*Collect. Math.* **40**, 1 (1989), 19–30


# The unitary analogue of Pillai's arithmetical function

LASZLO TOTH

*3900 Satu Mare, str. N. Golescu 5, Romania*

Revised version received 2/OCT/89

ABSTRACT

A unitary analogue of Pillai's arithmetical function is introduced, and an asymptotic formula is proved.

## 1. Introduction

Pillai's arithmetical function [7] is defined by

$$P(n) = \sum_{k=1}^{n} (k, n),$$

where $(k, n)$ denotes the g. c. d. of $k$ and $n$. It is easy to show that

$$P(n) = \sum_{d|n} d\varphi\left(\frac{n}{d}\right), \tag{1.1}$$

where $\varphi(n)$ is the Euler totient function, that is $P(n)$ represents the Dirichlet convolution of the multiplicative arithmetical functions $E(n) = n$ and $\varphi(n)$, thus it is also multiplicative [6, §4.4, problem 6].

Formula (1.1) furnishes the following asymptotic estimate [9]

$$\sum P(n) = \frac{3}{\pi^2}\, x^2 \log x + O(x^2), \tag{1.2}$$

and by partial summation we obtain

$$\sum_{n\leq x} \frac{P(n)}{n} = \frac{6}{\pi^2}\, x\log x + O(x), \tag{1.3}$$

which implies that $P(n)/n$ behaves like $6\log n/\pi^2$ or for $k \leq n$ the average value of $(k,n)$ is $6\log n/\pi^2$ [1].

It is well known that a divisor $d$ of an integer $n$ is called unitary if $n = de$ and $(d,e) = 1$, notation $d \| n$. Following Cohen [2], let $(k,n)_*$ denote the greatest divisor of $k$ which is a unitary divisor of $n$.

In this paper we introduce the function $P^*(n)$ defined by

$$P^*(n) = \sum_{k=1}^{n} (k,n)_*,$$

which is the unitary analogue of the Pillai function $P(n)$. We show that the function $P^*(n)$ is multiplicative (corollary 3.1) and we establish an asymptotic formula for the summatory function of $P^*(n)$ (theorem 3.2).

In the last part of the paper we investigate a slightly more general function, namely

$$P_r^*(n) = \sum_{k=1}^{n} (k,n)_*^r,$$

and give an asymptotic formula for its summatory function in the case $r > 1$ (theorem 4.2).

Our method is purely elementary and is based on lemma 2.1 instead of the usual formula of corollary 2.1. A similar procedure has been adopted by Cohen [3].

## 2. Prerequisites

Let $\varphi^*(n)$ denote, as usual, the unitary analogue of the Euler function, that is $\varphi^*(n)$ represents the number of positive integers $k \leq n$ with $(k,n)_* = 1$ (ie, $k$ is semiprime to $n$ [2]). The multiplicative function

$$\mu^*(n) = (-1)^{\omega(n)},$$

where $\omega(n)$ denotes the number of distinct prime factors of $n$, is the unitary analogue of the Möbius function $\mu(n)$ and we have

$$\sum_{d\|n} \mu^*(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases} \tag{2.1}$$

$$\varphi^*(n) = \sum_{d\|n} d\mu^*\left(\frac{n}{d}\right). \tag{2.2}$$

We note that $\varphi^*(n)$ is multiplicative, being the unitary convolution of two multiplicative functions [2, lemma 6.1] and for $n = p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$,

$$\varphi^*(n) = (p_1^{a_1} - 1)(p_2^{a_2} - 1)\cdots(p_t^{a_t} - 1). \tag{2.3}$$

We need the following familiar estimates,

$$\sum_{n\le x} n^s = \frac{x^{s+1}}{s+1} + O(x^s), \qquad s \ge O, \tag{2.4}$$

$$\sum_{n\le x} \frac{1}{n} = O(\log x), \tag{2.5}$$

$$\sum_{n\le x} \frac{1}{n^s} = O\left(x^{1-s}\right), \qquad O < s < 1, \tag{2.6}$$

$$\sum_{n> x} \frac{1}{n^s} = O\left(x^{1-s}\right), \qquad s > 1. \tag{2.7}$$

*Remark 2.1.* A trivial consequence of (2.4) is the estimate

$$\sum_{n\le x} n^s = \frac{x^{s+1}}{s+1} + O\left(x^{s+\varepsilon}\right), \tag{2.8}$$

valid for each $\varepsilon$, $O \le \varepsilon < 1$ and $s \ge O$, which is a starting point of our treatment.

**Lemma 2.1**

*For each* $\varepsilon$, $O \le \varepsilon < 1$ *and* $s \ge O$

$$\sum_{\substack{n\le x\\ (n,k)=1}} n^s = \frac{x^{s+1}\varphi(k)}{(s+1)k} + O\left(x^{s+\varepsilon}\sigma_{-\varepsilon}(k)\right), \tag{2.9}$$

where σ (k) is the sum of th

*Proof.* By (2.8) we have

$$\begin{aligned}\sum_{\substack{n\le x\\(n,k)=1}} n^s &= \sum_{n\le x} n^s \sum_{\substack{d|(n,k)\\ de=n}} \mu(d)\\ &= \sum_{d|k} d^s\mu(d) \sum_{e\le x/d} e^s\\ &= \sum_{d|k} d^s\mu(d)\left\{\frac{x^{s+1}}{(s+1)d^{s+1}} + O\left(\left(\frac{x}{d}\right)^{s+\varepsilon}\right)\right\}\\ &= \frac{x^{s+1}}{s+1}\sum_{d|k}\frac{\mu(d)}{d} + O\left(x^{s+\varepsilon}\sum_{d|k} d^{-\varepsilon}\right)\\ &= \frac{x^{s+1}}{s+1}\,\frac{\varphi(k)}{k} + O\big(x^{s+\varepsilon}\sigma_{-\varepsilon}(k)\big). \ \square\end{aligned}$$

**Corollary 2.1** ($\varepsilon = O$)

*For* $s \ge O$

$$\sum_{\substack{n\le x\\(n,k)=1}} n^s = \frac{x^{s+1}\varphi(k)}{(s+1)k} + O\big(x^s\tau(k)\big), \tag{2.10}$$

*where* $\tau(k)$ *denotes the number of divisors of* $k$.

The proof of the subsequent lemma follows easily by (2.6).

**Lemma 2.2**

*For each* $s$ *and* $\varepsilon$, $s \ge O$, $\varepsilon > O$, $s+\varepsilon < 1$

$$\sum_{n\le x}\frac{\sigma_{-\varepsilon}(n)}{n^{s+\varepsilon}} = O\big(x^{1-s-\varepsilon}\big). \tag{2.11}$$

Let $J(n)$ denote the Jordan totient function of second order defined by

$$J(n) = n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right) \qquad [4,\ \text{p. } 147].$$

**Lemma 2.3** [2, lemma 5.1]

$$\sum_{\substack{n\leq x\\(n,k)=1}} \frac{\mu(n)}{n^2} = \frac{6k^2}{\pi^2 J(k)} . \tag{2.12}$$

## 3. The function $P^*(n)$

First of all we establish the unitary analogue of formula (1.1).

**Theorem 3.1**

$$P^*(n) = \sum_{d\|n} d\varphi^*\left(\frac{n}{d}\right) . \tag{3.1}$$

*Proof.* Write the set $A = \{1,2,\ldots,n\}$ as $A = \bigcup_{d\|n} A_d$, where $k \in A_d$ if and only if $(k,n)_* = d$, $d\|n$, $d|k$ and the subsets $A_d$ are mutually disjoint. Hence $k = jd$, $1 \leq j \leq n/d$ and $(j,n/d)_* = 1$, that is the subset $A_d$ contains exactly $\varphi^*(n/d)$ elements and we obtain (3.1). □

**Corollary 3.1**

*The function $P^*(n)$ is multiplicative.*

*Proof.* Using the above theorem, $P^*(n)$ is the unitary convolution of two multiplicative functions and it is multiplicative [2, lemma 6.1]. □

In order to obtain an asymptotic formula for the summatory function of $P^*(n)$ we need the following lemmas too. Let

$$\psi(n) = n\prod_{p|n}\left(1+\frac{1}{p}\right)$$

[illegible]

**Lemma 3.1**

*The series*

$$\sum_{n=1}^{\infty} \frac{\varphi(n)\mu^*(n)}{n^2\psi(n)}$$

*is absolutely convergent and its sum is given by*

$$\alpha \equiv \prod_p \left(1 - \frac{1}{(p+1)^2}\right), \tag{3.2}$$

*the product being extended over all primes $p$.*

*Proof.* The absolute convergence of the series follows by

$$\left|\frac{\varphi(n)\mu^*(n)}{n^2\psi(n)}\right| = \frac{n\prod_{p|n}\left(1-\frac{1}{p}\right)}{n^3\prod_{p|n}\left(1+\frac{1}{p}\right)} < \frac{1}{n^2}.$$

The general term is a multiplicative function of $n$, thus the series can be expanded into an infinite product of Euler type [4, § 17.4].

$$\begin{aligned}
\alpha &= \prod_p \left(\sum_{i=0}^{\infty} \frac{\varphi(p^i)\mu^*(p^i)}{p^{2i}\psi(p^i)}\right) \\
&= \prod_p \left(1 - \frac{p-1}{p^2(p+1)} - \frac{p(p-1)}{p^5(p+1)} - \frac{p^2(p-1)}{p^8(p+1)} - \cdots\right) \\
&= \prod_p \left(1 - \frac{p-1}{p^2(p+1)}\left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right)\right) \\
&= \prod_p \left(1 - \frac{p-1}{p^2(p+1)\left(1 - \frac{1}{p^2}\right)}\right) \\
&= \prod_p \left(1 - \frac{1}{(p+1)^2}\right). \ \Box
\end{aligned}$$

**Lemma 3.2.**

$$\sum_{\substack{n\le x\\(n,k)=1}} n\varphi(n) = \frac{2kx^3}{\pi^2\psi(k)} + O\big(x^2\log x\,\tau(k)\big). \tag{3.3}$$

*Proof.* By (2.10) for $s=2$ we have

$$\begin{aligned}
\sum_{\substack{n\le x\\(n,k)=1}} n\varphi(n) &= \sum_{\substack{n\le x\\(n,k)=1}} n \sum_{de=n}\mu(d)e\\
&= \sum_{\substack{de=n\le x\\(n,k)=1}} \mu(d)de^2\\
&= \sum_{\substack{d\le x\\(d,k)=1}} \mu(d)d \sum_{\substack{e\le x/d\\(e,k)=1}} e^2\\
&= \sum_{\substack{d\le x\\(d,k)=1}} \mu(d)d\left\{\frac{\varphi(k)}{3k}\left(\frac{x}{d}\right)^3 + O\left(\left(\frac{x}{d}\right)^2\tau(k)\right)\right\}\\
&= \frac{\varphi(k)x^3}{3k}\sum_{\substack{d\le x\\(d,k)=1}} \frac{\mu(d)}{d^2} + O\left(x^2\tau(k)\sum_{d\le x}\frac{1}{d}\right)\\
&= \frac{\varphi(k)x^3}{3k}\sum_{\substack{d=1\\(d,k)=1}}^{\infty} \frac{\mu(d)}{d^2} + O\left(x^3\sum_{d>x}\frac{1}{d^2}\right) + O\left(x^2\tau(k)\sum_{d\le x}\frac{1}{d}\right).
\end{aligned}$$

Now (3.3) follows by relations (2.12), (2.7) and (2.5). □

The following formula is interesting by itself.

**Lemma 3.3**

$$\sum_{n\le x}\varphi(n)\varphi^*(n) = \frac{2\alpha}{\pi^2}\,x^3 + O\big(x^2\log^3 x\big), \tag{3.4}$$

where $\alpha$ is given by (3.2).

*Proof.* Using (2.2) we deduce

$$\varphi(n)\varphi^*(n) = \varphi(n) \sum_{\substack{de=n\\(d,e)=1}} \mu^*(d)e = \sum_{\substack{de=n\\(d,e)=1}} \varphi(d)\mu^*(d)\varphi(e)e,$$

and by lemma 3.2

$$\begin{aligned}\sum_{n\le x} \varphi(n)\varphi^*(n) &= \sum_{d\le x} \varphi(d)\mu^*(d) \sum_{\substack{e\le x/d\\(e,d)=1}} e\varphi(e)\\ &= \sum_{d\le x} \varphi(d)\mu^*(d) \left\{ \frac{2d}{\pi^2\psi(d)}\left(\frac{x}{d}\right)^3 + O\left(\left(\frac{x}{d}\right)^2 \log\frac{x}{d}\,\tau(d)\right)\right\}\\ &= \frac{2x^3}{\pi^2} \sum_{d\le x} \frac{\varphi(d)\mu^*(d)}{d^2\psi(d)} + O\left(x^2\log x \sum_{d\le x} \frac{\varphi(d)\tau(d)}{d^2}\right)\\ &= \frac{2x^3}{\pi^2} \sum_{d=1}^{\infty} \frac{\varphi(d)\mu^*(d)}{d^2\psi(d)} + O\left(x^3 \sum_{d>x}\frac{1}{d^2}\right) + O\left(x^2\log x\sum_{d\le x}\frac{\tau(d)}{d}\right),\end{aligned}$$

where the main term is $(2\alpha/\pi^2)x^3$ by Lemma 3.1, the first $O$-term is $O(x^2)$ by (2.7) and using that

$$\sum_{n\le x}\frac{\tau(n)}{n} = \sum_{de\le x}\frac{1}{de} \le \left(\sum_{d\le x}\frac{1}{d}\right)^2 = O(\log^2 x)$$

by (2.5), the second $O$-term is $O(x^2\log^3 x)$ which completes the proof. □

By partial summation we immediately have

**Lemma 3.4**

$$\sum_{n\le x}\frac{\varphi(n)\varphi^*(n)}{n^3} = \frac{6\alpha}{\pi^2}\log x + O(1), \tag{3.5}$$

*where $\alpha$ is defined by (3.2).*

**Theorem 3.2**

$$\sum_{n\leq x} P^*(n) = \frac{3\alpha}{\pi^2}\, x^2 \log x + O(x^2). \tag{3.6}$$

*where $\alpha$ is given by (3.2).*

*Proof.* Using (3.1) and lemma 2.2 for $s = 1$ and $O < \varepsilon < 1$

$$\begin{aligned}\sum_{n\leq x} P^*(n) &= \sum_{\substack{de=n\leq x\\(d,e)=1}} \varphi^*(d)e\\ &= \sum_{d\leq x}\varphi^*(d) \sum_{\substack{e\leq x/d\\(e,d)=1}} e\\ &= \sum_{d\leq x}\varphi^*(d)\left\{\frac{\varphi(d)}{2d}\left(\frac{x}{d}\right)^2 + O\left(\left(\frac{x}{d}\right)^{1+\varepsilon}\sigma_{-\varepsilon}(d)\right)\right\}\\ &= \frac{x^2}{2}\sum_{d\leq x}\frac{\varphi(d)\varphi^*(d)}{d^3} + O\left(x^{1+\varepsilon}\sum_{d\leq x}\frac{\sigma_{-\varepsilon}(d)}{d^{\varepsilon}}\right).\end{aligned}$$

Now by lemma 3.4 and (2.11) for $s = 0$ we obtain

$$\begin{aligned}\sum_{n\leq x} P^*(n) &= \frac{x^2}{2}\left(\frac{6\alpha}{\pi^2}\log x + O(1)\right) + O\left(x^{1+\varepsilon}x^{1-\varepsilon}\right)\\ &= \frac{3\alpha}{\pi^2}\, x^2 \log x + O(x^2).\end{aligned}$$

and the proof is complete. □

*Remark 3.1.* By partial summation, theorem 3.2 gives

$$\sum_{n\leq x} \frac{P^*(n)}{n} = \frac{6\alpha}{\pi^2}\, x \log x + O(x). \tag{3.7}$$

Hence $P^*(n)/n$ behaves like $6\alpha \log n/\pi^2$ or for $k \leq n$ the average value of $(k,n)_*$ is $6\alpha \log n/\pi^2$.

*Remark 3.2.* Using formula (2.10) instead of (2.9) the remaining term of (3.6) becomes $O(x^2 \log x)$ and we obtain only the almost trivial $\sum_{n\leq x} P^*(n) = O(x^2 \log x)$.

## 4. The function $P_r^*(n)$

Using the same arguments as in the proofs of theorem 3.1 and corollary 3.1 we have for the function

$$P_r^*(n) = \sum_{k=1}^{n} (k,n)_*^r,$$

where $r$ is an arbitrary real or complex number, the following results.

**Theorem 4.1**

$$P_r^*(n) = \sum_{d\|n} d^r \varphi^* \left(\frac{n}{d}\right) . \tag{4.1}$$

**Corollary 4.1**

*The function $P_r^*(n)$ is multiplicative.*

*Remark 4.1.* Let $f(n)$ be an arbitrary arithmetical function and $P_f^*(n)$ be defined by

$$P_f^*(n) = \sum_{k=1}^{n} f\big((k,n)_*\big).$$

Then, similarly to (4.1), we have

$$P_f^*(n) = \sum_{d\|n} f(d)\varphi^* \left(\frac{n}{d}\right),$$

representing the unitary analogue of Cesàro's formula [4, p. 127]. If $f(n)$ is multiplicative, then the function $P_f^*(n)$ is also multiplicative.

**Lemma 4.1**

*For $r > 1$ the series*

$$\sum_{n=1}^{\infty} \frac{\varphi(n)\varphi^*(n)}{n^{r+2}}$$

*is absolutely convergent and its sum is given by*

$$\alpha_r \equiv \zeta(r)\zeta(r+1)\prod_p \left(1 - \frac{3}{p^{r+1}} + \frac{1}{p^{r+2}} + \frac{1}{p^{2r+1}}\right), \tag{4.2}$$

where $\zeta(r)$ is the Riemann zeta function

*Proof.*

$$\frac{\varphi(n)\varphi^*(n)}{n^{r+2}} \le \frac{1}{n^r}, \qquad r > 1,$$

thus the series is absolutely convergent and its sum can be evaluated by expanding it into an infinite product of Euler type, similarly to the proof of lemma 3.1. □

**Theorem 4.2**

*For $r > 1$*

$$\sum_{n \le x} P_r^*(n) = \frac{\alpha_r}{r+1}\, x^{r+1} + O(A_r(x)), \tag{4.3}$$

*where $\alpha_r$ is given by (4.2) and $A_r(x) = x^r$, $x^2 \log^2 x$, $x^2$ according as $r > 2$, $r = 2$ or $r < 2$.*

*Proof.* Using (4.1) and (2.9) we get

$$\begin{aligned}
\sum_{n \le x} P_r^*(n) &= \sum_{\substack{de = n \le x \\ (d,e)=1}} \varphi^*(d) e^r \\
&= \sum_{d \le x} \varphi^*(d) \sum_{\substack{e \le x/d \\ (d,e)=1}} e^r \\
&= \sum_{d \le x} \varphi^*(d) \left\{ \frac{\varphi(d)}{(r+1)d} \left(\frac{x}{d}\right)^{r+1} + O\left(\left(\frac{x}{d}\right)^{r+\varepsilon} \sigma_{-\varepsilon}(d)\right)\right\} \\
&= \frac{x^{r+1}}{r+1} \sum_{d \le x} \frac{\varphi(d)\varphi^*(d)}{d^{r+2}} + O\left( x^{r+\varepsilon} \sum_{d \le x} \frac{\varphi^*(d)\sigma_{-\varepsilon}(d)}{d^{r+\varepsilon}} \right) \\
&= \frac{x^{r+1}}{r+1} \sum_{d=1}^{\infty} \frac{\varphi(d)\varphi^*(d)}{d^{r+2}} + O\left( x^{r+1} \sum_{d > x} \frac{1}{d^r} \right) + O\left( x^{r+\varepsilon} \sum_{d \le x} \frac{\sigma_{-\varepsilon}(d)}{d^{r-1+\varepsilon}} \right),
\end{aligned}$$

where the main term is $(\alpha_r/r+1)x^{r+1}$ by lemma 4.1, the first remaining term is $O(x^2)$ by (2.7) and for the second remaining term we have: for $r > 2$ choose $\varepsilon = 0$ and obtain

$$O\left( x^r \sum_{d \le x} \frac{\tau(d)}{d^{r-1}} \right) = O(x^r);$$

for $r = 2$ and $\varepsilon = 0$ again it is

$$O\left( x^2 \sum \frac{\tau(d)}{d} \right) = O(x^2 \log^2 x);$$

for $1 < r < 2$ choose $\varepsilon > 0$, $r-1+\varepsilon < 1$ and get $O(x^{r+\varepsilon}x^{1-r+1-\varepsilon}) = O(x^2)$ by lemma 2.2 ($s = r-1$) and the proof is complete. □

*Remark 4.2.* An asymptotic formula for the sum $\sum_{n\le x} P_r(n)$, where

$$P_r(n) = \sum_{k=1}^{n} (k,n)^r, \qquad r > 1$$

has been obtained by Alladi [1], see also [9].

## References


1. K. Alladi, On generalized Euler functions and related totients, in *New Concepts in Arithmetic Functions*, Matscience Report 83, Madras, 1975.
2. E. Cohen, Arithmetical functions associated with the unitary divisors of an integer, *Math. Z.* 74 (1960), 66–80.
3. E. Cohen, Remark on a set of integers, *Acta Sci. Math. Szeged* **25** (1964), 179–180.
4. L. E. Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952.
5. K. Nageswara Rao, On the unitary analogues of certain totients, *Monatsh. Math.* **70** (1966), 149–154.
6. I. Niven and H. Zuckerman, *An Introduction to the Theory of Numbers*, 4th edition, Wiley, New York, 1980.
7. S. S. Pillai, On an arithmetic function, *J. Annamalai Univ.* 2 (1933), 243–248.
8. R. Sivaramakrishnan, On the three extensions of Pillai's arithmetic function $\beta(n)$, *Math. Student* **39** (1971), 187–190.
9. E. Teuffel, Aufgabe 599, *Elem. Math.* **25** (1970), 65.