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Containing ℓ_1 or c_0 and best approximation

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Abstract

The purpose of this paper is to obtain sufficient conditions, for a Banach space X to contain or exclude c_0 or ℓ_1, in terms of the sets of best approximants in X for the elements in the bidual space.

Let X be a closed subspace of a Banach space Y. If $y \in Y$, we denote by $P_X(y)$, the set of best approximants of y in X, that is,

$$P_X(y) = \{x \in X : \|y - x\| = \|y + X\|\}.$$

The set $P_X(y)$ is closed, convex and bounded and it may be empty. If it contains exactly (at least) one element, for every y in Y, then X is said to be a Chebyshev (proximinal) subspace of Y.

The closed ball with center x and radius r is denoted by $B_X(x, r)$. The closure of a set A is denoted $\bar{A}$ and its interior by A^i. X° will denote the annihilator of X in Y^*. Let us say that X has the $1\frac{1}{2}$-ball property in Y if X is a proximinal subspace of Y and the following equality holds for all y in Y,

$$\|y\| = d(0, P_X(y)) + \|y + X\|,$$

where $d(0, P_X(y))$ denotes the distance from the origin to $P_X(y)$. (This is equivalent [7, Cor. 4] to the definition given by D. Yost in [13]). It is known that the Banach spaces ℓ_1 and c_0 has the $1\frac{1}{2}$-ball property in their biduals [13, Lemma 2.6; 14, Th. 4,5,1 and Lemma 9]. However, whereas ℓ_1 is a Chebyshev subspace in its bidual, for $X = c_0$ and for for every F in $X^{**} = \ell_\infty$ we have

$$B_X^i(0, 2\|F + X\|) \subset P_X(F) - P_X(F).$$

The assertions are, in a certain sense, opposite; observe that

$$0 \in P_X(F) - P_X(F) \subset B_X(0, 2\|F + X\|),$$

for any Banach space X which is proximinal in its bidual and for all F in X^{**}.

Given $0 \leq t \leq 2$, we say that a (automatically closed) proximinal subspace X of a Banach space Y has the I(t)-property in Y, if for every $y \in Y \backslash X$, we have

$$B_X^i(0, t\|y + X\|) \subset P_X(y) - P_X(y).$$

P. Harmand and A. Lima [8. Th. 3.5] proved that if X has the I(2)-property in its bidual (this is equivalent, [9, Th. 1.2 and Prop. 1.5], to the fact that X is an M-ideal in its bidual), then it contains a subspace isomorphic to c_0. Moreover, A. Lima showed in [9, Th. 2.6] that such a space X is an Asplund space, so X does not contain ℓ_1.

On the other hand, G. Godefroy proved in [4, Th. 4] that if X is an L-summand in its bidual (that is, there exists a closed subspace Y in X^{**}, such that $X^{**} = j_X(X) + Y$ and

$$\|j_X(x) + y\| = \|x\| + \|y\|,$$

for $x \in X$ and $y \in Y$) then X is w.s.c.. In particular, X has a subspace isomorphic to ℓ_1, and no subspace of X is isomorphic to c_0 (see [5] for more information about the geometry of the Banach spaces containing or not ℓ_1).

These observations allow us to glimpse that there is some relation between the fact that a (non-reflexive) Banach space contains subspaces isomorphic to ℓ_1 or c_0 and the "size" of the difference sets of the sets of best approximants of the elements of the bidual in the space. The object of this paper is to illuminate this relation.

Given a bounded subset K of a Banach space X, we write

$$\|K\| := \sup\{\|x\| : x \in K\}$$

while $\delta(K)$ will denote the diameter of K.

Now we recall the definition of the property (u), which was introduced by A. Pelczynski in [11]: A Banach space X has the property (u) if for every F in the w*-sequential closure of X in X^{**}, there exists a W.C.U.-series $\sum y_k$ (that is for every f in X^*, $\sum f(y_k)$ converges) such that F is the w*-limit of the sequence $\{\sum_{k=1}^n y_k\}$.

More concretely, we prove the following result:

Theorem

Let X be a non-reflexive Banach space.

(1) If X has the I(t)-property in its bidual, then,

a) if $t > 0$, no subspace of X is isomorphic to ℓ_1 ,

b) if $t > 1$ X has the property (u). In particular X has a subspace isomorphic to c_0 .

(2) If X has the $1\frac{1}{2}$-ball property in its bidual, then

a) if $t < 2$ and $\delta(P_X F)) \leq t\|F + X\|$ for every F in X^{**}, no subspace of X is isomorphic to c_0,

b) if $t < 1$ and $F \in X^{**}\backslash X$ is such that $\delta(P_X(F)) \leq t\|F + X\|$, F is not of first Baire class. In particular if X does not contain ℓ_1, then $\delta(P_X(F)) \geq \|F + X\|$ for every $F \in X^{**}\backslash X$.

The proof of the first lemma is inspired in the one of [13, Th. 4].

Lemma 1

Let X be a subspace of a Banach space Y and $0 \leq t \leq 2$. If X has the I(t)-property in Y, then, for every f in Y^*, we have

$$\|P_{X^\circ}(f)\| \leq \|f\| + (1 - t)\|f + X\|.$$

Proof. We claim that X has the I(t)-property in Y if, and only if, X is a proximinal subspace in Y and the following assertion holds for all x in X and y in Y with $\|x\| < \|y + X\|$,

$$B_Y(y + (t/2)x, \|y + X\|) \cap B_Y(y - (t/2)x, \|y + X\|) \cap X \neq \emptyset.$$

Indeed, if $t = 0$, both assertions are equivalent to the proximinality. Suppose then that $t > 0$. If X has the I(t)-property in Y and $x \in X$, $y \in Y$ with $\|x\| < \|y + X\|$, then $tx = z - z'$, with z and z' in $P_X(y)$. So

$$\left\|y + \frac{tx - z - z'}{2}\right\| = \|y - z'\| = \|y + X\| = \|y - z\| = \left\|y - \frac{tx + z + z'}{2}\right\|,$$

hence $(z + z')/2$ is the desired element in the above intersection.

Conversely, fix $y \in Y$, $x \in X$, such that $\|x\| < t\|y + X\|$. By assumption there exists z in the intersection

$$B_Y\left(y + \frac{x}{2}, \|y + X\|\right) \cap B_Y\left(y - \frac{x}{2}, \|y + X\|\right) \cap X$$

and so, $x = x/2 + z - (z - x/2) \in P_X(y) - P_X(y)$, and this proves the claim.

Now, fix x in X, y in Y, such that $\|x\| < 1 = \|y + X\|$. By the claim, there exists z in X, satisfying

$$\left\| y \pm \frac{t}{2} x - z \right\| \leq 1.$$

If $f \in Y^*$ and we take g in $P_{X^0}(f)$, then

$$2g(y) + t(f-g)(x) = f\left(y + \frac{t}{2}x - z\right) + (2g - f)\left(y - \frac{t}{2}x - z\right)$$

so

$$2\Re g(y) + t\Re(f-g)(x) \leq \|f\| + \|2g - f\|.$$

According to the identification of X^0 with $(Y/X)*$ and since

$$\|g - f\| = \|(g-f)/X\|,$$

we have

$$2\|g\| \leq \|f\| + (1-t)\|f - g\| + \|g\|,$$

that is

$$\|g\| \leq \|f\| + (1-t)\|f + X\|$$

for all f in Y^* and g in $P_{X^0}(f)$, as required. □

Remarks. Let X be a subspace of a Banach space Y with the I(t)-property in Y. It is clear that if, for f in Y^*, f^1 and f^2 are two Hahn-Banach extensions of $f_{|X}$, then $f^1 - f^2 = g - h$, with g, h in $P_{X^0}(f)$. (In particular $g - h \in P_{X^0}(f - h)$). Hence, applying the above lemma we have

$$\|f^1 - f^2\| \leq (2-t)\|f + X\|.$$

If $t = 2$, then we obtain that X has the unique Hahn-Banach extension property in Y, which is proved in [14, Th. 4]. On the other hand, it is easy to prove that under the additional assumption that X has the $1\frac{1}{2}$-ball property in Y, the inequality in the lemma actually is equivalent to the I(t)-property.

In what follows, for a prefixed Banach space X, π will denote the projection on X^{***} with range X^* and kernel X° (the annihilator of X in X^{***}). If we fix τ in X^{***} and w in X°, it is clear that

$$\|\tau - (\tau - \pi\tau)\| = \|\pi\tau\| = \|\pi(\tau - w)\| \leq \|\tau - w\|$$

so, $\tau - \pi \in P_{X^\circ} f(\tau)$. On the other hand, if we assume that X has the I(t)-property in X^{**}, by lemma 1, we have

$$\|\tau - \pi\tau\| \leq \|\tau\| + (1-t)\|\pi\tau\|. \tag{+}$$

In the next result, which is an obvious adaptation of the proof of [6, Th. 1], we emphasize the fact that $t > 1$.

Lemma 2

Let X be a Banach space, F in X^{**} and $0 < \varepsilon \leq 1$. If

$$\|\tau - \pi\tau\| \leq \|\tau\| - \varepsilon\|\pi\tau\|,$$

for all τ in X^{***}, then F/B_{X^*} is the difference of two bounded lower semicontinuous functions on (B_{X^*}, w^*). In particular X has the property (u).

Proof. First of all, we claim that

$$B_{X^{***}}(0,1) \subset \text{co}\left(\frac{1}{\varepsilon} B_{X^\circ}(0,1) \cup B_{X^\circ}(0,1)\right),$$

indeed, for each $\tau \in B_{X^{***}}(0,1)\backslash X$, writing

$$\alpha := \varepsilon\,\|\pi\tau\|,$$

$$u := \left(\varepsilon\,\|\pi\tau\|\right)^{-1}\pi\tau$$

and

$$v := \left(1 - \varepsilon\,\|\pi\tau\|\right)^{-1}\left(\tau\|\tau - \pi\tau\|\right),$$

one has $\tau = \alpha u + (1-\alpha)v$ and, by the assumed inequality $\|v\| \leq 1$, as required. Now, it is enough to observe that in the proof of [6, Lemma 2], the assumption of that X is an M-ideal in X^{**} is only used to imply the statement (2), but this statement is not a necessary condition for its proof, indeed, this condition can be relaxed by only assuming the condition (+). Hence, the first statement of our lemma 2 is consequence of [6, Lemma 3], which is an independent (with the condition of M-ideal) topological argument. On the other hand, the second one follows by repeating the proof from Godefroy and Li [6, pp. 366–367] word for word. □

Proof of first part of the theorem. If we assume that X has the I(t)-property, by lemma 1 and the observation (+), we have that

$$\|\tau - \pi\tau\| \leq (2-t)\|\tau\|$$

for all τ in X^{***}, hence $\|1-\pi\| \leq 2-t$. Therefore, according to [2, Prop. 2], if $t>0$, X contains no subspace isomorphic to ℓ_1. The first one of b) follows from the condition (+) and the lemma 2. The second one follows from a) and a classical result of Pelczynski [12, Prop. 2]. □

For the assertion 2) we will use the following result, which is classical.

Lemma 3

Let X be a Banach space, Z be a subspace of X and $G \in Z^{**}$. Then

$$\|G+Z\| \leq 2\|G+X\| \leq 2\|G+Z\|.$$

Proof. Let x be in X. By the Hahn-Banach theorem it is easily seen, that $\|x+Z\| = \|x+Z^{**}\|$. Let G be in Z^{**}. We put $a = \|G+X\|$. Let $\beta > 0$ be given. Then there exists $x_\beta \in X$ such that $\|x_\beta - G\| < a+\beta$. By the above remark there exists $z_\beta \in Z$ such that $\|x_\beta - z_\beta\| < a+\beta$. Therefore: $\|G - z_\beta\| < 2a + 2\beta$ and the lemma is proven. □

Proof of the part (2) of the theorem. 2)-a) Assume that X has a subspace isomorphic to c_0 and fix $\varepsilon > 0$. According to [10, Prop. 2e3], there exists a subspace Z of X, and a linear isomorphism T from c_0 to Z such that

$$\|z\| \leq \|T(z)\| \leq (1+\varepsilon)\|z\|$$

for all $z \in c_0$. On the other hand, putting $F = (1,1,\ldots,1,\ldots)$ in ℓ_∞ and $z = (2,0,\ldots,0,\ldots)$ in c_0 we have that

$$\|f\| = \|F + c_0\| = 1 = \|F - z\| \leq \frac{\|z\|}{2}.$$

Therefore, for every x, y in X, we have

$$2 = \|z\| \leq \|T(z)\| \leq \|T(z) - x\| + \|x - y\| + \|y\|.$$

Now, if we note $G = T^{**}(F)$ (the norm of $G - T(z)$ and G are less than or equal to $1+\varepsilon$) and if we take infimum in x, y, with $x, y \in P_X(G)$, we obtain

$$2 \leq d\big(T(z), P_X(G)\big) + d\big(0, P_X(G)\big) + k\big(P_X(G)\big)$$

and by the $1\frac{1}{2}$-ball property of X and the bound on the diameter

$$\begin{aligned} 2 &\leq \|G - T(z)\| - \|G + X\| + \|G\| - \|G + X\| + t\,\|G + X\| \\ &\leq 2(1+\varepsilon) + (t-2)\|G + X\|, \end{aligned}$$

that is

$$(2-t)\|G + X\| \leq 2\varepsilon$$

Therefore, according to lemma 3 and the inequality

$$\|G + Z\| \geq \|F + c_0\| = 1,$$

we have

$$2 - t \leq 4\varepsilon,$$

in contradiction with the choice of t and the arbitrarity of ε.

(2)-b) Assume $t < 1$ and take F in X^{**} such that

$$\delta(P_X(F)) \leq t\,\|F + X\|.$$

It is clear that

$$\big\|P_X(F)\big\| \leq \delta\big(P_X(F)\big) + d\big(0, P_X(F)\big) \leq t\|F + X\| + d\big(0, P_X(F)\big).$$

By assumption, X has the $1\frac{1}{2}$-ball property in X^{**} and $t < 1$, so

$$\|P_X(F)\| \leq \|F\| + (t-1)\|F + X\| \leq F\|.$$

Now, fix $z \in P_X(F)$. It is clear that

$$B_X\big(z, (1-t)\|F + X\|\big) \subset \bigcap_{x \in X} B_X\big(x, \|F - x\|\big).$$

Indeed, if $u \in B_X(z, (1-t)\|F + X\|)$, since $z - x \in P_X(F - x)$

$$\begin{aligned} \|u - x\| &\leq \|u - z\| + \|z - x\| \\ &\leq (1-t)\|F + X\| + \|F - x\| + (t-1)\|F + X\| \\ &= \|F - x\|. \end{aligned}$$

Hence

$$F \notin \Big\{F \in X^{**} : \bigcap_{x \in X} B_X(x, \|F - x\|) \text{ contains at most one element}\Big\},$$

and therefore, by [4, Prop. 3 and rem.], F is not of first Baire class. $\square$

Remarks and questions. It is known [13, Cor. 2.3] that the real Banach space $C(K)$, K compact, has the $1\frac{1}{2}$-ball property in its bidual, so our theorem gives information about the set of best approximants. Indeed, with the same arguments of the theorem, it is easy to prove that there exist F in X^{**}, $X = C([0,1])$, and x, y in $P_X(F)$, such that $\|x - y\| = 2\,\|F + X\|$.

There are Banach spaces that have the I(t)-property in their biduals which are not M-ideals in their biduals. For example, let us denote X_α the space $c_0 \times c_0$ normed by

$$\|(x,y)\|_\alpha = \max\left\{\|x\|, \|y\| + \alpha\,\|x\|, (1+\alpha)\|y\|\right\}$$

If $0 < \alpha \leq 1$, it can be verified that X_α has the I($2 - 2\alpha$)-property in its bidual but is not an M-ideal in its bidual. Of course, the assertion (1) of the theorem is trivial for these examples ($0 \leq \alpha \leq 1$) (this class of Banach space satisfies a lot of the geometric properties of the Banach spaces which are M-ideals in their biduals [1]). However [3] a suitable iteration in the above renorming ($c_0(X_{\alpha_n})$ for an appropiate sequence $\{\alpha_n\}$) allow to build a dense subspace Z of c_0 and a new norm in Z, $\|\cdot\|$, such that $[Z, \|\cdot\|]$ has the I(t)-property (for some $t > 1$). But, unfortunately we do not know if this space is isomorphic to a Banach space Y, which is an M-ideal in its bidual.

Analogously, there are Banach spaces (a suitable product of a Banach space, which is an L-summand in its bidual, and a reflexive Banach space) that verify the hypothesis 2-a) (it is easy to prove, with the same arguments of the theorem, that in such spaces there are no Fréchet-differentiable points of the norm in X) or 2-b).

Hence, it is natural to ask:

Does exist a Banach space which has the I(t)-property in its bidual (satisfies the hypothesis 2-a) or 2-b)) and which is not isomorphic to an M-ideal in its bidual (to a suitable product as above)?

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