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Sturdy bands of semigroups

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Abstract

In this paper we consider band compositions determined by two transitive systems of monomorphisms, called the sturdy bands of semigroups.

A.H. Clifford [3] introduced semilattices of groups and determined their structure in terms of what we now call *strong semilattices* of groups. Strong semilattices of groups, i.e. semilattice compositions determined by one transitive system of homomorphisms, are also considered by many authors. In [1] and [2] authors gave the construction of a band of monoids and of an inflation of a band of monoids by two systems of homomorphisms, which must not be transitive. In this paper we consider band compositions of semigroups determined by transitive systems of homomorphisms, called the *strong bands of semigroups*. Strong bands of monoids and groups are considered by B. M. Schein [8]. For example, B. M. Schein [8] proves that a semigroup S is a strong band of groups if and only if S is orthodox and a band of groups (called also the orthocryptogroup). For other characterizations of orthocryptogroups see [5], [6] or [9]. By Proposition 1 and Theorem 1 we give generalizations of Petrich's results [7, p 87–88, 4, p. 98]. For some related results we refer to [6] and [5].

For undefined notions and notations we refer to [4] or [7].

Let I be a band. To each $i \in I$ we associate a semigroup S_i such that $S_i \cap S_j = \emptyset$ if $i \neq j$. Let $\leq_1$ and $\leq_2$ be quasiorders (i.e. reflexive and transitive binary relations) on I defined in the following way:

$$i \leq_1 j \iff ji = i, \qquad i \leq_2 j \iff ij = i.$$

Let φ_{ij} and ψ_{ij} be homomorphisms of S_j into S_i over $\leq_1$ and $\leq_2$ respectively, for which the following properties hold:

(1) For every $i \in I$, φ_{ii} and ψ_{ii} are the identical automorphisms of S_i;
(2) $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$, $i \leq_1 j \leq_1 k$;
(3) $\psi_{ij} \circ \psi_{jk} = \psi_{ik}$, $i \leq_2 j \leq_2 k$;
(4) $\varphi_{kj,k} \circ \psi_{ki} = \psi_{kj,j} \circ \varphi_{ji}$, $j \leq_1 i$, $k \leq_2 i$.

Define a multiplication $*$ on $S = \bigcup_{i \in I} S_i$ by:

$$s_i * s_j = \varphi_{ij,i}(s_i)\, \psi_{ij,j}(s_j), \qquad s_i \in S_i,\ s_j \in S_j.$$

Then $(S, *)$ is a semigroup and a band I of semigroups S_i, $i \in I$ [8]. If $j \leq_1 i$ and $j \leq_2 i$, then by (4) and by (1) it follows that

$$\psi_{ji} = \varphi_{ji}. \tag{4*}$$

DEFINITION 1. A semigroup S is a *strong band of semigroups* if it can be constructed in the previous way, and we denote it by $S = [I; S_i, \varphi_{ij}, \psi_{ij}]$. If all φ_{ij} and ψ_{ij} are one-to-one, then we say that S is a *sturdy band of semigroups* S_i, $i \in I$, and we denote it by $S = \langle I; S_i, \varphi_{ij}, \psi_{ij} \rangle$.

If $S = [I; S_i, \varphi_{ij}, \psi_{ij}]$ is a strong matrix of semigroups, then, as in [8], we obtain that all φ_{ij} and ψ_{ij} are one-to-one. In the general case, when I is an arbitrary band, this is not true. Now we will consider strong bands for which all φ_{ij} and ψ_{ij} are one-to-one, i.e. sturdy bands of semigroups.

Proposition 1

On $S = \langle I; S_i, \varphi_{ij}, \psi_{ij} \rangle$ define a relation ρ by:

$$a \rho b \iff \varphi_{ij,i}(a) = \psi_{ij,j}(b) \ \wedge\ \psi_{ji,i}(a) = \varphi_{ji,j}(b),$$

$a \in S_i$, $b \in S_j$. Then ρ is a congruence and S is a subdirect product of I and S/ρ.

Conversely, let $S \subseteq I \times T$ be a subdirect product of a band I and a semigroup T such that the following condition holds:

$$\begin{cases} (i \leq_1 j \ \wedge\ (j,u) \in S \Longrightarrow (i,u) \in S \\ (i \leq_2 j \ \wedge\ (j,u) \in S) \Longrightarrow (i,u) \in S, \end{cases} \tag{5}$$

for all $i,j \in I$, $u \in T$. Then S is a sturdy band of semigroups $S_i = (\{i\} \times T) \cap S$, $i \in I$.

Proof. Let $S = \langle I; S_i, \varphi_{ij}, \psi_{ij} \rangle$. It is clear that ρ is reflexive and symmetric. We will prove the transitivity. Let $a\rho b$ and $b\rho c$, $a \in S_i$, $b \in S_j$, $c \in S_k$. Then

(6) $\varphi_{ij,i}(a) = \psi_{ij,j}(b),$

(7) $\psi_{ji,i}(a) = \varphi_{ji,j}(b),$

(8) $\varphi_{jk,j}(b) = \psi_{jk,k}(c),$

(9) $\psi_{kj,j}(b) = \varphi_{kj,k}(c).$

Now,

$$\begin{aligned}
\varphi_{jkij,jki} \circ \psi_{jki,ki} \circ \varphi_{ki,k}(c) &= \varphi_{jkij,jki} \circ \varphi_{jki,jk} \circ \psi_{jk,k}(c) && \text{(by (4))}\\
&= \varphi_{jkij,jk} \circ \varphi_{jk,j}(b) && \text{(by (8),(2))}\\
&= \varphi_{jkij,j}(b) && \text{(by (2))}\\
&= \psi_{jkij,j}(b) && \text{(by (4*))}\\
&= \psi_{jkij,ij} \circ \psi_{ij,j}(b) && \text{(by (3))}\\
&= \psi_{jkij,ij} \circ \varphi_{ij,i}(a) && \text{(by (6))}\\
&= \varphi_{jkij,jki} \circ \psi_{jki,i}(a) && \text{(by (4))}\\
&= \varphi_{jkij,jki} \circ \psi_{jki,ki} \circ \psi_{ki,i}(a). && \text{(by(3))}
\end{aligned}$$

Since $\psi_{jki,ki}$ and $\varphi_{jkij,jki}$ are one-to-one, then we have that

$$\psi_{ki,i}(a) = \varphi_{ki,k}(c).$$

In a similar way we prove that

$$\varphi_{ik,i}(a) = \psi_{ik,k}(c).$$

Thus, $a\rho c$, so ρ is transitive.

Let $a\rho b$, $a \in S_i$, $b \in S_j$ and let $c \in S_k$. Then (6) and (7) hold, so

$$\begin{aligned}
\psi_{jikjk,ikjk} \circ \varphi_{ikjk,ik} \circ \varphi_{ik,i}(a) &= \varphi_{jikjk,jik} \circ \psi_{jik,ik} \circ \varphi_{ik,i}(a) && \text{(by (4))}\\
&= \varphi_{jikjk,jik} \circ \varphi_{jik,ji} \circ \psi_{ji,i}(a) && \text{(by (4))}\\
&= \varphi_{jikjk,jik} \circ \varphi_{jik,ji} \circ \varphi_{ji,j}(b) && \text{(by (7))}\\
&= \varphi_{jikjk,jikj} \circ \varphi_{jikj,j}(b) && \text{(by (2))}\\
&= \varphi_{jikjk,jikj} \circ \psi_{jikj,j}(b) && \text{(by (4*))}\\
&= \psi_{jikjk,jk} \circ \varphi_{jk,j}(b) && \text{(by (4))}\\
&= \psi_{jikjk,ikjk} \circ \psi_{ikjk,jk} \circ \varphi_{jk,j}(b). && \text{(by (3))}
\end{aligned}$$

Since $\psi_{jikjk,ikjk}$ is one-to-one, then

$$\varphi_{ikjk,ik} \circ \varphi_{ik,i}(a) = \psi_{ikjk,jk} \circ \varphi_{jk,j}(b). \tag{10}$$

Moreover,

$$\begin{aligned}\varphi_{ikjk,ik} \circ \psi_{ik,k}(c) &= \psi_{ikjk,kjk} \circ \varphi_{kjk,k}(c) && \text{(by (4))}\\ &= \psi_{ikjk,kjk} \circ \psi_{kjk,k}(c) && \text{(by (4*))}\\ &= \psi_{ikjk,k}(c) && \text{(by (3))}\\ &= \psi_{ikjk,jk} \circ \psi_{jk,k}(c), && \text{(by (3))}\end{aligned}$$

so

$$\varphi_{ikjk,ik} \circ \psi_{ik,k}(c) = \psi_{ikjk,jk} \circ \psi_{jk,k}(c). \tag{11}$$

Now we have that

$$\begin{aligned}\varphi_{ikjk,ik}(a * c) &= \varphi_{ikjk,ik}(\varphi_{ik,i}(a) \cdot \psi_{ik,k}(c))\\ &= \varphi_{ikjk,ik} \circ \varphi_{ik,i}(a) \cdot \varphi_{ikjk,ik} \circ \psi_{ik,k}(c)\\ &= \psi_{ikjk,jk} \circ \varphi_{jk,j}(b) \cdot \psi_{ikjk,jk} \circ \psi_{jk,k}(c) \quad \text{(by (10, (11))}\\ &= \psi_{ikjk,jk}(\varphi_{jk,j}(b) \cdot \psi_{jk,k}(c))\\ &= \psi_{ikjk,jk}(b * c).\end{aligned}$$

Thus,

$$\varphi_{ikjk,ik}(a * c) = \psi_{ikjk,jk}(b * c),$$

and, similarly,

$$\psi_{jkik,ik}(a * c) = \varphi_{jkik,jk}(b * c).$$

Therefore, $a * c \, \rho \, b * c$. In a similar way we prove that $c * a \, \rho \, c * b$, so ρ is a congruence.

Let η be the band congruence on S whose classes are the various S_i. Then it is clear that

$$\rho \cap \eta = \varepsilon,$$

where ε is the identical relation on S. Therefore, S is a subdirect product of $S/\eta \cong I$ and S/ρ [4, Proposition II 1.4].

Conversely, let $S \subseteq I \times T$ be a subdirect product of a band I and a semigroup T such that (5) holds. For each $i \in I$, let $S_i = (\{i\} \times T) \cap S$. Then it is clear that S is a band I of semigroups S_i, $i \in I$.

Define functions φ_{ij}, $i \le_1 j$, and ψ_{ij}, $i \le_2 j$, by:

$$\begin{aligned} \varphi_{ij}((j,u)) &= (i,u), \qquad (j,u) \in S_j \ i \le_1 j, \\ \psi_{ij}((j,u)) &= (i,u), \qquad (j,u) \in S_j \ i \le_2 j. \end{aligned}$$

By (5) it follows that $(i,u) \in S$, so $(i,u) \in S_i$ in both cases. It is easy to prove that φ_{ij} and ψ_{ij} are homomorphisms and that conditions (1)–(4) hold. Let $(i,u) \in S_i$ and $(j,v) \in S_j$. Then

$$\varphi_{ij,i}\big((i,u)\big)\psi_{ij,j}\big((j,v)\big) = (ij,u)(ij,v) = (ij,uv) = (i,u)(j,v)$$

$$\big(= (i,u) * (j,v)\big).$$

Hence, S is a strong band of semigroups S_i, $i \in I$.

Let $i \le_1 j$ and let $(j,u),(j,v) \in S_j$ such that

$$\varphi_{ij}\big((j,u)\big) = \varphi_{ij}\big((j,v)\big).$$

Then we have that $(i,u) = (i,v)$, i.e. $u = v$. Hence, $(j,u) = (j,v)$, so φ_{ij} is one-to-one. In a similar way we prove that any ψ_{ij}, $i \le_2 j$, is one-to-one. Therefore, S is a sturdy band of semigroups S_i, $i \in I$. □

Theorem 1

S is a sturdy band of groups if and only if S is regular and a subdirect product of a band and a group.

Proof. Let $S = \langle I; S_i, \varphi_{ij}, \psi_{ij}\rangle$ be a sturdy band of groups G_i, $i \in I$, and let e_i be an identity element of G_i, $i \in I$. Then by Proposition 1 we have that S is a subdirect product of I and a semigroup S/ρ. Since S is regular and S/ρ is a homomorphic image of S, then S/ρ is regular. It is clear that every idempotent from S/ρ is of the form $e\rho$, where e is an idempotent from S, i.e. $e = e_i$ for some $i \in I$. Moreover, for every $i,j \in I$ we have that

$$\begin{aligned} \varphi_{ij,i}(e_i) &= e_{ij} = \psi_{ij,j}(e_j), \\ \psi_{ji,i}(e_i) &= e_{ji} = \varphi_{ji,j}(e_j), \end{aligned}$$

so $e_i \rho e_j$. Hence, S/ρ has only one idempotent, so S/ρ is a group.

Conversely, let S be a regular semigroup and S be a subdirect product of a band I and a group G. Let e be an identity element of G. Assume that $S \subseteq I \times G$.

Let $i \in I$. Then there exists $a \in G$ such that $(i,a) \in S$. Since S is regular, then there exists $(j,b) \in S$ such that

$$(i,a) = (i,a)(j,b)(i,a) \qquad \text{and} \qquad (j,b) = (j,b)(i,a)(j,b).$$

Thus

$$(i,a) = (iji, aba) \qquad \text{and} \qquad (j,b) = (jij, bab),$$

i.e. $i = iji$, $a = aba$, $j = jij$ and $b = bab$. So $b = a^{-1}$. Now we have that

$$(i,e) = (ijji, abba) = (i,a)(j,b)^2(i,a) \in S.$$

Moreover, it is clear that (i,e) is the identity element of a semigroup $G_i = (\{i\} \times G) \cap S$. Also, we have that

$$(i,b) = (i,e)(j,b)(i,e) \in S$$

and

$$(i,a)(i,b)(i,a) = (i,aba) = (i,a).$$

Therefore, G_i is regular and since G_i contains only one idempotent, then G_i is a group.

Let $i,j \in I$ and let $i \leq_1 j$, $(j,a) \in S$. Then $ji = i$, so

$$(i,a) = (ji, ae) = (j,a)(i,e) \in S.$$

In a similar way we prove that by $i \leq_2 j$ and $(j,a) \in S$ it follows that $(i,a) \in S$. Hence, (5) holds, so by Proposition 1 we have that S is a sturdy band of groups G_i, $i \in I$. □

Corollary

S is a sturdy band of periodic groups if and only if S is a subdirect product of a band and a periodic group. □

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