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The approximation property of order (p,q) in Banach spaces

J. C. DÍAZ

E. T. S. Ingenieros Agrónomos, Finca Alameda del Obispo, 14004 Córdoba, Spain

J. A. LÓPEZ MOLINA AND M. J. RIVERA(*)

E. T. S. Ingenieros Agrónomos, Camino de Vera, 46022 Valencia, Spain

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ABSTRACT

We introduce the approximation property of order (p, q) in Banach spaces (in short AP_{pq}) to study topological properties of the space $D_{q'p'}(E, F)$ of (q', p')-dominated operators between the Banach spaces E and F. After some equivalent formulations of the AP_{pq}, we characterize the reflexivity of $D_{q'p'}(E, F)$ when E has the AP_{qp} or F' has the AP_{pq} and we give sufficient conditions for $E\hat{\otimes}_{\alpha_{pq}}F$ and $E\hat{\otimes}_{\alpha'_{pq}}F$ to be weakly sequentially complete.

Introduction

In 1955, Grothendieck [5] introduced the notion of approximation property (in short AP) for Banach spaces. Twenty years later Saphar introduced a weakened version of it, namely the approximation property of order $p \geq 1$ (in short AP_p). All Banach

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spaces have the AP_2. The first examples of Banach spaces without AP_p for $p \neq 2$, and of Banach spaces with AP_p for $p \neq 2$ but without AP are due to Reinov [15]. These properties are closely related to the tensor topologies g_p of Saphar [17] and they are useful to study topological properties of the space $\prod_p(E, F)$ of p-absolutely summing maps from the Banach space E into the Banach space F [18]. On the other hand the tensor topology g_p is a particular case of the tensor topologies α_{pq} of Lapresté [11, 2, 3]. Then it is natural to consider approximation properties of order (p, q) in relation with the topology α_{pq}. This new property will provide the natural setting to investigate topological properties of the space $D_{q'p'}(E, F)$ of (q', p')-dominated operators between the Banach spaces E and F.

In section 1, we introduce the approximation property of order (p, q) (in short AP_{pq}) of Banach spaces and we give some characterizations of it. In section 2 we provide applications to the space $D_{q'p'}(E, F)$. Then we characterize the reflexivity of $D_{q'p'}(E, F)$ if E has the AP_{pq} or F' has the AP_{pq}, giving applications to the reflexivity of $\ell^u \hat{\otimes}_{\alpha_{pq}} \ell^v$, and we study sufficient conditions for the weak sequential completeness of $E \hat{\otimes}_{\alpha'_{pq}} F$ and $E \hat{\otimes}_{\alpha_{pq}} F$.

1. Terminology and notations

The notation about real or complex Banach and locally convex spaces is standard [10]. Throughout the paper, E and F will be always real or complex Banach spaces. B_E will be the closed unit ball of E, E' will be the dual Banach space of E and I_E and J_E will be, respectively, the identity map on E and the canonical inclusion of E into its bidual E''. If $T \in \mathcal{L}(E, F)$, the transposed map will be denoted by T'. If $p \in [1, \infty]$, p' will be the conjugate number given by $1/p + 1/p' = 1$, where $1/\infty = 0$ as usual.

We follow the book of Piestch [14] for the definitions of operator ideals and of the classes of operators from E into F. Concerning general definitions and properties of tensor norms, we refer the reader to [2, 3, 6, 7]. Given $p \in [1, \infty]$ and a sequence (x_i) (finite or not) of elements of E, we put

$$\varepsilon_p((x_i)) = \sup_{x' \in B_{E'}} \left(\sum_i |\langle x_i, x' \rangle|^p \right)^{1/p},$$

$$\pi_p((x_i)) = \left(\sum_i \|x_i\|^p \right)^{1/p}$$

for $p<\infty$, and

$$\varepsilon_\infty\big((x_i)\big)=\pi_\infty\big((x_i)\big)=\sup_i\|x_i\|.$$

Then we define

$$\ell^p(E)=\big\{(x_i)\in E^{\mathbb{N}}:\varepsilon_p\big((x_i)\big)<\infty\big\}$$

and

$$\ell^p[E]=\big\{(x_i)\in E^{\mathbb{N}}:\pi_p\big((x_i)\big)<\infty\big\}.$$

If $(\lambda_i)\in\mathbb{K}^{\mathbb{N}}$ and $p\in[1,\infty]$, we put

$$\pi_p\big((\lambda_i)_{i=n}^m\big)=\pi_p\big((\lambda_i')\big)$$

where $\lambda_i'=\lambda_i$ if $i=n,n+1,\dots,m$ and $\lambda_i'=0$ in other case.

Given a pair of numbers $p,q\in[1,\infty]$, we shall always consider a number $r\in[1,\infty]$ such that

$$1/r+1/q'+1/p'=1. \tag{1}$$

Then, the tensor topology α_{pq} of Lapreste on the tensor product $E\otimes F$ is defined by the norm

$$\alpha_{pq}(u)=\inf\pi_r\big((\lambda_i)\big)\varepsilon_{q'}\big((x_i)\big)\varepsilon_{p'}\big((y_i)\big),\qquad u\in E\otimes F$$

where the inf is taken over all the representations of u in the form

$$u=\sum_{i=1}^n\lambda_ix_i\otimes y_i,\qquad \lambda_i\in\mathbb{K},\ x_i\in E,\ y_i\in F,\ i=1,\dots,n.$$

The space $E\otimes F$ with this norm is denoted by $E\otimes_{\alpha_{pq}}F$ and its completion by $E\hat{\otimes}_{\alpha_{pq}}F$. By [11], (see also [3]), every element of $E\hat{\otimes}_{\alpha_{pq}}F$ can be written in the form

$$z=\sum_{i=1}^\infty\lambda_ix_i\otimes y_i \tag{2}$$

with $(\lambda_i)\in\ell^r$ ($(\lambda_i)\in c_0$ if $r=\infty$), $(x_i)\in\ell^{q'}(E)$ and $(y_i)\in\ell^{p'}(F)$.

There are natural continuous linear maps

$$\Phi_{pq}:E\hat{\otimes}_{\alpha_{pq}}F\longrightarrow\mathcal{L}(E',F)$$

$$\Psi_{pq}:E'\hat{\otimes}_{\alpha_{pq}}F\longrightarrow\mathcal{L}(E,F)$$

such that, if z is given by (2), for every $x' \in E'$ and $y' \in F'$

$$\langle \Phi_{pq}(z)(x'), y' \rangle = \sum_{i=1}^{\infty} \lambda_i \langle x_i, x' \rangle \langle y_i, y' \rangle$$

and analogously for Ψ_{pq}. The set $\Psi_{pq}(E' \hat{\otimes}_{\alpha_{pq}} F)$ is the set $N_{pq}(E, F)$ of the (p, q)-compact operators from E into F, which will be always provided with the quotient norm.

A map T from E into F is called (p, q)-dominated if there is $M \rangle 0$ such that

$$\pi_{r'}\big(\langle T(x_i), y_i' \rangle\big) \leq M\, \varepsilon_p\big((x_i)\big)\, \varepsilon_q\big((y_i')\big) \tag{3}$$

for every $(x_i) \in \ell^p(E)$ and every $(y_i') \in \ell^q(F')$, where $r, p, q \in [1, \infty]$ are such that $1/r + 1/p + 1/q = 1$. The set $D_{pq}(E, F)$ of the (p, q)-dominated operators is normed by $\pi_{pq}(T) = \inf M$, where the inf is taken over all M such that (3) holds. It is known [11, 2, 3] that

$$\big(E \hat{\otimes}_{\alpha_{pq}} F\big)' = D_{q'p'}(E, F').$$

If $q = 1$, $\alpha_{pq} = \alpha_{p1}$ is the topology g_p of Saphar and

$$\big(E \hat{\otimes}_{g_p} F\big)' = \prod_{p'}(F, E'),$$

the space of p'-absolutely summing operators from F into E' [17]. The norm of $T \in \prod_{p'}(E, F)$ will be denoted by $\pi_{p'}(T)$.

The dual tensor norm of α_{pq} will be denoted by α'_{pq}. The dual space $(E \hat{\otimes}_{\alpha'_{pq}} F)'$ is the space $I_{pq}(E, F')$ of (p, q)-factorable operators from E into F' [2, 3]. The norm of $T \in I_{pq}(E, F')$ will be denoted by $I_{pq}(T)$.

Finally, we refer the reader to [4] for questions related to the Radon-Nikodym properties of a Banach space, and to [19, 20] for Schauder bases and decompositions.

2. The approximation property of order (p,q)

Given E and F and numbers p, q which define a tensor norm α_{pq}, besides the normed topology, we shall consider on $D_{q'p'}(E, F)$ two other useful Hausdorff topologies. The first one is the topology σ_0 induced by the weak topology $\sigma\big(D_{q'p'}(E, F''), E \hat{\otimes}_{\alpha_{pq}} F'\big)$. The second one is the topology $T_{q'p'}$ defined by the family of seminorms

$$\big\{P_{(x_i),(y_i')} : (x_i) \in \ell^{q'}(E),\ (y_i') \in \ell^{p'}(F')\big\}$$

where

$$P_{(x_i),(y_i')}(T) = \pi_{r'}\big((\langle T(x_i), y_i' \rangle)\big)$$

for every $T \in D_{q'p'}(E, F)$. It is easy to show that $T_{q'p'}$ is finer than σ_0 and coarser than the norm topology on $D_{q'p'}(E, F)$. We have

Proposition 1

Let $1 \leq p,q < \infty$ *and let* r *be defined by (1). If* $1 < r < \infty$, *the topological dual of* $[D_{q'p'}(E,F), T_{q'p'}]$ *is the quotient of* $E\hat{\otimes}_{\alpha_{pq}}F'$ *by the absolute polar of* $D_{q'p'}(E,F) \subset (E\hat{\otimes}_{\alpha_{pq}}F')'$.

Proof. We take the orthogonal set H of $D_{q'p'}(E,F)$ in $E\hat{\otimes}_{\alpha_{pq}}F'$, which is closed since $D_{q'p'}(E,F)$ can be looked as a subset of $\big(E\hat{\otimes}_{\alpha_{pq}}F'\big)'$. We form the quotient

$$Q = \big(E\hat{\otimes}_{\alpha_{pq}}F'\big)/H.$$

Let P be the canonical projection from $E\hat{\otimes}_{\alpha_{pq}}F'$ onto Q.

Clearly $\sigma(D_{q'p'}(E,F),Q)$ is coarser than $T_{q'p'}$. On the other hand, given $(x_i) \in \ell^{q'}(E)$ and $(y_i') \in \ell^{p'}(F')$, the map A from ℓ^r into $E\hat{\otimes}_{\alpha_{pq}}F'$ such that

$$A\big((\lambda_i)\big) = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i', \qquad (\lambda_i) \in \ell^r,$$

is continuous and hence weakly continuous. Then if B_r is the closed unit ball of ℓ^r, $A(B_r)$ is $\sigma\big(E\hat{\otimes}_{\alpha_{pq}}F', D_{q'p'}(E,F'')\big)$-compact and $P\big(A(B_r)\big)$ is $\sigma(Q, D_{q'p'}(E,F))$-compact. Then for every $T \in D_{q'p'}(E,F))$

$$\begin{aligned} P_{(x_i),(y_i')}(T) &= \pi_{r'}\left(\left(\langle T(x_i), y_i'\rangle\right)\right) \\ &= \sup\left\{\left|\sum_{i=1}^{\infty} \lambda_i \langle T(x_i), y_i'\rangle\right| : (\lambda_i) \in B_r\right\} \\ &= \sup\left\{|\langle T, w\rangle| : w \in P\big(A(B_r)\big)\right\} \end{aligned}$$

and $T_{q'p'}$ is coarser than the Mackey topology $\tau\big(D_{q'p'}(E,F'),Q\big)$. Then

$$\big[D_{q'p'}(E,F), T_{q'p'}\big]' = Q$$

as sets. □

Corollary 1

Let $1 \leq p,q < \infty$ *and let* r *be defined by (1). If* $1 < r < \infty$ *and* F *is reflexive, for every* E *we have*

$$\big[D_{q'p'}(E,F'), T_{q'p'}\big]' = E\hat{\otimes}_{\alpha_{pq}}F.$$

Theorem 1

Let $1 \leq p, q < \infty$ and let r be defined by (1). If $1 < r < \infty$, for a Banach space E the following assertions are equivalent:

1) *For every F, the canonical map*

$$\Psi_{pq} : F'\hat{\otimes}_{\alpha_{pq}} E \longrightarrow \mathcal{L}(F, E)$$

is one to one.

2) *For every F, the canonical map*

$$\Phi_{pq} : F\hat{\otimes}_{\alpha_{pq}} E \longrightarrow \mathcal{L}(F', E)$$

is one to one.

3) *For every F, $E' \otimes F$ is σ_0-dense in $D_{p'q'}(E, F)$.*
4) *For every F, $E' \otimes F$ is $T_{p'q'}$-dense in $D_{p'q'}(E, F)$.*

If $r = \infty$, 1), 2) and 3) are equivalent and 4) implies any one of them.

Proof. 1) $\Longrightarrow$ 2) If

$$\Phi_{pq}\left(\sum_{i=1}^{\infty} \lambda_i y_i \otimes x_i\right) = 0 \in \mathcal{L}(F', E),$$

with $(\lambda_i) \in \ell^r$ ($(\lambda_i) \in c_0$ if $r = \infty$), $(y_i) \in \ell^{q'}(F)$, $(x_i) \in \ell^{p'}(E)$, since F' is complemented in F''', we have

$$\Psi_{pq}\left(\sum_{i=1}^{\infty} \lambda_i J_F(y_i) \otimes x_i\right) = 0 \in \mathcal{L}(F''', E)$$

and by 1)

$$0 = \sum_{i=1}^{\infty} \lambda_i J_F(y_i) \otimes x_i \in F''\hat{\otimes}_{\alpha_{pq}} E.$$

But the isometry [7, satz 1.12; 3, 2.4]

$$J_F \otimes I_E : F \otimes_{\alpha_{pq}} E \longrightarrow F'' \otimes_{\alpha_{pq}} E$$

can be isometrically extended to the respective completions. Then

$$\sum_{i=1}^{\infty} \lambda_i y_i \otimes x_i = 0.$$

2) $\Longrightarrow$3) Given F, by 2) the map

$$\Phi_{pq} : F'\hat{\otimes}_{\alpha_{pq}} E \longrightarrow F'\hat{\otimes}_{\varepsilon} E \subset \mathcal{L}(F'', E),$$

is injective. Let $z \in F'\hat{\otimes}_{\alpha_{pq}} E$ be such that

$$\langle z, x' \otimes y'' \rangle = 0, \qquad x' \otimes y'' \in E' \otimes F'' \subset D_{p'q'}(E, F'') = \big(F'\hat{\otimes}_{\alpha_{pq}} E\big)'.$$

Then, for every $T \in (F'\hat{\otimes}_{\varepsilon} E)'$ we have

$$\begin{aligned}
\big|\langle \Phi_{pq}(z), T\rangle\big| &\le \|T\|\, \varepsilon\big(\Phi_{pq}(z)\big) \\
&= \|T\| \sup\big\{\big|\langle \Phi_{pq}(z), v^{\circ\circ} \otimes u^{\circ}\rangle\big| : v^{\circ\circ} \in V^{\circ\circ},\ u^{\circ} \in U^{\circ}\big\} \\
&= \|T\| \sup\big\{\big|\langle z, \Phi'_{pq}(v^{\circ\circ} \otimes u^{\circ})\rangle\big| : v^{\circ\circ} \in V^{\circ\circ},\ u^{\circ} \in U^{\circ}\big\} \\
&= \|T\| \sup\big\{\big|\langle z, u^{\circ} \otimes v^{\circ\circ}\rangle\big| : v^{\circ\circ} \in V^{\circ\circ},\ u^{\circ} \in U^{\circ}\big\} \\
&= 0.
\end{aligned}$$

Then $\Phi_{pq}(z) = 0$ and hence $z = 0$. Therefore, $E' \otimes F''$ is $\sigma\big((F'\hat{\otimes}_{\alpha_{pq}} E)', F'\hat{\otimes}_{\alpha_{pq}} E\big)$ dense in $(F'\hat{\otimes}_{\alpha_{pq}} E)' = D_{p'q'}(E, F'')$. But $E' \otimes F$ is $\sigma\big((F'\hat{\otimes}_{\alpha_{pq}} E)', F'\hat{\otimes}_{\alpha_{pq}} E\big)$ dense in $E' \otimes F''$ by the representation with series of the elements $z \in F'\hat{\otimes}_{\alpha_{pq}} E$ and the theorem of bipolars. Hence, $E' \otimes F$ is σ_0-dense in $D_{p'q'}(E, F) \subset D_{p'q'}(E, F'')$.

3) $\Longrightarrow$ 4) (If $1 < r < \infty$). Let

$$Q = \big[D_{p'q'}(E, F), T_{p'q'}\big]'.$$

By proposition 1 and the definition of σ_0 we obtain that $E' \otimes F$ is $\sigma(D_{p'q'}(E, F), Q)$-dense in $D_{p'q'}(E, F)$. Then $E' \otimes F$ is $T_{p'q'}$-dense in $D_{p'q'}(E, F)$.

4) $\Longrightarrow$3) It is inmediate since $T_{p'q'}$ is finer than σ_0.

3) $\Longrightarrow$1) Given F, every $z \in F'\hat{\otimes}_{\alpha_{pq}} E$ has a representation

$$z = \sum_{i=1}^{\infty} \lambda_i y'_i \otimes x_i$$

with $(\lambda_i) \in \ell^r\big((\lambda_i) \in c_0 \text{ if } r = \infty\big)$, $(y'_i) \in \ell^{q'}(F')$ and $(x_i) \in \ell^{p'}(E)$. Then, for every $x' \in E'$,

$$\begin{aligned}
\left\|\sum_{i=n}^{m} \lambda_i \langle x_i, x'\rangle y'_i\right\| &= \sup\left\{\left|\sum_{i=n}^{m} \lambda_i \langle x_i, x'\rangle \langle y'_i, y''\rangle\right| : y'' \in B_{F''}\right\} \\
&\le \pi_r\big((\lambda_i)_{i=1}^{n}\big)\pi_{q'}\big(((\langle x_i . x'\rangle)_{i=n}^{m}\big)\, \varepsilon_{p'}\big((y'_i)\big)
\end{aligned}$$

is arbitrarily small when n and m increases. Then we get

$$\sum_{i=1}^{\infty} \lambda_i \langle x_i, x' \rangle y'_i \in F', \qquad x' \in E'.$$

Now, suppose that z is such that $\Psi_{pq}(z) = 0$. Then for every $y \in F$ and every $x' \in E'$ we have

$$\sum_{i=1}^{\infty} \lambda_i \langle y'_i, y \rangle \langle x_i, x' \rangle = 0.$$

Since $J_F(F)$ is $\sigma(F'', F')$-dense in F'' we have

$$\sum_{i=1}^{\infty} \lambda_i \langle y'_i, y'' \rangle \langle x_i, x' \rangle = 0, \qquad x' \in E,\ y'' \in F''. \tag{4}$$

This implies that $z^t \in E \hat{\otimes}_{\alpha_{pq}} F'$ vanishes on $E' \otimes F'' \subset D_{p'q'}(E, F'')$. Since $E \hat{\otimes}_{\alpha_{pq}} F'$ is a subspace of $E \hat{\otimes}_{\alpha_{pq}} F'''$, by 3) we obtain that

$$\langle z^t, T \rangle = 0, \qquad T \in D_{p'q'}(E, F'') = \big(E \hat{\otimes}_{\alpha_{pq}} F'\big)'.$$

Then $z^t = 0$ and $z = 0$. □

The theorem 1 motivates the following definition

DEFINITION 1. We say that a Banach space E has the approximation property of order (p, q) (in short the AP_{pq}), $1 \leq p, q < \infty$, if for every Banach space F the canonical map

$$\Phi_{pq} : F \hat{\otimes}_{\alpha_{pq}} E \longrightarrow \mathcal{L}(F', E)$$

is injective.

Remark 1. If $p = q = 1$, since $\alpha_{11} = \pi$, the projective tensor topology, the AP_{11} is the classical approximation property of Grothendieck. If $p \geq 1$ and $q = 1$, since $\alpha_{p1} = g_p$, the AP_{p1} is the Saphar-Reinov's approximation property of order p. The theorem 1 gives some characterizations of the AP_{pq} in Banach spaces.

Remark 2. By a result of Grothendieck [5, Remarque 10, 2°, p. 95; 3, 5.6], if E has the AP then E has the AP_{pq} for all $p, q \in [1, \infty[$.

Proposition 2

Suppose that $1 < p, q < \infty$. If E has the AP_p, E has the AP_{pq} .

Proof. Given a Banach space F, a map $T \in D_{p'q'}(E,F)$, sequences $(x_i) \in \ell^{p'}(E)$ and $(y_i') \in \ell^{q'}(F')$ and $\varepsilon > 0$, by Kwapien's theorem, [14, Theorem 17.4.3] there are a reflexive Banach space M and linear maps $A \in \prod_{p'}(E,M)$ and $B \in \mathcal{L}(M,F)$ such that $T = BA$ and $B' \in \prod_{q'}(F',M')$. Since E has the AP_p, by a result of Bourgain and Reinov [1, Lemma 7] there exists $z \in E' \otimes M$ such that

$$\pi_{p'}\big((A-z)(x_i)\big) \leq \varepsilon.$$

Then $Bz \in E' \otimes F$ and $\big(B'(y_i')\big) \in \ell^{q'}[F']$. By the Hölder's inequality we obtain

$$\begin{aligned} P_{(x_i),(y_i')}(T - Bz) &= \pi_{r'}\big((\langle (BA - Bz)(x_i), y_i'\rangle)\big) \\ &= \pi_r\big((\langle (A - z)(x_i), B'(y_i')\rangle)\big) \\ &= \sup\left\{ \left| \sum_{i=1}^{\infty} \alpha_i \langle (A-z)(x_i), B'(y_i')\rangle \right| : \pi_r\big((\alpha_i)\big) \leq 1 \right\} \\ &\leq \sup\left\{\pi_r\big((\alpha_i)\big) : \pi_r\big((\alpha_i)\big) \leq 1\right\} \pi_{p'}\big((A-z)(x_i)\big)\pi_{q'}\big((B'(y_i'))\big) \\ &\leq \varepsilon\, \pi_{q'}\big((B'(y_i')\big). \end{aligned}$$

By 4) of theorem 1, E has the AP_{pq} .□

Proposition 3

Every E has the AP_{pq} for each $1 < p,q \leq 2$.

Proof. In this case α_{pq} is a tensor norm equivalent to α_{22} [3, Proposition 1.8]. Saphar shows that every E has the AP_2 [18, théorème 3.5]. By the proposition 2, E has the $\mathrm{AP}_{22} = \mathrm{AP}_{pq}$. This result is also a consequence of the fact that α_{22} is totally accesible (see [3], corollary of theorem 9.4 and the remarks that follow to proposition 5.6). □

Remark 3. If $1 < p < 2$ there is a Banach space E without the AP_p [15]. However this space has the AP_{pq} if $1 < q \leq 2$.

3. Applications

We give some results in which approximation porperties of some order appear in a natural way. The first one is to study the reflexivity of the spaces $D_{q'p'}(E,F)$ and $E\hat{\otimes}_{\alpha_{pq}}F$.

Theorem 2

Let E and F be reflexive spaces such that E has the AP_{qp} or F' has the AP_{pq} for some pair $1 < p,q < \infty$. Then $E\hat{\otimes}_{\alpha_{pq}}F'$ and $D_{q'p'}(E,F)$ are reflexive if and only if

$$D_{q'p'}(E,F) = E'\hat{\otimes}_{\alpha'_{pq}}F$$

and

$$I_{pq}(E',F') = N_{pq}(E',F').$$

Proof. First we suppose that E has the AP_{qp}.

Sufficient condition: We have

$$\left(E\hat{\otimes}_{\alpha_{pq}}F'\right)' = D_{q'p'}(E,F) = E'\hat{\otimes}_{\alpha'_{pq}}F$$

and

$$\left(E'\hat{\otimes}_{\alpha'_{pq}}F\right)' = I_{pq}(E',F') = N_{pq}(E',F').$$

But the map $T \mapsto T'$ from $N_{pq}(E',F')$ to $N_{qp}(F,E)$ is an isometry onto, E and F being reflexive. By the AP_{qp} of E,

$$N_{qp}(F,E) = F'\hat{\otimes}_{\alpha_{pq}}E,$$

which is isometric to $E\hat{\otimes}_{\alpha_{pq}}F'$. Then

$$N_{pq}(E',F') = E\hat{\otimes}_{\alpha_{pq}}F'$$

and $E\hat{\otimes}_{\alpha_{pq}}F'$ and $D_{q'p'}(E,F)$ are reflexive.

Necessary condition: $E'\hat{\otimes}_{\alpha'_{pq}}F$ is a subspace of $D_{q'p'}(E,F)$ [3, Theorem 9.4]. Since F and $E\hat{\otimes}_{\alpha_{pq}}F'$ are reflexive,

$$D_{q'p'}(E,F)' = E\hat{\otimes}_{\alpha_{pq}}F' = \left[D_{q'p'}(E,F),\sigma_0\right]'.$$

But E has the AP_{qp}. Then, by the condition 3) of theorem 1, we get that $E'\otimes F$ is σ_0-dense in $D_{q'p'}(E,F)$ and hence it is dense in $D_{q'p'}(E,F)$. Then

$$E'\hat{\otimes}_{\alpha'_{pq}}F = D_{q'p'}(E,F).$$

Now as above, for every $1 < p,q < \infty$

$$\begin{aligned} N_{pq}(E',F') &= E\hat{\otimes}_{\alpha_{pq}}F' \\ &= \left(E\hat{\otimes}_{\alpha_{pq}}F'\right)'' \\ &= \left(D_{q'p'}(E,F)\right)' \\ &= \left(E'\hat{\otimes}_{\alpha'_{pq}}F\right)' \\ &= I_{pq}\left(E',F'\right). \end{aligned}$$

Finally, if F' has the AP_{pq} , the proof is analogous by the remark that

$$N_{pq}\left(E',F'\right) = E\hat{\otimes}_{\alpha_{pq}}F',$$

$D_{p'q'}(F',E')$ is isometric to $D_{q'p'}(E,F)$ (by transposition) and [7, p. 16]

$$F\hat{\otimes}_{\alpha'_{pq}}E' = E'\hat{\otimes}_{\alpha'_{pq}}F. \ \square$$

Proposition 4

Let us suppose that

$$D_{q'p'}\left(E,F'\right) = E'\hat{\otimes}_{\alpha'_{pq}}F'$$

for $1 < p,q$. If E' and F' are separable, $E\hat{\otimes}_{\alpha_{pq}}F$ is reflexive if and only if it is weakly sequentially complete.

Proof. Since

$$\left(E\hat{\otimes}_{\alpha_{pq}}F\right)' = D_{q'p'}(E,F') = E'\hat{\otimes}_{\alpha'_{pq}}F'$$

is separable, the topology $\sigma\left((E\hat{\otimes}_{\alpha_{pq}}F)'', D_{q'p'}(E,F')\right)$ is metrizable on the bounded sets of $(E\hat{\otimes}_{\alpha_{pq}}F)''$. If $E\hat{\otimes}_{\alpha_{pq}}F$ is weakly sequentially complete, we must have

$$\left(E\hat{\otimes}_{\alpha_{pq}}F\right)'' = E\hat{\otimes}_{\alpha_{pq}}F. \ \square$$

In view of theorem 2 and proposition 4, it is interesting to know sufficient conditions for the equality

$$D_{q'p'}(E,F) = E'\hat{\otimes}_{\alpha'_{pq}}F$$

to hold.

Proposition 5

Let us suppose that E is such that E' has the Radon-Nikodym property and E'' has the AP_q for some $1 < q < \infty$. Then for every F and every $1 < p < \infty$,

$$D_{q'p'}(E,F) = E'\hat{\otimes}_{\alpha'_{pq}} F.$$

Proof. Given $T \in D_{q'p'}(E,F)$, by Kwapien's theorem [14, Theorem 17.4.3], there exists a Banach space M and mappings $A \in \Pi_{q'}(E,M)$ and $B \in \mathcal{L}(M,F)$ such that $B' \in \Pi_{p'}(F',M')$ and $T = BA$. Since E' has the Radon-Nikodym property, A is a quasi-q'-nuclear operator [12, p. 228]. By a result of Saphar [18, Théorème 4] there is a sequence $(A_n)_{n=1}^{\infty}$ in $E' \otimes M$ which converges to A in $\Pi_{q'}(E,M)$. Since $E'\hat{\otimes}_{\alpha'_{pq}} F$ is a subspace of $D_{q'p'}(E,F)$ we have

$$\alpha'_{pq}\big(BA_n - BA_m\big) = \pi_{q'p'}\big(B(A_n - A_m)\big) \leq \pi_{p'}^{\mathrm{dual}}(B)\pi_{q'}\big(A_n - A_m\big)$$

(where $\pi_{p'}^{\mathrm{dual}}(B)$ is the norm of B' in $\Pi_{p'}(F',M')$ [14, Theorem 17.4.3]). Then there exists $z \in E'\hat{\otimes}_{\alpha'_{pq}} F$ such that $z = \lim_{n\to\infty} BA_n$ in this space. On the other hand

$$\pi_{q'p'}\big(BA_n - T\big) \leq \pi_{p'}^{\mathrm{dual}}(B)\pi_{q'}\big(A_n - A\big)$$

and the continuously extended inclusion

$$J : E'\hat{\otimes}_{\alpha'_{pq}} F \longrightarrow D_{q'p'}(E,F)$$

is an isometry. Then

$$J(Z) = \lim_{n\to\infty} BA_n$$

in $D_{q'p'}(E,F)$. But this limit is T. Then J is onto. □

Remark 4. The same proof can be used to show that the proposition 5 also holds when E' has the AP and the Radon-Nikodym property. Now every $A \in \Pi_{q'}(E,M)$ is quasi-q'-nuclear [12, p. 228] and it can be approximated in $\Pi_{q'}(E,M)$ by linear maps of finite rank [13, Sätze 26 and 43].

Proposition 6

Let F be complemented in F'' and such that F'' has the Radon-Nikodym property and F''' has the AP_p for some $1 < p < \infty$. Then for every E and every $1 < q < \infty$,

$$D_{q'p'}(E,F) = E'\hat{\otimes}_{\alpha'_{pq}} F.$$

Proof. If $T \in D_{q'p'}(E, F)$, by proposition 5,

$$T' \in D_{q'p'}(F', E') = F''\hat{\otimes}_{\alpha'_{pq}} E'$$

(because given $A \in \Pi_{q'}(X, Y)$, we have $A'' \in \Pi_{q'}(X'', Y'')$). Then $T'' \in E'\hat{\otimes}_{\alpha'_{pq}} F''$ is the limit in this space of a sequence $(z_n)_{n=1}^{\infty}$ of elements of $E' \otimes F''$. Let P be a projection from F'' onto F. We have $T = PT''J_E$ and $Pz_nJ_E \in E' \otimes F$. Since

$$D_{q'p'}(T - Pz_nJ_E) \leq \|P\|\, D_{q'p'}(T'' - z_n)\|J_E\| = \|P\|\, \alpha'_{pq}(T'' - z_n),$$

we obtain $T \in E'\hat{\otimes}_{\alpha'_{pq}} F$ as in proposition 5. □

Remark 5. As in the remark 4, the proposition 6 also holds if F'' has the Radon-Nikodym property and the AP.

As a consequence of theorem 2 and propositions 2, 5 and 6 we obtain

Corollary 2

If E and F are reflexive spaces such that E has the AP_q for some $1 < q < \infty$, (resp. F' has the AP_p for some $1 < p < \infty$), then for every $1 < p < \infty$ (resp. for every $1 < q < \infty$), $D_{q'p'}(E, F)$, $E\hat{\otimes}_{\alpha_{pq}} F'$ and $E'\hat{\otimes}_{\alpha'_{pq}} F$ are reflexive if and only if

$$I_{pq}(E', F') = N_{pq}(E', F').$$

EXAMPLE 1. On reflexivity of $\ell^u\hat{\otimes}_{\alpha_{pq}}\ell^v$. Let $1 < u, v, p, q < \infty$. We have:

(i) If $u < 2$ and $v \leq 2$ (or $u \leq 2$ and $v < 2$), then $\ell^u\hat{\otimes}_{\alpha_{pq}}\ell^v$ is reflexive for every p and q.

(ii) If $u \geq 2$ and $v \geq 2$, $\ell^u\hat{\otimes}_{\alpha_{pq}}\ell^v$ is always not reflexive.

(iii) if $u < 2$ and $v > 2$ and $u < q$, (or $u > 2$, $v < 2$ and $v < p$), $\ell^u\hat{\otimes}_{\alpha_{pq}}\ell^v$ is reflexive.

(iv) If $u < 2$, $v > 2$ and $u \geq q$ and $p \leq 2$ (or $u > 2$, $v < 2$, $v \geq p$ and $q \leq 2$), $\ell^u\hat{\otimes}_{\alpha_{pq}}\ell^v$ is reflexive.

Proof. (i) If $u \leq 2$ and $v \leq 2$, $\ell^u \hat{\otimes}_{\alpha_{pq}} \ell^v$ is isomorphic to $l^u \hat{\otimes}_\varepsilon \ell^v$ [3, Proposition 10.2]. By [9, Proposition 3.7], $\ell^u \hat{\otimes}_{\alpha_{pq}} \ell^v$ is reflexive if $u < 2$ and $v \leq 2$ or $u \leq 2$ and $v < 2$ and $\ell^u \hat{\otimes}_{\alpha_{pq}} \ell^v$ is not reflexive if $u = v = 2$.

(ii) Let $u \geq 2$ and $v \geq 2$. Let J be the composition of the natural inclusion maps

$$\ell^{u'} \longrightarrow \ell^2 \longrightarrow \ell^2 \longrightarrow \ell^v.$$

Since the identity map on ℓ^2 is (p,q)-factorable [14, Theorem 22.1.11], we have $J \in I_{pq}(\ell^{u'}, \ell^v)$. However $J \notin N_{pq}(\ell^{u'} \ell^v)$ since J is not compact. By corollary 2, $\ell^u \hat{\otimes}_{\alpha_{pq}} \ell^v$ is not reflexive.

(iii) Now, let us suppose $u < 2$, $v > 2$ and $u' > q'$. If $T \in I_{pq}(\ell^{u'}, \ell^v)$, by [14, Theorem 19.4.6], there are a probability space (Ω, μ) and linear maps $A \in \mathcal{L}(\ell^{u'}, L^{q'}(\Omega, \mu))$ and $C \in \mathcal{L}(L^p(\Omega, \mu), \ell^v)$ such that $T = CIA$ where I is the inclusion map from $L^{q'}(\Omega, \mu)$ into $L^p(\Omega, \mu)$. By a result of Rosenthal [16, Theorem A.2], A is compact. Since ℓ^u has the AP, there is a sequence $(A_n)_{n=1}^\infty$ in $\ell^u \otimes L^{q'}(\Omega, \mu)$ such that

$$\lim_{n \to \infty} \|A - A_n\| = 0.$$

Then $CIA_n \in I_{pq}(\ell^{u'}, \ell^v)$ for every $n \in \mathbb{N}$ and

$$I_{pq}(T - CIA_n) = I_{pq}(CI(A - A_n)) \leq \|C\| \, \|A - A_n\|.$$

Then $(CIA_n)_{n=1}^\infty$ is a Cauchy sequence in $\ell^u \otimes \ell^v$ with the norm induced by $I_{pq}(\ell^{u'}, \ell^v)$. But this norm is α_{pq} by the proposition 6. Hence $T \in \ell^u \hat{\otimes}_{\alpha_{pq}} \ell^v$ and $T \in N_{pq}(\ell^{u'}, \ell^v)$. By corollary 2, $\ell^u \hat{\otimes}_{\alpha_{pq}} \ell^v$ is reflexive.

(iv) If $2 < u' \leq q'$ and $p \leq 2$, by [3, Proposition 1.8], $\ell^u \hat{\otimes}_{\alpha_{pq}} \ell^v$ is isomorphic to $\ell^u \hat{\otimes}_{\alpha_{22}} \ell^v$. If $T \in I_{22}(\ell^{u'}, \ell^v)$, there exists a localizable measure space (Ω, μ) and maps $A \in \mathcal{L}(\ell^{u'}, L^2(\Omega, \mu))$ and $B \in \mathcal{L}(L^2(\Omega, \mu), \ell^v)$ such that $T = BA$ [3, 4.6]. By the above quoted result of Rosenthal, A is compact and the proof ends like in a former case.

The other cases follow by transposition of the tensor norm α_{pq}. □

Remark 6. The last example shows, in particular, that $\mathcal{L}(\ell^2, \ell^2) \neq N_{pq}(\ell^2, \ell^2)$. Then, the Radon-Nikodym property of E, E', F and F' does not imply the equality $I_{pq}(E, F) = N_{pq}(E, F)$ for $1 < p, q < \infty$.

The next applications concern the weak sequential completeness of $E \hat{\otimes}_{\alpha'_{pq}} F$ and $E \hat{\otimes}_{\alpha_{pq}} F$.

Theorem 3

Let $1 < p,\ q < \infty$. *Assume that* E *or* F *has a finite dimensional unconditional Schauder decomposition (shortly an FDUSD). Then* $E\hat{\otimes}_{\alpha'_{pq}}F$ *is weakly sequentially complete if and only if* E *and* F *are weakly sequentially complete.*

Proof. Let us assume that E has an FDUSD. We can suppose without loss of generality that this FDUSD of E is also monotone [20, Proposition 15.3]. Since E is weakly sequentially complete, the FDUSD is also boundedly complete (see the comments of [20, p. 534] and [19, Chapter II, Corollary 17.3.b]. Then there exists a Banach space G such that $E = G'$ [20, Theorem 15.14] and E has the Radon-Nikodym property [4, Theorem 1, p. 79]. By remark 4 we have

$$E\hat{\otimes}_{\alpha'_{pq}}F'' = G'\hat{\otimes}_{\alpha'_{pq}}F'' = D_{q'p'}(G, F'').$$

From now on, the proof is the same of [8, Theorem 2].

If F has an FDUSD, the proof follows by transposition of α'_{pq}. □

EXAMPLE 2. If $1 < p,q < \infty$ and $1 \leq u,v < \infty$, the space $\ell^u\hat{\otimes}_{\alpha_{pq}}\ell^v$ is weakly sequentially complete but in general, it is not reflexive (this is a consequence of proposition 5 and the results of example 1).

Theorem 4

Suppose that E *and* F *are such that* E *has an FDUSD and*

$$I_{qp}(F', E) = N_{qp}(F', E)$$

for $1 < p,q < \infty$. *Then* $E\hat{\otimes}_{\alpha_{pq}}F$ *is weakly sequentially complete if and only if* E *and* F *are weakly sequentially complete.*

Proof. Since E has the AP_{qp}, the proof is the same of [8, Theorem 1]. □

Remark 7. The hypothesis $I_{qp}(F', E) = N_{qp}(F', E)$ holds if every linear map from F' into every $L^{p'}(\Omega, \mu)$ is compact; for example if $F = \ell^v$ with $1 < v < 2$ and $v < p < \infty$ (see the proof of example 1).

Corollary 3

If E *is weakly sequentially complete and* $1 < v < 2$, $v < p < \infty$ *and* $1 < q < \infty$, $E\hat{\otimes}_{\alpha_{pq}}\ell^v$ *is weakly sequentially complete.*

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