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The p-adic differential equation $y' = \omega y$ in the closed unit disk

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ABSTRACT

Let $\mathbb{K}$ be a complete ultrametric algebraically closed field. When D is a set in $\mathbb{K}$, we denote by $H(D)$ (resp. $H_b(D)$) the set of the analytic elements in D (resp. the bounded analytic elements in D). Let $\mathcal{B} = (b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ such that $1 < |b_{n+1}| < |b_n|$ and $\lim_{n\to\infty} |b_n| = 1$ and let

$$\Lambda_\rho(\mathcal{B}) = \mathbb{K} \setminus \left(\cup_{n=1}^{\infty} d(b_n, \rho^-)\right)$$

First we characterize the elements $f \in \cap_{\rho>0} H_b(\Lambda_\rho(\mathcal{B}))$ which are Meromorphic Products [S2] in the form $\prod_{n=1}^{\infty}(x - a_n/x - b_n)$. Next, we translate the results of analytic extension through a T-filter [S3] into terms of Meromorphic Products. We then apply that to the extension of a differential equation $y' = \omega y$ defined in the disk $d(0,1)$ ($|x| \leq 1$) to $K \setminus \mathcal{B}$; y has an extension in the form $\lambda \prod_{n=1}^{\infty}(x - a_n/x - b_n)$ while ω has the extension $\sum_{n=1}^{\infty}((1/x - a_n - 1/x - b_n))$.

As a consequence we can characterize the $\omega \in H(d(0,1))$ such that the equation $y' = \omega y$ has solutions in $H(d(0,1))$: these ω are the series $\sum_{n=1}^{\infty}((1/x - a_n) - (1/x - b_n))$. When $\mathbb{K} = \mathbb{C}_p$ such a series may be equal to a constant $\lambda \in \mathbb{C}_p$ in $d(0,1)$ if and only if $|\lambda| < \rho^{-1/p-1}$.

Introduction and results

Let $(\mathbb{K}, |\cdot|)$ be a complete ultrametric algebraically closed field.

For $a \in \mathbb{K}$ and $r > 0$, we denote

$$d(a,r) = \{x \in \mathbb{K} : |x-a| \leq r\},$$

$$d(a,r^-) = \{x \in \mathbb{K} : |x-a| < r\},$$

and

$$C(a,r) = \{x \in \mathbb{K} : |x-a| = r\}.$$

For $a \in \mathbb{K}$ and $r, s \in \mathbb{R}_+$ with $0 < r < s$, we denote

$$\Gamma(a,r,s) = \{x \in \mathbb{K} : r < |x-a| < s\}.$$

Let A be a closed non necessarily bounded set in $\mathbb{K}$ and let $R_b(A)$ be the algebra of the rational functions h with no pole in A, such that $|h(x)|$ has a finite upper bound in A. The algebra $R_b(A)$ admits the norm $\|\cdot\|_A$ of the uniform convergence on A as a norm of $\mathbb{K}$-algebra and its completion denoted by $H_b(A)$ is the Banach algebra of the bounded analytic elements on A. If A is bounded, then $R_b(A)$ is the set of all the rational functions with no pole in A and $H_b(A)$ is the algebra $H(A)$ of all the analytical elements on A [E1, E2].

If A has a hole $T = d(a,\rho^-)$ and if A' is the image of A by an inversion of center a, it is easily seen that $H(A)$ is isometrically isomorphic to $H(A' \cup \{a\})$ [E2]. Therefore, we may apply to the algebras $H_b(A)$ most of the results already known for the Banach algebras $H(D)$ (with D closed and bounded). If $\omega \in H_b(A)$ we will denote by $\mathcal{E}(\omega)$ the equation $y' = \omega y$ with $y \in H_b(A)$.

In all of the following, we will denote by $\mathcal{B}$ *an injective sequence* $(b_n)_{n\in\mathbb{N}^*}$ *in* $\mathbb{K}$ *such that* $1 < |b_{n+1}| \leq |b_n|$ *for all* n *and* $\lim |b_n| = 1$; we will denote

$$\hat{\mathcal{B}} = \{b_1, \ldots, b_n, \ldots\},$$

and for every $\rho \in]0,1[$ we will denote

$$\Lambda_\rho(\mathcal{B}) = \mathbb{K} \setminus \left(\bigcup_{n=1}^{\infty} d(b_n, \rho^-) \right)$$

and denote by $\mathcal{F}_\rho$ the pierced decreasing filter of center 0, of diameter 1 [E3].

If $\mathcal{F}_\rho$ is a T-filter on $\Lambda_\rho(\mathcal{B})$ for some $\rho \in]0,1[$, then $\mathcal{F}_\rho$ is a T-filter on $\Lambda_\rho(\mathcal{B})$ for every $\rho \in]0,1[$, [E4, S1].

Let $f \in \bigcap_{\rho>0} H_b(\Lambda_\rho(\mathcal{B}))$. A pole b_n of f will be called a *pole of order* q if $(x - b_n)^q f(x)$ has a limit $l \neq 0$ when x approaches b_n ($q \in \mathbb{N}^*$). Then b_n will be called a *simple pole* if $q = 1$.

We call a *Meromorphic Product associated to the sequence* $\mathcal{B}$ a function f defined in $\mathbb{K} \setminus \hat{\mathcal{B}}$ in the form

$$f = \prod_{n=1}^{\infty} \left(\frac{x - a_n}{x - b_n} \right),$$

where $(a_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{K}$ such that

$$\lim_{n \to \infty} (a_n - b_n) = 0 \qquad [\mathrm{S}_2, \mathrm{S}_3].$$

Then we know that such a Meromorphic Product belongs to $H_b(\Lambda_\rho(\mathcal{B}))$ whenever $\rho > 0$ [S_1, S_2].

When τ is the Meromorphic Product

$$\tau = \prod_{n=1}^{\infty} \frac{x - a_n}{x - b_n},$$

for every $\rho > 0$ there is an integer $q(\rho)$ such that $d(a_n, \rho^-) = d(b_n, \rho^-)$ for every $n > q(\rho)$; we will denote

$$M_\rho(\tau) = \Lambda_\rho(\mathcal{B}) \setminus \left(\bigcup_{n=1}^{q(\rho)} d(a_n, \rho^-) \right)$$

and then τ is invertible in $H_b(M_\rho(\tau))$.

As $\mathcal{B}$ is an injective sequence we can easily characterize the Meromorphic Products among the functions defined in $\mathbb{K} \setminus \{b_1, \ldots, b_n, \ldots\}$.

Theorem 1

Let f be a function defined in $\mathbb{K} \setminus \mathcal{B}$. The following conditions α) and β) are equivalent.

α) *i) f belongs to $H_b(\Lambda_\rho(\mathcal{B}))$ for all $\rho > 0$,*

ii) each b_n is a simple pole for f,

iii) $\lim_{|x| \to \infty} f(x) = 1$,

iv) f is not identically null on $d(0, 1)$.

β) *f is a Meromorphic Product associated to $\mathcal{B}$.*

Theorem 1 enables us to translate the analytic extension through a T-filter made in [S3] in terms of Meromorphic Products.

Theorem 2

Let g belong to $H(d(0,1))$. Assume that $\mathcal{F}_\rho$ is a T-filter on $\Lambda_\rho(\mathcal{B})$. There exists a Meromorphic Product $\bar{g}$ associated to $\mathcal{B}$ whose restriction to $d(0,1)$ is equal to g.

In Theorems 3 and 4 $\mathbb{K}$ is supposed to have characteristic zero. These results will be generalized to infraconnected sets in a further article [S5].

Now Theorem 2 applied to a solution g of a differential equation $y' = \omega y$ in $d(0,1)$ provides a $\overline{g}$ and a $\overline{\omega}$ so that $\mathcal{E}(\omega)$ extends to $\Lambda_\rho(\mathcal{B})$.

Theorem 3

Let $\omega \in H(d(0,1))$ and assume that the equation $\mathcal{E}(\omega)$ $y' = \omega y$ has a non identically null solution $g \in H(d(0,1))$. Assume that $\mathcal{F}_\rho$ is a T-filter on $\Lambda_\rho(\mathcal{B})$ and let

$$g(x) = \prod_{n=1}^{\infty} \left(\frac{x - a_n}{x - b_n}\right)$$

be a Meromorphic Product associated to $\mathcal{B}$ whose restriction to $d(0,1)$ is equal to g, as obtained in Theorem 2.

The series

$$\omega(x) = \sum_{n=1}^{\infty} \left(\frac{1}{x - a_n} - \frac{1}{x - b_n}\right)$$

converges in $H_b(M_\rho(\bar{g}))$ and it satisfies $\bar{\omega}(x) = \omega(x)$ for all $x \in d(0,1)$.

The space of the solutions y of the equations

$$\mathcal{E}(\bar{\omega})\ y' = \bar{\omega} y$$

defined in $H_b(M_\rho(\bar{g}))$ is generated by $\bar{g}$, and these solutions belong to $H_b(\Lambda_\rho(\mathcal{B}))$.

On the other hand, thanks to Theorem 3 we can characterize the $\omega \in H(D)$ such that the equation $\mathcal{E}(\omega)$ has solutions g in $H(D)$.

Theorem 4

Let D be $d(0,1)$ and let $\omega \in H(D)$. The following assertions are equivalent:

a) There exist sequences $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{K}\setminus d(0,1)$ with $\lim_{n\to\infty} |b_n| = 1$ and $\lim_{n\to\infty} |a_n - b_n| = 0$, such that

$$\omega = \sum_{n=1}^{\infty} \frac{1}{x - a_n} - \frac{1}{x - b_n}.$$

b) The equation $\mathcal{E}(\omega)$ has solution g in $H(D)$, different from zero.

Corollary

Let $\lambda \in \mathbb{C}_p$. The following assertions a) and b) are equivalent.

a) *There exist sequences (a_n) and (b_n) in $\mathbb{C}_p \setminus (0,1)$ with $\lim_{n\to\infty} |b_n| = 1$ and $\lim_{n\to\infty} |a_n - b_n| = 0$ such that*

$$\sum_{n=1}^{\infty} \frac{1}{x-a_n} - \frac{1}{x-b_n} = \lambda,$$

whenever $x \in d(0,1)$.

b) $|\lambda| < p^{-1/(p-1)}$.

Proof of the Theorems

Let "Log" be a logarithm function of base $a > 1$ and let v be the valuation of $\mathbb{K}$ defined by $v(x) = -\operatorname{Log}|x|$.

When D is a closed infraconnected set of diameter $R \in [0, +\infty]$, for $g \in H(D)$, $a \in D$ and $\mu \geq -\operatorname{Log} R$, we define

$$v_a(g,\omega) = \lim_{\substack{v(x)\to\mu \\ v(x)\neq\mu \\ x\in D}} v\big(g(x)\big) \qquad [\mathrm{E}_3,\ \mathrm{G},\ \mathrm{E}_1].$$

When $a = 0$ we only write $v(g,\mu)$ instead of $v_0(g,\mu)$. The properties of the functions $v_a(g,\mu)$ were given in [E3,G] and recalled in many papers like [E6]. Also the increasing and decreasing filters were defined in [E3] and recalled in [E6]. The T-filters were defined in [E4].

The proof of Theorem 1 will use the following Proposition that is an obvious application of the properties of the Mittag-Leffler series for an analytic element [A, R].

Proposition 1

The two following conditions are equivalent.

a) *f belongs to $H_b(\Lambda_\rho(\mathcal{B}))$ for all $\rho > 0$, each b_n is a simple pole for f and $\lim_{|x|\to\infty} f(x) = a$.*

b)

$$f(x) = a + \sum_{n=1}^{\infty} \frac{\alpha_n}{x-b_n},$$

with $\lim_{n\to\infty} \alpha_n = 0$.

Proof of Theorem 1. Suppose first that β) is true and let us show it that implies α). We already know that $f \in H_b(\Lambda_\rho(\mathcal{B}))$ for all $\rho > 0$ and each b_n is a simple pole. Finally, it is easily seen that $\lim_{|x|\to\infty} f(x) = 1$. [S$_2$].

Suppose now that α) is true and let us show β).

Since $\lim_{|x|\to\infty} f(x) = 1$, we know that there exists $R > |b_1|$ such that $|f(x)| = 1$ for all $x \in \mathbb{K} \setminus d(0,R)$, and then f has no zero and no pole in $\mathbb{K} \setminus d(0,R)$.

Since f is not identically null in $d(0,1)$ f is not anulled by the only pierced filter of $\Lambda_\rho(\mathcal{B})$, hence f is quasi-invertible in $H(d(0,R) \cap \Lambda_\rho(\mathcal{B}))$ [E$_3$].

Let $D_\rho = d(0,R) \cap \Lambda_\rho(\mathcal{B})$. Suppose first that f has no zero in $d(0,1)$. $|f(x)|$ is then equal to a constant $C \neq 0$ in $d(0,1)$.

Actually we will prove that the relation $|f(x)| = C$ remains true in a set in the form $d(0,r) \cap \Lambda_\rho(\mathcal{B})$ with $r > 1$. Indeed let $h \in R(D_\rho)$ be such that

$$\|f - h\|_{D_\rho} < C \tag{1}$$

$|h(x)|$ is clearly equal to C in $d(0,1)$. Since h has neither any zero nor any pole in $d(0,1)$ it does exist $r > 1$ such that it still has no zero and no pole in $d(0,r)$ and then $|h(x)| = C$ does hold in $d(0,r) \cap \Lambda_\rho(\mathcal{B})$.

Let $t(\rho)$ be an integer such that $|b_{t(\rho)}| < r$. Since f has no zero in $\Lambda_\rho(\mathcal{B}) \cap d(0,r)$ it is easily seen that for every $n \geq t(\rho)$, $d(b_n, \rho)$ contains as many zeros as many poles of f (the zeros are counted according to their multiplicity order).

As we may take ρ as small as we want, it is easily seen that

$$\lim_{n\to\infty}(b_n - a_n) = 0$$

and then the product

$$\prod_{n=1}^{\infty}\left(\frac{x-a_n}{x-b_n}\right)$$

converges in $H(\Lambda_\rho(\mathcal{B}))$.

On the other hand, since f is bounded and f has no zero and no pole in $\mathbb{K} \setminus d(0,R)$, we see that $v(f,\mu)$ is definitively constant for $\mu \leq -\operatorname{Log} R$. As it is already constant for $\mu \geq -\operatorname{Log} r$, we see that f has as many zeros as poles in $\Gamma(0,r,R)$.

The poles are $b_1, \ldots, b_{t(\rho)-1}$. Hence we may write the zeros $a_1, \ldots, a_{t(\rho)-1}$ (in repeating q times a zero of order q) and we see the Meromorphic Product

$$h(x) = \prod_{n=1}^{\infty}\left(\frac{x-a_n}{x-b_n}\right)$$

has exactly the same zeros and the same poles as f. But then it is easily seen that f/h is a constant.

Indeed, h is invertible in $H_b(M_\rho(h))$ so that $f(x)/h(x)$ has a Mittag-Leffler series in the infraconnected set $M_\rho(h)$ [R]. But f/h has no pole in $\mathbb{K}$ and then its Mittag-Leffler series is reduced to a constant α. Finally $\alpha = 1$, because

$$\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} h(x) = 1.$$

Now let us consider the general case when f has a finite number of zeros in $d(0,1)$. Then f factorizes in the form $P(x)g(x)$ where P is a monic polynomial (of degree q), all the zeros of which are in $d(0,1)$ and g is an element of $H_b(\Lambda_\rho(\mathcal{B}))$ such that $g(x) \neq 0$ for all $x \in d(0,1)$ and

$$\lim_{|x|\to\infty} x^q g(x) = 1.$$

Let Q be a q-degree monic polynomial whose zeros belong to $\mathbb{K} \setminus d(0,1)$. Obviously we have

$$\lim_{|x|\to\infty} Q(x)g(x) = 1$$

again. Then $Q(x)g(x)$ is a Meromorphic Product associated to $\mathcal{B}$.

Since P and Q have the same degree,

$$f(x) = \frac{P(x)}{Q(x)}(Q(x)g(x))$$

is also a Meromorphic Product associated to $\mathcal{B}$ and that ends the proof of Theorem 1. □

Proof of Theorem 2. Since $\mathcal{F}_\rho$ is a T-filter, the Theorem 8 of [S4] shows that there exists $h \in H_b(\Lambda_\rho(\mathcal{B}))$ in the form

$$h = \sum_{j=1}^{\infty} \frac{\theta_j}{x - b_j}$$

whose restriction to $d(0,1)$ is equal to g and $\Lambda_\rho(\mathcal{B})$ has a T-filter $\mathcal{F}_\rho$. The sequence of holes $T_n = d(b_n, \rho^-)$ then makes an idempotent T-sequence $(T_n, 1)_{n\in\mathbb{N}^*}$ [S1]. Now Theorem 5 of [S3] shows there exists $\bar{g} \in H_b(\Lambda_\rho(\mathcal{B}))$ whose restriction to $d(0,1)$ is also equal to g, whose poles are simple, and whose limit is equal to 1 when $|x|$ goes to ∞.

By proposition 1, g is in the form

$$g = 1 + \sum_{n=1}^{\infty} \frac{a_n}{x - b_n};$$

and by Theorem 1, g is then a Meromorphic Product associated to $\mathcal{B}$. □

Proof of Theorem 3. By Theorem 2 there exists a Meromorphic Product $\bar{g}$ associated to $\mathcal{B}$, in the form

$$\bar{g} = \prod_{n=1}^{\infty} \left(\frac{x - a_n}{x - b_n} \right)$$

whose restriction to $d(0,1)$ is equal to g.

Since $\lim_{n\to\infty}(a_n - b_n) = 0$, it is easily seen that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{x - a_n} - \frac{1}{x - b_n} \right) = \sum_{n=1}^{\infty} \frac{b_n - a_n}{(x - a_n)(x - b_n)}$$

is convergent in $H_b(M_\rho(g))$.

Consider now the sequence

$$g_n = \prod_{j=1}^{n} \frac{x - a_j}{x - b_j}$$

that converges to $\bar{g}$ in $H_b(\Lambda_\rho(\mathcal{B}))$. By Corollary of [E5] we know that the sequence $\bar{g}'_n$, converges to g'.

But

$$g'_n = g_n \sum_{j=1}^{n} \left(\frac{1}{x - a_j} - \frac{1}{x - b_j} \right)$$

and therefore at the limit, $\bar{g}' = \bar{g}\bar{\omega}$. In particular, in $d(0,1)$, $\bar{\omega}(x) = \omega(x)$.

In addition, $\bar{g}$ is invertible in each algebra $H_b(\Lambda_\rho(\mathcal{B}))$ hence by results of [E6], $\bar{g}$ generates the linear space of the solutions of the equation $\mathcal{E}(\bar{\omega})$ in $H_b(\Lambda_\rho(\mathcal{B}))$ and that completes the poof of Theorem 3. □

Proof of Theorem 4. First let us assume a) is satisfied and let $g(x)$ be the Meromorphic Product

$$\prod_{n=1}^{\infty} \left(\frac{x - a_n}{x - b_n} \right).$$

It belongs to $H(d(0,1))$ and it is a solution of the equation $\mathcal{E}(\omega)$ different from zero, hence b) is true. Now, we suppose b) is true, and let g be a solution of $\mathcal{E}(\omega)$ (different from zero) that belongs to $H(D)$.

By Theorem 4, there exist sequences (b_n) and (a_n) in $\mathbb{K} \setminus d(0,1)$ with

$$\lim_{n\to\infty} |b_n| = 1, \qquad \lim_{n\to\infty} a_n - b_n = 0,$$

such that

$$\prod_{n=1}^{\infty}\left(\frac{x-a_n}{x-b_n}\right) = g(x)$$

whenever $x \in D(0,1)$, and then

$$\frac{g'(x)}{g(x)} = \sum_{n=1}^{\infty}\left(\frac{1}{x-a_n} - \frac{1}{x-b_n}\right),$$

hence a) is satisfied. □

Proof of the Corollary. We know that the solutions of the differential equation $y' = \lambda y$ in the disk $d(0,r)$ are in the form $A\exp(\lambda x)$ and then the series

$$\sum_{n\in\mathbb{N}} \frac{(\lambda x)^n}{n!}$$

is convergent if and only if $|x| < |\lambda|^{-1}p^{-1/(p-1)}$. But then, $\exp(\lambda x) \in H(d(0,1))$ if and only if $|\lambda| < p^{-1/(p-1)}$.

Now by Theorem 4, $\mathcal{E}(\lambda)$ has solutions if and only if a) is satisfied hence a) and b) are equivalent. □

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