over $\mathbb{C}^2$. The results can be extended to several complex variables.

Corresponding to an $f \in \mathcal{F}$, the maximum modulus $M = M_f$ and the maximum term $\mu = \mu_f$ on $\mathbb{R}^2$ are defined [2, pp. 100] as

$$M(\sigma) = M_f(\sigma_1, \sigma_2) = \max\left\{|f(s_1, s_2)| : s_1, s_2 \in \mathbb{C},\ \Re s_1 = \sigma_1,\ \Re s_2 = \sigma_2\right\}$$

$$\mu(\sigma) = \mu_f(\sigma_1, \sigma_2) = \max_{(m,n)\in\mathbb{N}^2}\left\{|a_{m,n}| \exp(\sigma_1\lambda_m + \sigma_2\mu_n)\right\}$$

where $\mathbb{N}$ is the set of natural numbers. The following two lemmas are due to Sarkar [2, pp. 101].

**Lemma A**

*Let $f \in \mathcal{F}$ be of finite order. Then $\rho = (\rho_1, \rho_2) \gg (0,0)$ is an order point of $f$ if and only if*

$$\limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log\log M(\sigma)}{\log(e^{\sigma_1\rho_1} + e^{\sigma_2\rho_2})} = 1, \qquad \sigma \in \mathbb{R}^2.$$

**Lemma B**

*Let $f \in \mathcal{F}$ be of finite order. Then $\tau = (\tau_1, \tau_2) \gg (0,0)$ is a type point of $f$ if and only if*

$$\limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log M(\sigma)}{\tau_1 e^{\sigma_1\rho_1} + \tau_2 e^{\sigma_2\rho_2}} = 1, \qquad \sigma \in \mathbb{R}^2.$$

The geometric means of $|f(s_1,s_2)|$ are defined as

$$(1.4)\quad G(\sigma_1,\sigma_2)=\exp\left\{\lim_{(T_1,T_2)\to\infty}\frac{1}{4T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}\log\left|f(\sigma_1+it_1,\sigma_2+it_2)\right|dt_1\,dt_2\right\}$$

$$(1.5)\qquad g_k(\sigma_1,\sigma_2)=\exp\left\{\frac{k^2}{e^{k\sigma_1}e^{k\sigma_2}}\int_0^{\sigma_1}\int_0^{\sigma_2}\log G(x_1,x_2)e^{kx_1}e^{kx_2}\,dx_1\,dx_2\right\}$$

where $k$ is a positive number.

Salimov [3] has given the definitions of $R$-order and $R$-type of functions $f\in\mathcal{F}$. He has defined the geometric means of the functions $f\in\mathcal{F}$ as above and obtained some results about them. In this paper, following the definitions of order and type [2], we have studied the properties of geometric means of the functions $f\in\mathcal{F}$, and obtained some inequalities which are the best possible.

## 2. Theorem 1

**Theorem 1**

*For $f(s_1,s_2)$, $f\in\mathcal{F}$, we have*

$$(2.1)\qquad \lim_{(\sigma_1,\sigma_2)\to\infty}\frac{g_k(\alpha_1\sigma_1,\alpha_2\sigma_2)}{g_k(\sigma_1,\sigma_2)e^{k\sigma_1(1-\alpha_1)}e^{k\sigma_2(1-\alpha_2)}}=0$$

*where $\alpha_1$, $\alpha_2$ $(0<\alpha_1,\alpha_2<1)$ are constants.*

We first prove the following lemma.

**Lemma 1**

*For $f(s_1,s_2)$, $f\in\mathcal{F}$, and $0<\sigma_1'<\sigma_1<\bar\sigma_1$, $0<\sigma_2'<\sigma_2<\bar\sigma_2$, we have*

$$\begin{aligned}
(e^{k\sigma_1}-e^{k\bar\sigma_1})(e^{k\bar\sigma_2}-e^{k\sigma_2'})\log G(\bar\sigma_1,\sigma_2')+(e^{k\sigma_2}-e^{k\bar\sigma_2})(e^{k\bar\sigma_1}-e^{k\sigma_1'})\log G(\sigma_1',\bar\sigma_2)\\
+(e^{k\sigma_1}-e^{k\bar\sigma_1})(e^{k\sigma_2}-e^{k\bar\sigma_2})\log G(\sigma_1,\sigma_2)\\
\le\left\{e^{k\sigma_1+k\sigma_2}\log g_k(\sigma_1,\sigma_2)-e^{k\bar\sigma_1+k\bar\sigma_2}\log g_k(\bar\sigma_1,\bar\sigma_2)\right\}\\
\le(e^{k\sigma_1}-e^{k\bar\sigma_1})(e^{k\bar\sigma_2}-1)\log G(\sigma_1,\sigma_2)\\
+(e^{k\sigma_2}-e^{k\bar\sigma_2})(e^{k\bar\sigma_1}-1)\log G(\sigma_1,\sigma_2)\\
+(e^{k\sigma_1}-e^{k\bar\sigma_1})(e^{k\sigma_2}-e^{k\bar\sigma_2})\log G(\sigma_1,\sigma_2)
\end{aligned}$$

*where $k$ is any positive number.*

*Proof of Lemma 1.* $G(x_1, x_2)$ is an increasing function of $x_1$ and $x_2$ [3] and therefore we have from (1.5)

$$
\begin{aligned}
e^{k\sigma_1+k\sigma_2}&\log g_k(\sigma_1,\sigma_2) - e^{k\bar\sigma_1+k\bar\sigma_2}\log g_k(\bar\sigma_1,\sigma_2)\\
&= k^2\int_0^{\sigma_1}\int_0^{\sigma_2}\log G(x_1,x_2)\,e^{kx_1+kx_2}\,dx_1\,dx_2\\
&\quad - k^2\int_0^{\bar\sigma_1}\int_0^{\bar\sigma_2}\log G(x_1,x_2)\,e^{kx_1+kx_2}\,dx_1\,dx_2\\
&= k^2\int_{\bar\sigma_1}^{\sigma_1}\int_0^{\bar\sigma_2}\log G(x_1,x_2)\,e^{kx_1+kx_2}\,dx_1\,dx_2\\
&\quad + k^2\int_0^{\bar\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}\log G(x_1,x_2)\,e^{kx_1+kx_2}\,dx_1\,dx_2.\\
&\quad + k^2\int_{\bar\sigma_1}^{\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}\log G(x_1,x_2)\,e^{kx_1+kx_2}\,dx_1\,dx_2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(2.2)\qquad e^{k\sigma_1+k\sigma_2}&\log g_k(\sigma_1,\sigma_2) - e^{k\bar\sigma_1+k\bar\sigma_2}\log g_k(\sigma_1,\sigma_2)\\
&\le (e^{k\sigma_1}-e^{k\bar\sigma_1})(e^{k\bar\sigma_2}-1)\log G(\sigma_1,\bar\sigma_2)\\
&\quad + (e^{k\sigma_2}-e^{k\bar\sigma_2})(e^{k\bar\sigma_1}-1)\log G(\sigma_1,\sigma_2)\\
&\quad + (e^{k\sigma_1}-e^{k\bar\sigma_1})(e^{k\sigma_2}-e^{k\bar\sigma_2})\log G(\sigma_1,\sigma_2).
\end{aligned}
$$

Also

$$
\begin{aligned}
(2.3)\qquad e^{k\sigma_1+k\sigma_2}&\log g_k(\sigma_1,\sigma_2) - e^{k\bar\sigma_1+k\bar\sigma_2}\log g_k(\sigma_1,\bar\sigma_2)\\
&\ge k^2\int_{\bar\sigma_1}^{\sigma_1}\int_{\sigma_2'}^{\bar\sigma_2}\log G(x_1,x_2)\,e^{kx_1+kx_2}\,dx_1\,dx_2\\
&\quad + k^2\int_{\sigma_1'}^{\bar\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}\log G(x_1,x_2)\,e^{kx_1+kx_2}\,dx_1\,dx_2\\
&\quad + k^2\int_{\bar\sigma_1}^{\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}\log G(x_1,x_2)e^{kx_1+kx_2}\,dx_1\,dx_2\\
&\ge (e^{k\sigma_1}-e^{k\bar\sigma_1})(e^{k\bar\sigma_2}-e^{k\sigma_2'})\log G(\sigma_1,\sigma_2')\\
&\quad + (e^{k\sigma_2}-e^{k\bar\sigma_2}(e^{k\bar\sigma_1}-e^{k\sigma_1'})\log G(\sigma_1',\sigma_2)\\
&\quad + (e^{k\sigma_1}-e^{k\bar\sigma_1})(e^{k\sigma_2}-e^{k\bar\sigma_2})\log G(\sigma_1,\sigma_2).
\end{aligned}
$$

Combining (2.2) and (2.3), we obtain the lemma. □

*Proof of Theorem 1*. If we put $\bar{\sigma}_1 = \alpha_1\sigma_1$, $\sigma_1' = \beta_1\sigma_1$, $\bar{\sigma}_2 = \alpha_2\sigma_2$, $\sigma_2' = \beta_2\sigma_2$, where $\beta_1 < \alpha_1, \beta_2 < \alpha_2$ in Lemma 1, we get

$$\begin{aligned}
&\left(e^{k\sigma_1} - e^{k\sigma_1\alpha_1}\right)\left(e^{k\sigma_2\alpha_2} - e^{k\sigma_2\beta_2}\right)\log G(\alpha_1\sigma_1, \beta_2\sigma_2) \\
&+\left(e^{k\sigma_2} - e^{k\sigma_2\alpha_2}\right)\left(e^{k\sigma_1\alpha_1} - e^{k\sigma_1\beta_1}\right)\log G(\beta_1\sigma_1, \alpha_2\sigma_2) \\
&+\left(e^{k\sigma_1} - e^{k\sigma_1\alpha_1}\right)\left(e^{k\sigma_2} - e^{k\sigma_2\alpha_2}\right)\log G(\alpha_1\sigma_1, \alpha_2\sigma_2) \\
&\qquad \le e^{k\sigma_1 + k\sigma_2}\log g_k(\sigma_1, \sigma_2) - e^{k\sigma_1\alpha_1 + k\sigma_2\alpha_2}\log g_k(\alpha_1\sigma_1, \alpha_2\sigma_2) \\
&\qquad \le \left(e^{k\sigma_1} - e^{k\sigma_1\alpha_1}\right)\left(e^{k\sigma_2\alpha_2} - 1\right)\log G(\alpha_1, \alpha_2\sigma_2) \\
&\qquad\quad + \left(e^{k\sigma_2} - e^{k\sigma_2\alpha_2}\right)\left(e^{k\sigma_1\alpha_1} - 1\right)\log G(\alpha_1\sigma_1, \sigma_2) \\
&\qquad\quad + \left(e^{k\sigma_1} - e^{k\sigma_1\alpha_1}\right)\left(e^{k\sigma_2} - e^{k\sigma_2\alpha_2}\right)\log G(\sigma_1, \sigma_2).
\end{aligned}$$

By dividing the whole inequality by $e^{k\sigma_1\alpha_1}e^{k\sigma_2\alpha_2}$, we obtain

$$\begin{aligned}
&\left(e^{k\sigma_1(1-\alpha_1)} - 1\right)\left(1 - e^{k\sigma_2(\beta_2-\alpha_2)}\right)\log G(\alpha_1\sigma_1, \beta_2\sigma_2) \\
&+\left(e^{k\sigma_2(1-\alpha_2)} - 1\right)\left(1 - e^{k\sigma_1(\beta_1-\alpha_1)}\right)\log G(\beta_1\sigma_1, \alpha_2\sigma_2) \\
&+\left(e^{k\sigma_1(1-\alpha_1)} - 1\right)\left(e^{k\sigma_2(1-\alpha_2)} - 1\right)\log G(\alpha_1\sigma_1, \alpha_2\sigma_2) \\
&\qquad \le \left(e^{k\sigma_1(1-\alpha_1)}e^{k\sigma_2(1-\alpha_2)}\right)\log g_k(\sigma_1, \sigma_2) - \log g_k(\alpha_1\sigma_1, \alpha_2\sigma_2) \\
&\qquad \le \left(e^{k\sigma_1(1-\alpha_1)} - 1\right)\left(1 - e^{-k\sigma_2\alpha_2}\right)\log G(1\sigma_1, \alpha_2\sigma_2) \\
&\qquad\quad + \left(e^{k\sigma_2(1-\alpha_2)} - 1\right)\left(1 - e^{-k\sigma_1\alpha_1}\right)\log G(\alpha_1\sigma_1, \sigma_2) \\
&\qquad\quad + \left(e^{k\sigma_1(1-\alpha_1)} - 1\right)\left(1 - e^{k\sigma_2(1-\alpha_2)} - 1\right)\log G(\sigma_1, \sigma_2).
\end{aligned}$$

By taking limits on both sides we get

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \frac{g_k(\alpha_1\sigma_1, \alpha_2\sigma_2)}{\{g_k(\sigma_1, \sigma_2)\}^{e^{k\sigma_1(1-\alpha_1)}e^{k\sigma_2(1-\alpha_2)}}} = 0. \ \square$$

## 3. Theorem 2

**Theorem 2**

*Let the Dirichlet series $f(s_1, s_2)$, $f \in \mathcal{F}$, be of order $\rho = (\rho_1, \rho_2)$ and type $\tau = (\tau_1, \tau_2)$. Then*

$$\limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log\log G(\sigma_1, \sigma_2)}{\log(e^{\sigma_1\rho_1} + e^{\sigma_2\rho_2})} \le 1. \tag{3.1}$$

$$\limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log G(\sigma_1,\sigma_2)}{\tau_1 e^{\sigma_1\rho_1} + \tau_2 e^{\sigma_2\rho_2}} \le 1, \tag{3.2}$$

$$\limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log\log g_k(\sigma_1,\sigma_2)}{\log(e^{\sigma_1\rho_1} + e^{\sigma_2\rho_2})} \le 1, \tag{3.3}$$

$$\limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log g_k(\sigma_1,\sigma_2)}{\tau_1 e^{\sigma_1\rho_1} + \tau_2 e^{\sigma_2\rho_2}} \le 1. \tag{3.4}$$

The above results follow directly from the definitions (1.4), and (1.5), and lemmas A and B.

Let us set, for $\rho_1 = \rho_2 = \rho$, $0 < \rho < \infty$,

$$\begin{aligned} &\limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log g_k(\sigma_1,\sigma_2)}{e^{\sigma_1\rho} + e^{\sigma_2\rho}} = a \\ &\liminf_{(\sigma_1,\sigma_2)\to\infty} \frac{\log g_k(\sigma_1,\sigma_2)}{e^{\sigma_1\rho} + e^{\sigma_2\rho}} = b \qquad (0 < b \le a < \infty). \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\limsup_{(\sigma_1,\sigma_2)\to\infty} \frac{\log G(\sigma_1,\sigma_2)}{e^{\sigma_1\rho} + e^{\sigma_2\rho}} = c \\ &\liminf_{(\sigma_1,\sigma_2)\to\infty} \frac{\log G(\sigma_1,\sigma_2)}{e^{\sigma_1\rho} + e^{\sigma_2\rho}} = d \qquad (0 < d \le c < \infty). \end{aligned} \tag{3.6}$$

## 4. Theorem 3

**Theorem 3**

*Let $f(s_1,s_2)$, $f \in \mathcal{F}$, be of order $\rho = (\rho_1,\rho_2)$, $0 < \rho < \infty$. Then for $\rho_1 = \rho_2 = \rho$*

$$\frac{kd}{\rho+k} \le b \le a \le \frac{kc}{\rho+k} \tag{4.1}$$

$$c \le \frac{a(\rho+2k)^{(\rho+2k)/k}}{4k^2\rho^{\rho/k}} \tag{4.2}$$

$$c + \frac{\rho d(3\rho + 4k)}{4k(\rho + k)} \le \frac{a(\rho + 2k)^{(\rho+2k)/k}}{4k^2 \rho^{\rho/k}} \,. \tag{4.3}$$

*Proof.* If $A$ is a constant, then from (1.5), we have for $h > 0$, and $\rho_1 = \rho_2 = \rho$

$$\log g_k\left(\sigma_1 + \frac{h}{\rho}, \sigma_2 + \frac{h}{\rho}\right) = \frac{k^2}{e^{k(\sigma_1 + h/\rho)} e^{k(\sigma_2 + h/\rho)}}$$

$$\times \int_0^{\sigma_1 + h/\rho} \int_0^{\sigma_2 + h/\rho} e^{k(x_1 + x_2)} \log G(x_1, x_2)\, dx_1\, dx_2. \tag{4.4}$$

And this expression is

$$\begin{aligned} = \frac{k^2}{e^{k(\sigma_1 + h/\rho)} e^{k(\sigma_2 + h/\rho)}} &\left\{ \int_0^{\sigma_1^0} \int_0^{\sigma_2^0} + \int_0^{\sigma_1^0} \int_{\sigma_2^0}^{\sigma_2} + \int_0^{\sigma_1^0} \int_{\sigma_2}^{\sigma_2 + h/\rho} \right. \\ &+ \int_{\sigma_1^0}^{\sigma_1} \int_0^{\sigma_2^0} + \int_{\sigma_1}^{\sigma_1 + h/\rho} \int_0^{\sigma_2^0} + \int_{\sigma_1^0}^{\sigma_1} \int_{\sigma_2^0}^{\sigma_2} + \int_{\sigma_1^0}^{\sigma_1} \int_{\sigma_2}^{\sigma_2 + h/\rho} \\ &+ \left. \int_{\sigma_1}^{\sigma_1 + h/\rho} \int_{\sigma_2^0}^{\sigma_2} + \int_{\sigma_1}^{\sigma_1 + h/\rho} \int_{\sigma_2}^{\sigma_2 + h/\rho} \right\} e^{k(x_1, x_2)} \log G(x_1, x_2)\, dx_1\, dx_2 \end{aligned} \tag{4.5}$$

$$\begin{aligned} < \frac{A}{e^{k\sigma_1} e^{k\sigma_2}} &+ \frac{(e^{k\sigma_1^0} - 1)}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left[ (e^{k\sigma_2} - e^{k\sigma_2^0}) \log G(\sigma_1^0, \sigma_2) \right] \\ &+ (e^{kh/\rho} - 1) e^{k\sigma_2} \log G(\sigma_1^0, \sigma_2 + h/\rho) \\ &+ \frac{(e^{k\sigma_2^0} - 1)}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left[ (e^{k\sigma_1} - e^{k\sigma_1^0}) \log G(\sigma_1, \sigma_2^0) \right. \\ &+ \left. (e^{kh/\rho} - 1) e^{k\sigma_1} \log G(\sigma_1 + h/\rho, \sigma_2^0) \right] \\ &+ \frac{(c + \varepsilon) k^2}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \int_{\sigma_1^0}^{\sigma_1} \int_{\sigma_2^0}^{\sigma_2} \left( e^{(k+\rho)x_1} e^{kx_2} + e^{(k+\rho)x_2} e^{kx_1} \right) dx_1\, dx_2 \\ &+ \frac{(c + \varepsilon) k (e^{kh/\rho} - 1)}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left[ e^{k\sigma_2} \int_{\sigma_1^0}^{\sigma_1} \left( e^{(k+\rho)x_1} + e^{kx_1} e^{\rho\sigma_2} e^{h} \right) dx_1 \right. \\ &+ \left. e^{k\sigma_1} \int_{\sigma_2^0}^{\sigma_2} \left( e^{kx_2} e^{\rho\sigma_1} e^{h} + e^{(k+\rho)x_2} \right) dx_2 \right] \\ &+ \frac{(e^{kh/\rho} - 1)^2}{e^{2kh/\rho}} \log G(\sigma_1 + h/\rho, \sigma_2 + h/\rho). \end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&= \frac{A}{e^{k\sigma_1} e^{k\sigma_2}} + \frac{e^{k\sigma_1^0} - 1}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left[ (e^{k\sigma_2} - e^{k\sigma_2^0}) \log G(\sigma_1^0, \sigma_2) \right. \\
&\quad \left. + (e^{kh/\rho} - 1) e^{k\sigma_2} \log G(\sigma_1^0, \sigma_2 + h/\rho) \right] \\
&\quad + \frac{e^{k\sigma_2^0} - 1}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left[ (e^{k\sigma_1} - e^{k\sigma_1^0}) \log G(\sigma_1, \sigma_2^0) \right. \\
&\quad \left. + (e^{kh/\rho} - 1) e^{k\sigma_1} \log G(\sigma_1 + h/\rho, \sigma_2^0) \right] \\
&\quad + \frac{(c+\varepsilon)k^2}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left[ \frac{e^{(k+\rho)\sigma_1} - e^{(k+\rho)\sigma_1^0})(e^{k\sigma_2} - e^{k\sigma_2^0})}{k(k+\rho)} \right. \\
&\quad \left. + \frac{e^{(k+\rho)\sigma_2} - e^{(k+\rho)\sigma_2^0})(e^{k\sigma_1} - e^{k\sigma_1^0})}{k(k+\rho)} \right] \\
&\quad + \frac{(c+\varepsilon)k(e^{kh/\rho} - 1)}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left[ \frac{e^{k+\rho)\sigma_1} - e^{(k+\rho)\sigma_1^0})e^{k\sigma_2}}{(k+\rho)} \right. \\
&\quad + \frac{e^{(k+\rho)\sigma_2} e^{h}(e^{k\sigma_1} - e^{k\sigma_1^0})}{k} + \frac{e^{(k+\rho)\sigma_1} e^{h}(e^{(k\sigma_2} - e^{k\sigma_2^0})}{k} \\
&\quad \left. + \frac{e^{(k+\rho)\sigma_2} - e^{(k+\rho)\sigma_2^0})e^{k\sigma_1}}{(k+\rho)} \right] + \frac{(e^{kh/\rho} - 1)^2}{e^{2kh/\rho}} \log G(\sigma_1 + h/\rho, \sigma_2 + h/\rho).
\end{aligned}
$$

$$
\begin{aligned}
(4.8) \quad &= \frac{A}{e^{k\sigma_1} e^{k\sigma_2}} + \frac{e^{k\sigma_1^0} - 1}{e^{k\sigma_1} e^{2kh/\rho}} \\
&\quad \times \left[ \left(1 - e^{k\sigma_2^0} e^{-k\sigma_2}\right) \log G(\sigma_1^0, \sigma_2) + \left(e^{kh/\rho} - 1\right) \log G(\sigma_1^0, \sigma_2 + h/\rho) \right] \\
&\quad + \frac{e^{k\sigma_2^0} - 1}{e^{k\sigma_2} e^{2kh/\rho}} \left[ (1 - e^{k\sigma_1^0} e^{-k\sigma_1}) \log G(\sigma_1, \sigma_2^0) \right. \\
&\quad \left. + (e^{kh/\rho} - 1) \log G(\sigma_1 + h/\rho, \sigma_2^0) \right] \\
&\quad + \frac{c+\varepsilon}{(k+\rho)e^{2kh/\rho}} \left[ (e^{\rho\sigma_1} + e^{\rho\sigma_2}) - e^{\rho\sigma_1} e^{k\sigma_2^0} e^{-k\sigma_2} - e^{(k+\rho)\sigma_1^0} e^{-k\sigma_1} \right. \\
&\quad + e^{(k+\rho)\sigma_1^0} e^{-k\sigma_1} e^{k\sigma_2} e^{-k\sigma_2} - e^{\rho\sigma_2} e^{k\sigma_1^0} e^{-k\sigma_1} \\
&\quad \left. - e^{(k+\rho)\sigma_2^0} e^{-k\sigma_2} + e^{(k+\rho)\sigma_2^0} e^{-k\sigma_2} e^{k\sigma_1^0} e^{-k\sigma_1} \right] \\
&\quad + \frac{(c+\varepsilon)k(e^{kh/\rho} - 1)}{e^{2kh/\rho}} \left[ \frac{(e^{\rho\sigma_1} + e^{\rho\sigma_2}) - e^{(k+\rho)\sigma_1^0} e^{-k\sigma_1} - e^{(k+\rho)\sigma_2^0} e^{-k\sigma_2}}{(k+\rho)} \right.
\end{aligned}
$$

$$+ \frac{(e^{\rho\sigma_1} + e^{\rho\sigma_2})e^h - (e^{\rho\sigma_2} e^{k\sigma_1^0} e^{-k\sigma_1} + e^{\rho\sigma_1} e^{k\sigma_2^0} e^{-k\sigma_2})e^h}{k}\Bigg]$$

$$+ \frac{(e^{kh/\rho} - 1)^2}{e^{2kh/\rho}} \log G(\sigma_1 + h/\rho, \sigma_2 + h/\rho).$$

Hence, dividing both the sides by

$$(e^{\rho(\sigma_1 + h/\rho)} + e^{\rho(\sigma_2 + h/\rho)}) = (e^{\rho\sigma_1} + e^{\rho\sigma_2})e^h$$

and taking superior limits, we get

$$a \leq \frac{kc}{(k+\rho)e^{(2(k/\rho)+1)h}} + \frac{kc(e^{k(h/\rho)} - 1)}{e^{2kh/\rho}} \left[\frac{1}{(k+\rho)e^h} + \frac{1}{k}\right] + \frac{(e^{kh/\rho} - 1)^2 c}{e^{2kh/\rho}}. \tag{4.9}$$

Also, from (4.4), we have

$$\log g_k(\sigma_1 + h/\rho, \sigma_2 + h/\rho)$$

$$> \frac{k^2}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \int_0^{\sigma_1 + h/\rho} \int_0^{\sigma_2 + h/\rho} e^{k(x_1 + x_2)} \log G(x_1, x_2) dx_1\, dx_2 \tag{4.10}$$

$$= \frac{k^2}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left\{ \int_{\sigma_1^0}^{\sigma_1} \int_{\sigma_2^0}^{\sigma_2} + \int_{\sigma_1^0}^{\sigma_1} \int_{\sigma_2}^{\sigma_2 + h/\rho} + \int_{\sigma_1}^{\sigma_1 + h/\rho} \int_{\sigma_2^0}^{\sigma_2} \right.$$

$$\left. + \int_{\sigma_1}^{\sigma_1 + h/\rho} \int_{\sigma_2}^{\sigma_2 + h/\rho} \right\} e^{k(x_1 + x_2)} \log G(x_1, x_2) dx_1\, dx_2$$

$$> \frac{k^2}{e^{k\sigma_1} e^{k\sigma_2} e^{2kh/\rho}} \left[ \int_{\sigma_1^0}^{\sigma_1} \int_{\sigma_2^0}^{\sigma_2} e^{k(x_1 + x_2)} \log G(x_1, x_2) dx_1\, dx_2 \right. \tag{4.11}$$

$$+ \frac{e^{k\sigma_2}(e^{kh/\rho} - 1)}{k} \int_{\sigma_1^0}^{\sigma_1} e^{kx_1} \log G(x_1, \sigma_2) dx_1$$

$$+ \frac{e^{k\sigma_1}(e^{kh/\rho} - 1)}{k} \int_{\sigma_2^0}^{\sigma_2} e^{kx_2} \log G(\sigma_1, x_2) dx_2$$

$$\left. + \frac{e^{k\sigma_1} e^{k\sigma_2} (e^{kh/\rho} - 1)^2}{k^2} \log G(\sigma_1, \sigma_2) \right].$$

$$(4.12)\quad > \frac{k^2}{e^{k\sigma_1}e^{k\sigma_2}e^{2kh/\rho}}\Bigg[(d-\varepsilon)\int_{\sigma_1^0}^{\sigma_1}\int_{\sigma_2^0}^{\sigma_2}\big(e^{(k+\rho)x_1}e^{kx_2}$$
$$+ e^{(k+\rho)x_2}e^{kx_1}\big)dx_1\,dx_2$$
$$+\frac{(d-\varepsilon)(e^{kh/\rho}-1)}{k}\Bigg\{e^{k\sigma_2}\int_{\sigma_1^0}^{\sigma_1}\big(e^{(k+\rho)x_1}+e^{\rho\sigma_2}e^{kx_1}\big)dx_1$$
$$+e^{k\sigma_1}\int_{\sigma_2^0}^{\sigma_2}\big(e^{\rho\sigma_1}e^{kx_2}+e^{(k+\rho)x_2}\big)dx_2\Bigg\}\Bigg]$$
$$+\frac{(e^{kh/\rho}-1)^2}{e^{2kh/\rho}}\log G(\sigma_1,\sigma_2)$$

$$(4.13)\quad = \frac{k}{e^{k\sigma_1}e^{k\sigma_2}e^{2kh/\rho}}\Bigg[\frac{(d-\varepsilon)}{(k+\rho)}\Big\{\big(e^{(k+\rho)\sigma_1}-e^{(k+\rho)\sigma_1^0}\big)\big(e^{k\sigma_2}-e^{k\sigma_2^0}\big)$$
$$+\big(e^{(k+\rho)\sigma_2}-e^{(k+\rho)\sigma_2^0}\big)\big(e^{k\sigma_1}-e^{k\sigma_1^0}\big)\Big\}$$
$$+(d-\varepsilon)\big(e^{kh/\rho}-1\big)\Bigg\{\frac{e^{k\sigma_2}\big(e^{(k+\rho)\sigma_1}-e^{(k+\rho)\sigma_1^0}}{(k+\rho)}$$
$$+\frac{e^{k\sigma_2}e^{\rho\sigma_2}\big(e^{k\sigma_1}-e^{k\sigma_1^0}\big)}{k}+\frac{e^{k\sigma_1}e^{\rho\sigma_1}\big(e^{k\sigma_2}-e^{k\sigma_2^0}\big)}{k}$$
$$+\frac{e^{k\sigma_1}\big(e^{(k+\rho)\sigma_2}-e^{(k+\rho)\sigma_2^0}}{(k+\rho)}\Bigg\}\Bigg]+\frac{(e^{kh/\rho}-1)^2}{e^{2kh/\rho}}\log G(\sigma_1,\sigma_2)$$

$$(4.14)\quad = \frac{k}{e^{2kh/\rho}}\Bigg[\frac{(d-\varepsilon)}{(k+\rho)}\Bigg\{\big(e^{\rho\sigma_1}+e^{\rho\sigma_2}\big)-e^{\rho\sigma_1}e^{k\sigma_2^0}e^{-k\sigma_2}-e^{(k+\rho)\sigma_1^0}e^{-k\sigma_1}$$
$$+e^{(k+\rho)\sigma_1^0}e^{k\sigma_2^0}e^{-k\sigma_1}e^{-k\sigma_2}-e^{\rho\sigma_2}e^{k\sigma_1^0}e^{-k\sigma_1}-e^{(k+\rho)\sigma_2^0}e^{-k\sigma_2}$$
$$+e^{(k+\rho)\sigma_2^0}e^{k\sigma_1^0}e^{-k\sigma_1}e^{-k\sigma_2}\Bigg\}$$
$$+(d-\varepsilon)\big(e^{kh/\rho}-1\big)\Bigg\{\frac{e^{\rho\sigma_1}+e^{\rho\sigma_2}}{k+\rho}-\frac{e^{(k+\rho)\sigma_1^0}e^{-k\sigma_1}+e^{(k+\rho)\sigma_2^0}e^{-k\sigma_2}}{k+\rho}$$
$$-\frac{e^{\rho\sigma_2}-e^{k\sigma_1^0}e^{-k\sigma_1}+e^{\rho\sigma_1}e^{k\sigma_2^0}e^{-k\sigma_2}}{k}+\frac{e^{\rho\sigma_1}+e^{\rho\sigma_2}}{k}\Bigg\}\Bigg]$$
$$+\frac{(e^{kh/\rho}-1)^2}{e^{2kh/\rho}}\log G(\sigma_1\sigma_2).$$

Dividing both sides by

$$e^{\rho(\sigma_1+h/\rho)} + e^{\rho(\sigma_2+h/\rho)} = \left(e^{\rho\sigma_1} + e^{\rho\sigma_2}\right)e^{h}$$

and taking superior and inferior limits, respectively, we get

$$a \geq \frac{kd}{(k+\rho)e^{((2k/\rho)+1)h}} + \frac{(2k+\rho)(e^{kh/\rho}-1)d}{(k+\rho)e^{(2k/\rho+1)h}} + \frac{(e^{kh/\rho}-1)^2 c}{e^{(2k/\rho+1)h}}\,, \tag{4.15}$$

$$b \geq \frac{kd}{(k+\rho)e^{(2k/\rho+1)h}} + \frac{(2k+\rho)(e^{kh/\rho}-1)d}{(k+\rho)e^{2(k/\rho+1)h}} + \frac{(e^{kh/\rho}-1)^2 d}{e^{(2k/\rho+1)h}}\,. \tag{4.16}$$

If we put $h = 0$ in (4.9) and (4.16), we get (4.1). Also, from (4.15)

$$(e^{kh/\rho}-1)^2 c \leq a e^{(2k/\rho+1)h} - \frac{kd}{k+\rho} - \frac{(e^{kh/\rho}-1)(2k+\rho)d}{k+\rho}\,. \tag{4.17}$$

Therefore, for all $h > 0$

$$c \leq \frac{a e^{(2k/\rho+1)h}}{(e^{kh/\rho}-1)^2}\,. \tag{4.18}$$

The right hand side of the above inequality has the minimum value when

$$h = \rho/k \log(\rho + 2k/\rho).$$

therefore

$$c \leq \frac{a(\rho+2k)^{(\rho+2k)/k}}{4k^2\rho^{\rho/k}} \tag{4.2}$$

Also, from (4.17) and (4.2), we get

$$c + \frac{\rho d(3\rho+4k)}{4k(k+\rho)} \leq \frac{a(\rho+2k)^{(\rho+2k)/k}}{4k^2\rho^{\rho/k}}\,. \ \square \tag{4.3}$$

*Remark.* The result (4.1) has also been obtained by Salimov [3] following the definitions which he has adopted.

## References


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