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# A special type of triangulations in Numerical Nonlinear Analysis

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Abstract

To calculate the zeros of a map $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ we consider the class of triangulations of $\mathbb{R}^n$ so that a certain point belongs to a simplex of fixed diameter and dimension. In this paper two types of this new class of triangulations are constructed and shown to be useful to calculate zeros of piecewise linear approximations of $f$.

## 1. Introduction

In the following $\mathbb{R}$ denotes the set of real numbers, $D$ an open subset of $\mathbb{R}^{2n}$ and $C^\infty(D)$ the set of all mappings from $D$ into $\mathbb{R}^{2n}$ with derivatives of every order. To solve

$$F(u) = 0, \quad F : D \subseteq \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \quad F \in C^\infty(D),$$

with a finite number of zeros, let $G$ be a mapping from $D \subseteq \mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$, $G \in C^\infty(D)$ all whose zeros are known. We construct the homotopy

$$H : D \times [0,1] \longrightarrow \mathbb{R}^{2n}; \quad H(u,t) = (1-t) \times G(u) + t \times F(u).$$

We suppose that zero is a regular value for $H$, which is not a substancial restriction acoording to Brown's corollary of Sard's theorem [4]. Let us study the inverse image of zero, using the following theorems (with $m = 2n+1$, $p = 2n$, $M = D \times [0,1]$ and $N = \mathbb{R}^{2n}$).

**1-1-1.Theorem**

*If $H$ is a smooth map $H : M \to N$ from an $m$-manifold $M$ with boundary to a $p$-manifold $N$, where $m > p$, and $Y$ is a regular value for $H$ and for the restriction $H_{\delta M}$, then $H^{-1}(Y)$ is a smooth $(m-p)$-manifold with boundary. Furthermore the boundary $\delta(H^{-1}(Y))$ is the intersection of $H^{-1}(Y)$ with $\delta M$, and its dimension is $m - n - 1$ [4].*

**1-1-2. Theorem**

*Any smooth connected 1-manifold is diffeomorphic either to the circle $\mathbb{S}^1$ or to some real interval; that is, it is either a loop or a path [4].*

To calculate the zeros of $F$ we consider the connected component of $H^{-1}(0)$ from $t = 0$ to $t = 1$; $H^{-1}(0)$ is a union of paths and loops, according to (1-1-1) and (1-1-2). By differentiation of $H(u,t) = 0$ with respect to the variable $s$, $s$ being the arc and $\{(u,t)\}$ lying in a connected component of $H^{-1}(0)$, we obtain:

$$\sum_{i=1}^{2n} \frac{\partial H}{\partial u_i} \times \frac{du_i}{ds} + \frac{\partial H}{\partial t}\frac{dt}{ds} = 0.$$

It follows [2] that

$$\frac{du_i}{ds} = (-1)^i \times \det\left(H'_{-i}(u,t)\right), \qquad i = 1, ..., 2n, \tag{1}$$

$$\frac{dt}{ds} = (-1)^{2n+1} \times \det\left(H'_{-2n-1}(u,t)\right), \tag{2}$$

where $H'_{-i}$ is the Jacobian of $H$ with the $i$-th column deleted; so if the sign of $\det(H'_{-2n-1}(u,t)$ is constant, (2) implies that $t = t(s)$ is monotonous.

**1-1-3.** We want to solve $F^*(z) = 0$, where $F^* : D \subseteq \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic with a finite number of zeros ($\mathbb{C}$ denote the set of complex numbers). Let $G^*$ be a holomorphic mapping from $D \subseteq \mathbb{C}^n$ into $\mathbb{C}^n$ whose zeros are known. We construct the homotopy

$$H^* : D \times [0,1] \longrightarrow \mathbb{C}^n, \qquad H^* = (H_1^*, \dots H_n^*),$$

$$H^*(z,t) = (1-t) \times G^*(z) + t \times F^*(z).$$

Let $H : D \times [0,1] \to \mathbb{R}^{2n}$ be the mapping defined by $H = (\Re H_1^*, \Im H *_1, \dots, \Im H_n^*)$, and analogously

$$F : D \subseteq \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \qquad \text{with } F = (\Re F_1^*, \Im F_1^*, \dots, \Im F_n^*);$$

$$G : D \subseteq \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \qquad \text{with } G = (\Re G_1^*, \Im G_1^*, \dots, \Im G_n^*).$$

It is obvious that the equations $H^*(z,t) = 0$ and $H(u,t) = 0$ are equivalent.

**1-1-4. Theorem** [2]

*If $H^*$ is holomorphic, then* $\det(H'(u,t)) \geq 0$.

It follows that the paths of $H^{-1}(0)$ are monotonous with respect to $s$ and there are no loops. If we choose $G$ such that for $0 \leq t \leq 1$, each path starting in $(X^*, 0)$ with $G(X^*) = 0$, finishes in $(X^{**}, 1)$, $H(X^{**}, 1) = 0$. So, solving the problem of initial values (1), (2), $G(X^*) = 0$ we deduce that for each solution of $G(u) = 0$ we obtain a solution of $F(u) = 0$, and perhaps some paths which diverge to infinity. For a sufficient condition of non-divergence of the paths see [2, p. 349]. In the paragraph 1-2 we relate $H^{-1}(0)$ to $\theta$, $\theta$ being a piecewise linear approximation of $H$.

**1-2.** From the definition of topological degree $d(\theta, s, 0)$ at zero of a continuous mapping $\theta$ relative to an open bounded set $s$ [5] and the second section of theorem 1-3-5 [5, p. 16] we have: if $0 \notin \theta(\delta s)$ there is $\varepsilon > 0$ so that $d(H, s, 0) = d(\theta, s, 0)$ whenever $\|H - \theta\|_1 < \varepsilon$. It is also proved in [5] that if $d(H, s, 0) \neq 0$ there exists $X \in s$, so that $H(X) = 0$. Also if there is in $s$ an odd number of zeros of $\theta$, and $0 \notin \theta(\delta s)$ then $d(\theta, s, 0) \neq 0$; and, if there is no zero in $s$, $d(\theta, s, 0) = 0$.

We denote by $\delta s$ the border of $s$,

$$\|H\|_1 = \sup_{X \in s} \|DH(X)\| + \sup_{X \in s} \|H(X)\|.$$

So if $\theta$ and $H$ are sufficiently closed there is a tube for each connected component of $H^{-1}(0)$ that contains a connected component of $\theta^{-1}(0)$.

Our next theorem gives a sufficient condition to secure that $\theta$ is sufficiently close to $H$ (i.e. the triangulation is fine enough).

**1-2-1.Theorem**

*Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a $C^2(s)$ function and $\theta : D \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a piecewise linear approximation of $f$ relative to a regular triangulation of the domain of $f$, such that $\theta$ has a zero in the simplex $s$, and*

$$2M\varepsilon^2 < d_\infty(0, \theta(\delta s));$$

*$M$ being a upper bound for the second derivative of $f$ in $s$, and $\varepsilon$ the diameter of $s$. Then $f$ has also a zero in $s$.*

*Proof.* From the definition of topological degree, it follows that if $f$ has a zero in $s$, $d(f,s,0) = 1$, supposing that zero is not a critical value for $f$. We now apply the following two properties of the topological degree [1]:

1) if $d(f,s,0) \neq 0$ then $f^{-1}(0) \neq \emptyset$,
2) $d(\cdot,s,0)$ is constant on $\{f\} \in C(\bar{s})$ if

$$\sup_{X \in s} |f(X) - \theta(X)|_\infty < r,$$

with $r = d_\infty(0, \theta(\delta s))$.

We now calculate a bound for

$$\max_{X \in \bar{s}} \left| f(X) - \theta(X) \right|_\infty = \max_{X \in \bar{s}} \left( \left| f_1(X) - \theta_1(X) \right|, \ldots, \left| f_n(X) - \theta_n(X) \right| \right).$$

For this purpose we suppose that

$$\max_{X \in \bar{s}} \left| f(X) - \theta(X) \right|_\infty = \left| f_j(X^*) - \theta_j(X^*) \right|, \qquad X^* \in s,\ j \in 1, \ldots, n,$$

and we construct the function

$$f^*(X) : \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

$$f_k^*(X) = f_k^*(X, Y^*) = f_k(Y^*) + \sum_{i=1}^{n} \frac{\partial f_k(Y^*)}{\partial X_i} \times h_i, \qquad k = 1, \ldots, n;$$

with

$$X \in \bar{s}; \qquad s = \langle Y^0, \ldots, Y^n \rangle;$$

$$X = \sum_{i=0}^{n} \lambda_i Y^i; \qquad \sum_{i=0}^{n} \lambda_i = 1;\ \lambda_i \geq 0,\ i = 0, \ldots, n,$$

and

$$h_k = X_k - Y_k^* = \sum_{i=0}^{n} \lambda_i Y_k^* - Y_k^*, \qquad k = 1, \ldots, n.$$

To maximize $|f_j^*(X) - \theta_j(X)|$, $X \in \bar{s}$, we maximize

$$\left| \lambda_0 \left[ \frac{\partial f_j(Y^*)}{\partial X_1} Y_1^0 + \cdots + \frac{\partial f_j(Y^*)}{\partial X_n} Y_n^0 - Y_j^0 \right] + \cdots \right.$$
$$\left. + \lambda_n \left[ \frac{\delta f_j(Y^*)}{\delta X_1} Y_1^n + \cdots + \frac{\partial f_j(Y^*)}{\partial X_n} Y_n^n - Y_j^n \right] + \left[ f_j(Y^*) - \sum_{i=1}^{n} \frac{\partial f_j(Y^*)}{\partial X_i} Y_i^* \right] \right|,$$

or, in reduced form, to maximize $|A_0\lambda_0 + \cdots + A_n\lambda_n + B|$ with the conditions $\sum \lambda_i = 1$; $\lambda_i \geq 0$, $i = 0, \ldots, n$, with

$$A_k = \sum_{i=1}^{n} \frac{\delta f_j(Y^*)}{\delta X_i} \times Y_i^k - Y_j^k, \qquad k = 0, \ldots, n,$$

$$B = f_j(Y^*) - \sum_{i=1}^{n} \frac{\partial f_j(Y^*)}{\partial X_i} \times Y_i^*.$$

It is equivalent to calculate the

$$\max \begin{cases} |\max\{A_0, \ldots, A_n\} + B| \\ |\min\{A_0, \ldots, A_n\} + B| \end{cases}$$

so this maximum occurs for a $\lambda_i = 1$, $i \in \{0, \ldots, n\}$. Therefore

$$\max_{X \in \bar{s}} |f_j^*(X) - \theta_j(X)|$$

occurs at a vertex. Let this vertex be $Y^i$, we have

$$\theta_j(Y^i) = f_j(Y^i)$$

and

$$\max_{X \in \bar{s}} |f_j^*(X) - \theta_j(X)| = |f_j^*(Y^i - f_j(Y^i)|,$$

where

$$|f^*(Y^*, X) - f(X)|_\infty \leq M \times \varepsilon^2.$$

We have:

$$\begin{aligned}
\max_{X \in \bar{s}} |f(X) - \theta(X)|_\infty &\leq \max_{X \in \bar{s}} |f(X) - f^*(Y^*, X)|_\infty + \max_{X \in \bar{s}} |f^*(Y^*, X) - \theta(X)|_\infty \\
&\leq M\varepsilon^2 + |f^*(Y^*, Y^i) - f(Y^i)|_\infty + |f(Y^i) - \theta(Y^i)|_\infty \\
&\leq 2 \times M \times \varepsilon^2,
\end{aligned}$$

because

$$|f(Y^i) - \theta(Y^i)|_\infty = 0.$$

Being

$$\sup_{X \in s} |f(x) - \theta(X)|_\infty \leq \max_{X \in \bar{s}} |f(X) - \theta(X)|_\infty$$

it follows that:

$$\sup_{X \in s} |f(X) - \theta(X)|_\infty \leq 2M\varepsilon^2.$$

So, when $2 \times M \times \varepsilon^2 < d_\infty(0, \theta(\delta s))$, applying 1) and 2), if there is a zero of $\theta$ in $s$, there is also a zero of $f$ in $s$. $\square$

*Remarks.* 1) An $n$-simplex $s$ of $\mathbb{R}^n \times [0,1]$ is contained in a hyperplane; it follows that $H|_s$, $\theta|_s$ are mappings from $\mathbb{R}^n$ into $\mathbb{R}^n$.

2) If for each simplex $s$ of the triangulation, $\theta(s) \subset H(s)$ is verified, then the condition of Theorem 1-2-1 is necessary and sufficient.

### 1-2-2. Piecewise linear approximation

We need a domain of $\mathbb{R}^{2n} \times [0,1]$, whose closure contains the connected components of $H^{-1}(0)$, zero being a regular value for $H$; so that the intersections of the closure of these domains with the hyperplane $X_{2n+1} = 1$ are neighborhoods of the solutions of our problem. To obtain it, we construct a regular triangulation $K$ of $\mathbb{R}^{2n} \times [0,1]$, so that the point $X^*$ satisfying $G(X^*) = 0$, is contained in the interior of one of its $2n$-simplices $s$; and the diameter of the $2n$-simplices is less than the admissible error. We construct a piecewise-linear approximation $\theta$ of $H$ relative to the triangulation $K$:

$$\theta : D \times [0,1] \longrightarrow \mathbb{R}^{2n}$$

$$\theta(X) = \begin{cases} H(u,t) & \text{for } (u,t) \in \{\text{vértices } K\} \\ \sum_{i=0}^{k} \lambda_i \times H(Y^i) & \text{for } X = (u,t) \in \langle Y^0, \dots, Y^k \rangle, \end{cases}$$

$\lambda_i$ being the barycentric coordinates of $X$ concerning $\{Y^0, \dots, Y^k\}$. The study of $\theta^{-1}(0)$ is the usual one in this class of algorithms.

### 1-2-3. Construction of the algorithm

There are two ways:

1) We follow a conected component of $\theta^{-1}(0)$ lying on a connected component of $\theta^{-1}(0)$. The mesh of the triangulation is fine enough to guarantee the desirable approximation between the zeros of both functions. In the first step we know the existence of a zero of $G$ in the interior of a specific $2n$-simplex $s$, where there is also a zero of $\theta$ if the rank of the matrix

$$A = \begin{pmatrix} 1 & \dots & 1 \\ H(Y^0) & \dots & H(Y^2 n) \end{pmatrix},$$

is maximum, and $\lambda_i > 0$, $i = 0, \dots, 2n$ for

$$A \times (\lambda_0, \dots, \lambda_{2n})^t = (1, 0, \dots, 0)^t, \qquad s = \langle Y0, \dots, Y^{2n} \rangle.$$

Now it is known the existence of a zero of $\theta$ in $s$. We construct an algorithm following a connected component $m$ of $\theta^{-1}(0)$ and we verify if the $2n$-simplex $\tau_i$, that contains the intersection of $m$ with the hyperplane $t = 1$, contains a zero of $F$. For this verification we use Theorem 1-2-1.

2) In each $2n$-simplex $\tau_i$ which is intersected by a connected component of $\theta^{-1}(0)$, we verify the existence of a zero of $H$ with the aid of the Theorem 1-2-1. In the negative case we return to the former $2n$-simplex $\tau_{i-1}$ and change the mesh of the triangulation. We use for that a triangulation of $\tau_{i-1}$, and we use section (3) of Theorem 5-1-3 [5]: if $\tau_{i-1}$ is the disjoint union of the sets $\tau_{i-1,1}, \ldots, \tau_{i-1,N}$ then

$$d(H, \tau_{i-1}, 0) = \sum_{j=1}^{N} d(H, \tau_{i-1,j}, 0),$$

where

$$d(H, \tau_{i-1}, 0) = 1.$$

We start the study of each connected component of $\theta^{-1}(0)$ from a $2n$-simplex that contains in its interior a zero of $G$. In this kind of algorithms, the usual approach is, for a certain size of mesh and a certain point, to calculate the simplex that contains a certain point, which may not be a $2n$-simplex. This second approach is simpler: given a certain size of mesh and a certain point, the domain is triangulated so that the point is contained in a simplex whose dimension is one less the maximum feasible dimension. The details follow.

## 2. Regular triangulations

A simplicial complex $K$ consists of:

a) a set $\{Y\}$ whose elements are called vertices,

b) a set $\{s\}$ whose elements are called simplices and verify that: 1) one vertex is a simplex and, 2) any nonempty subset of a simplex is a simplex.

$|K|$ is the set of the mappings $\{\alpha\}$ from the vertices of $K$ into $[0,1]$ satisfying the following conditions: a) if $\alpha \in |K|$, the set of those $Y$ such that $\alpha(Y) \neq 0$ is a simplex of $K$, b) $\sum_{Y \in K} \alpha(Y) = 1$.

Define an open simplex $\langle s \rangle$ as the set of those $\alpha \in |K|$ such that $\alpha(Y) \neq 0 \Leftrightarrow Y \in s$.

The following definitions are given in [7]: A $j$-simplex $s$ is the relative interior of the convex hull of $j+1$ affinely independent points:

$$s = \langle Y_0, \ldots, Y_j \rangle = \left\{ \sum_{i=0}^{j} \lambda_i \times Y^i : \lambda_i > 0,\ i = 0, \ldots, j,\ \sum_{i=0}^{j} \lambda_i = 1 \right\}.$$

$G$ is a triangulation of a convex set $C \subset \mathbb{R}^n$ if: a) $G$ is a collection of $n$-simplices, b) the subsimplices of all the $n$-simplices of $G$ form a partition of $C$, and c) each $X \in C$ has a neighborhood that intersects only a finite number of simplices. We add the following definition: $G$ is regular if all its simplices have the same diameter.

Let $i$ be the linear extension to $|K|$ of the identity vertex mapping, and let $K$ be a simplicial complex with the following properties:

1) The vertices of $K$ are points of $\mathbb{R}^n$.
2) Each simplex is a subsimplex of a $n$-simplex. (1)
3) Each $(n-1)$-simplex is contained in two $n$-simplices.
4) The $(n+1)$-points that establish an $n$-simplex are affinely independent.
5) The diameter of the convex hull of the vertices of a simplex is fixed.

Then we have the following result.

**2-1. Theorem**

*The open simplex of vertices $\langle Y^0, \dots, Y^j \rangle$ may be identified with $i(\langle s \rangle)$, $\langle s \rangle \in |K|$.*

*Proof.* A vertex $Y^i$ as an element of $|K|$ is the characteristic function

$$Y^i(Y^j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

So for all $\alpha \in |K|$,

$$\alpha = \sum_{Y \in K} \alpha(Y) \times Y,$$

and for all $\alpha \in \langle s \rangle$,

$$\begin{aligned} i(\alpha) &= i\left(\sum_{i=0}^{j} \alpha(Y) \times Y\right) \\ &= i\left(\sum_{i=0}^{j} (\alpha(Y^i) \times Y^i\right) \\ &= \sum_{i=0}^{j} \alpha(Y^i) \times i(Y^i) \\ &= \sum_{i=0}^{j} \alpha(Y^i) \times Y^i, \end{aligned}$$

where

$$\sum_{i=0}^{j} \alpha(Y^i) = 1, \qquad \alpha(Y^i) > 0, \qquad i = 0, \ldots, j.$$

Hence, for each $\alpha$ belonging to $\langle s \rangle$, $i(\alpha)$ belongs to the interior of the convex hull of $\{Y^0, \ldots, Y^j\}$. Therefore $i(\langle s \rangle)$ is contained or is equal to the interior of the convex hull of $\{Y^0, \ldots, Y^j\}$. Reciprocally for each $Y$ belonging to the interior of the convex hull of $\{Y^0, ..., Y^n\}$,

$$Y = \sum_{i=0}^{j} \lambda_i \times Y^i,$$

with

$$\sum_{i=0}^{j} \lambda_i = 1, \qquad \lambda_i > 0, \ i = 0, \ldots, j.$$

Let $\beta \in |K|$ be such that

$$\beta(Y^i) = \begin{cases} \lambda_i, & i = 0, \ldots, j \\ \beta(Y^i) = 0, & i \neq 0, \ldots, j. \end{cases}$$

Then $\beta \in \langle s \rangle$, and

$$i(\beta) = i\left(\sum_{Y^i \in K} \alpha(Y^i) \times Y^i\right) = Y$$

hence $Y \in i(\langle s \rangle)$. Therefore the interior of the convex hull of $\{Y^0, \ldots, Y^j\}$ is equal to $i(\langle s \rangle)$. □

**2-2. Theorem**

*A regular triangulation of $\mathbb{R}^n$ consits of the set $\{i(\langle s \rangle)\}$, from the pair $(K, i)$.*

*Proof.* $\{i(\langle s^n \rangle)\} = G^n$ or set of $n$-simplices, $\{i(\langle s^{n-1} \rangle)\}$ or set of $(n-1)$-simplices, $\ldots$, $\{i(Y^i)\} = G^0$ or set of 0-simplices or vertices of the triangulation. $\{\{\langle s^i \rangle\} : i = 0, \ldots, n\}$ is a partition of $|K|$. As $i$ is a homeomorphism, $i(\{\{\langle s^i \rangle\} : i = 0, \ldots, n\})$ is a partition of $\mathbb{R}^n$. Since the number of simplices with a common vertex is finite, each $X \in \mathbb{R}^n$ has a neighborhood that intersects only a finite number of simplices. So $G^n$ is a triangulation. □

*1. First triangulation*

**2-3. Theorem**

*Given a point $X \in \mathbb{R}^n$, let $\{Y^0, \dots, Y^{n+1}\}$ be defined by*

$$Y_i^0 = X_i - (n - i + 1) \times \delta/(n+1), \qquad i = 1, \dots, n,$$

$$Y_{n+1}^0 = X_{n+1},$$

*and*

$$Y^i = Y^{i-1} + \delta \times u^i,$$

*where $u^i$ is the i-th unit vector. The interior of the convex hull of $\{Y^0, \dots, Y^{n+1}\}$ is a $(n+1)$-simplex $s$ of finite diameter $\delta$, and $X$ belongs to the interior of the subsimplex $\langle Y^0, \dots, Y^n \rangle$.*

*Proof.* The points

$$Y^0 = \left(Y0_1, \dots, Y_{n+1}^0\right),$$

$$Y^1 = Y^0 + \delta \times u^1 = \left(Y_1^0 + \delta, Y_2^0, \dots, Y_{n+1}^0\right),$$

$$Y^2 = Y^1 + \delta \times u^2 = \left(Y_1^0 + \delta, Y_2^0 + \delta, Y_3^0, \dots, Y_{n+1}^0\right),$$

$$\dots\dots\dots$$

$$Y^{n+1} = Y^n + \delta \times u^n = \left(Y_1^0 + \delta, \dots, Y_n^0 + \delta, Y_{n+1}^0 + \delta\right),$$

are affinely independent, because $\det B = \delta^{n+1}$, where $B$ is the column matrix of these points. It follows that the interior of the convex hull of these points is a simplex $s$. The diameter of $s$ is

$$\sup \left\{ \|Y^* - Y^{**}\|_\infty : Y^*, Y^{**} \in s \right\} = \delta.$$

We now calculate $Y^0$ from $X$ and $\delta$ with the condition that the barycentric coordinates of $X$ relative to $\{Y^0, \dots, Y^n\}$ are

$$\lambda_0 = \dots = \lambda_n = 1/(n+1), \qquad \sum_{i=0}^{n} \lambda_i = 1, \ \lambda_i > 0, \ i = 0, \dots, n.$$

This implies that $X$ belongs to the interior of the subsimplex $sb = \langle Y^0, \dots, Y^n \rangle$:

$$
\begin{aligned}
(X_1, \dots, X_n) &= \frac{1}{n+1}(Y_1^0, \dots, Y_n^0, X_{n+1}) + \frac{1}{n+1}(Y_1^0 + \delta, Y_2^0, \dots, X_{n+1}) + \cdots \\
&\quad + \frac{1}{n+1}(Y_1^0 + \delta, \dots, Y_n^0 + \delta, X_{n+1}), \\
X_1 &= \frac{1}{n+1}((n+1)Y0_1 + n\delta), \\
X_2 &= \frac{1}{n+1}((n+1)Y_2^0 + (n-1)\delta), \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
X_n &= \frac{1}{n+1}((n+1)Y_n^0 + \delta), \\
X_{n+1} &= Y_{n+1}^0.
\end{aligned}
$$

So,

$$
\begin{aligned}
Y_i^0 &= X_i - \frac{(n-i+1)\delta}{n+1}, \qquad i = 1, \dots, n, \\
Y_{n+1}^0 &= X_{n+1}. \ \square
\end{aligned}
$$

**2-4. Proposition**

*Given a point $Y^* \in \mathbb{R}^{n+1}$, let $K^0$ be the set*

$$
K^0 = \left\{ Y \in \mathbb{R}^{n+1} : Y = Y^* + \delta \sum_{i=1}^{n+1} k_i u^i, \ k_i \in \mathbb{Z}, \ i = 1, \dots, n+1, \ \delta \in \mathbb{R}^+ \right\},
$$

*where $u^i$ is the $i$-th unit vector.*

*The relative interior of the convex hull of $n+1$ points of $K^0$ that satisfy*

$$
Y^i = Y^{i-1} + \delta \times u^{\pi(i)}, \tag{2}
$$

*with $\pi$ a permutation of $\{1, \dots, n+1\}$, is a simplex $s^{n+1} = \langle Y^0, \dots, Y^{n+1} \rangle$. The set of all the simplices so constructed is a triangulation of $\mathbb{R}^{n+1}$.*

*Proof.* $\{K^0, \{s^{n+1}\}\}$ defines a simplicial complex $K$ with the properties:

1) $K^0$ are points of $\mathbb{R}^{n+1}$.
2) Each simplex is a subset of a $(n+1)$-simplex.
3) Each $n$-simplex is contained in two $(n+1)$-simplices.

Let us consider for this a $(n+1)$-simplex $(Y^0,\pi) = \langle Y^0,\dots,Y^{n+1}\rangle$. It has three types of $n$-subsimplices

$$\langle Y^0,\dots Y^{i-1},Y^{i+1},\dots,Y^{n+1}\rangle, \qquad i \neq 0,\ i \neq n+1,$$
$$\langle Y^1,\dots,Y^{n+1}\rangle,$$
$$\langle Y^0,\dots,Y^n\rangle.$$

Each of these subsimplices is a subsimplex only of $(Y^0,\pi)$ and another $(n+1)$-simplex:

First let us remark that, in a $(n+1)$-simplex $s^{n+1}$ with the ordination (2), two consecutives vertices have only one different coordinate, whose difference is $\delta$. Let be $sb = \langle Y^0,\dots,Y^{i-1},Y^{i+1},\dots,Y^{n+1}\rangle$, this subsimplex has only two consecutive vertices $Y^{i-1}$, $Y^{i+1}$ with two different coordinates, instead of one. So it is a subsimplex of the $(n+1)$-simplices that adds a vertex with one of these two coordinates increased. That is: $(Y^0,\pi)$ and $(Y^0,(\pi(1),\dots,\pi(i+1),\pi(i),\dots,\pi(n+1)))$.

Let $< Y^1,\dots,Y^{n+1} >= sb$. Two consecutive vertices have only one different coordinate, being $\delta$ the diference between the two. All the vertices of $sb$ have one equal coordinate. To construct a $(n+1)$-simplex from $\{Y^1,\dots,Y^{n+1}\}$ we add a vertex with this common coordinate diminished or increased. So $sb$, is only a subsimplex of

$$(Y^0,\pi) \qquad \text{and} \qquad \big(Y^0 + \delta \times u^{\pi(1)}; (\pi(2),\dots,\pi(n+1),\pi(1))\big).$$

Finally let $sb = \langle Y^0,\dots,Y^n\rangle$; with a similar argument we may prove that $sb$ is a subsimplex of

$$(Y^0,\pi) \qquad \text{and} \qquad \big(Y^0 - \delta \times u^{n+1}; (\pi(n+1),\pi(1),\dots,\pi(n))\big).$$

4) Given the pair $(Y^0,\pi)$, $\{Y^0,\dots,Y^{n+1}\}$ are affinely independent, because $\det B \neq 0$.

5) The infinite diameter of $(Y^0,\pi)$ is equal to $\delta$. □

*Remark.* We construct the 1-triangulation from $Y^*$ equal to $Y^0$ defined in 2-3; $Y^0$ is calculated from a certain diameter of mesh $\delta$ and any point $X$.

*2. Second triangulation*

Given a point $X \in \mathbb{R}^{n+1}$, let us construct the simplex $s^n$ that contains $X$, given in Proposition 2-3. We now write $Y^i = Y^{i-1} + \delta \times u^{\pi(i)}$, $i > 1$ in the form

$$Y^i = Y^{i-1} + s_{\pi(i)} \times u^{\pi(i)},$$

with $s$ the vector sign $(+1,\dots,+1)$. Let $Y^* = Y^0$.

**2-5 Theorem**

*Given a point $Y^*$, let $K^0$ be the set*

$$\left\{ Y \in \mathbb{R}^{n+1} : Y = Y^* + \delta \sum_{i=1}^{n+1} k_i \times u^i,\ k_i \in \mathbb{Z},\ i = 1, \dots, n+1,\ \delta \in \mathbb{R}^+ \right\}$$

*where $u^i$ is the $i$-th unit vector. Let $\{Y^0, \dots, Y^{n+1}\}$, $Y^i \in K^0$, $i = 1, \dots, n+1$ be the points*

*1) $Y^0 = Y^* + \delta(k_1, \dots, k_{n+1})^t$, $k_i \in \mathbb{Z}$, $i = 1, \dots, n+1$*

*2) $Y^i = Y^{i-1} + \delta s_{\pi(i)} u^{\pi(i)}$, $i = 1, \dots, n+1$,*

*with $\pi$ a permutation of $1, \dots, n+1$, and $s$ a $(n+1)$-vector sign.*

*The relative interior of the convex hull of these $n+2$ points is a $(n+1)$-simplex $s^{n+1}$.*

*The set $s^{n+1}$ of all the $(n+1)$-simplices so constructed is a triangulation of $\mathbb{R}^{n+1}$.*

*Proof.* With the symbol $(Y^0, \pi, s)$ we now denote the relative interior of the convex hull of $\{Y^0, \dots, Y^{n+1}\}$, formed with the rule

$$Y^i = Y^{i-1} + \delta s_{\pi(i)} u^{\pi(i)};$$

it is a simplex, because the rank of $B$ is maximum. The diameter of $(Y^0, \pi, s)$ is equal to $\delta$.

$\{K^0, \{s^{n+1}\}\}$ defines a complex with the properties (1); i.e.:

1) $K^0$ are points of $\mathbb{R}^{n+1}$.

2) Each simplex is a subset of a $(n+1)$-simplex.

3) Each $n$-simplex is contained in two $(n+1)$-simplices.

Let us consider a $(n+1)$-simplex $(Y^0, \pi, s) = \{Y^0, \dots, Y^{n+1}\}$. It has three types of $n$-simplices

$$\{Y^0, \dots, Y^{i-1}, Y^{i+1}, \dots, Y^{n+1}\}, \qquad i \neq 0, i \neq n+1,$$
$$\{Y^1, \dots, Y^{n+1}\},$$
$$\{Y^0, \dots, Y^n\}.$$

Each of these subsimplices is a subsimplex only of $(Y^0, \pi, s)$ and of other $(n+1)$-simplex: $\{Y^0, \dots, Y^{i-1}, Y^{i+1}, \dots, Y^{n+1}\}$, with a similar reasoning as 2-4 we conclude that it is a subsimplex of $(Y^0, \pi, s)$ and of $(Y^0; (\pi(1), \dots, \pi(i+1), \pi(i), \dots, \pi(n+1)); s)$.

Similarly $\{Y^0, \dots, Y^n\}$ is only a subsimplex of $(Y^0, \pi, s)$ and of $(Y^0, \pi, s - 2s_{\pi(n+1)}\delta u^{\pi(n+1)})$ and finally $\{Y^1, \dots, Y^{n+1}\}$ is a subsimplex of

$$(Y^0, \pi, s) \qquad \text{and} \qquad \left(Y^0 + 2s_{\pi(1)}\delta u^{\pi(1)}; \pi; s - 2s_{\pi(1)}\delta u^{\pi(1)}\right).$$

4) Given $(Y^0, \pi, s)$, the points $\{Y^0, \dots, Y^{n+1}\}$ are affinely independent.

5) The diameter of $(Y^0, \pi, s)$ is equal to $\delta$. □

*Remark.* There exists for each diameter of mesh and starting point one and only one affinity transforming the first triangulation in the Freudental's triangulation. Also, there exists for each diameter of mesh and starting point one and only one affinity transforming the second triangulation in the Tucker's triangulation.

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