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Tensor stable Fréchet and (DF)-spaces

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ABSTRACT

In this paper we introduce and investigate classes of Fréchet and (DF)-spaces which constitute a very general frame in which the problem of topologies of Grothendieck and some related dual questions have a positive answer. Many examples of spaces in these classes are provided, in particular spaces of sequences and functions. New counterexamples to the problems of Grothendieck are given.

1. Introduction

In his work about Fréchet and (DF)-spaces and tensor products in the fifties ([25], [26]) Grothendieck studied properties of function spaces from abstract considerations. The function spaces in question included spaces of sequences, analytic functions, infinitely differentiable functions, distributions and solutions spaces of certain linear partial differential equations among others. Tensor products constitute a very useful tool to treat function spaces and in particular it is important to know the structure of the bounded subsets in the projective tensor product of two Fréchet spaces and also the locally convex structure of the injective tensor product of duals of Fréchet spaces. Grothendieck posed the following questions:

(a) Let E and F be Fréchet spaces. Can every bounded subset B of the projective tensor product $E\widehat{\otimes}_\pi F$ be boundedly lifted, i.e. are there bounded subsets C, D of E and F respectively with $B \subset \overline{\Gamma(C \otimes D)}$? This is the so-called problem of topologies of Grothendieck ([26, question non resolue no. 2]).

(b) Let G and H be (DF)-spaces, is $G \otimes_\varepsilon H$ a (DF)-space ? ([26, question non resolue no. 10]).

(c) Let E be a Fréchet space and let G be a (DF)-space. Is $L_b(E, G)$ a (DF)-space ? ([26, question non resolue no. 7]).

Due to the recent work of the third author (see [35] and [36]) we know that the answer to these problems is negative. Partial positive answers were given in [18], [35] and [37]. Our purpose is to present a general frame in which the answer to all these questions of Grothendieck is positive. We shall introduce wide classes of Fréchet and (DF)-spaces, called (FG)-spaces and (DFG)-spaces respectively, such that the answer to problems (a), (b) and (c) is positive if we restrict our attention to these classes. In this way we properly extend the results of [18, 35, 37].

After this introduction, this article is divided in four sections. In section 2 we introduce the classes of (FG)-spaces and (DFG)-spaces and we study relations between them. Section 3 is devoted to provide examples of (FG)- and (DFG)-spaces. For instance Banach valued Köthe echelon spaces of order p, $1 \le p < \infty$, $p = 0$, the Fréchet spaces of measurable functions introduced by Reiher [34], $L_\rho(A)$, with absolutely continuous ρ (in particular the spaces $L_p((\mu_n)_{n\in\mathbb{N}})$ of Grothendieck [26], $1 \le p < \infty$, where μ_i are σ-finite measures ($i \in \mathbb{N}$)), Fréchet-Schwartz spaces with a finite dimensional decomposition and a continuous norm, the Fréchet space of continuous functions $CA_0(X)$, are examples of (FG)-spaces. (DFG)-spaces are, e.g., strong duals of (FG)-spaces, the weighted inductive limit of continuous functions ind $C(v_n)_0\,(X)$, and the projective hulls $C\bar{V}(X)$, $C\bar{V}_0(X)$ of the weighted inductive limits of spaces of continuous functions. In section 4 we give the main results. we prove that the answer to the three problems of Grothendieck mentioned above is positive for (FG) and (DFG)-spaces. We also treat the quasibarrelledness of $L_b(E, G)$ and $G \otimes_\varepsilon H$, where E is an (FG)-space and G and$H$ are (DFG)-spaces. At the end of this section we completely describe the quasibarrelledness of $L_b(\lambda_p(A), k_q(\bar{V}))$, where $\lambda_p(A)$ is a Köthe echelon space of order $1 \le p < \infty$, or $p = 0$ and $k_q(\bar{V})$ is a co-echelon space of order $1 \le q \le \infty$ or $q = 0$. All the positive results remain valid for complemented subspaces of (FG) or (DFG)-spaces. In particular for every Fréchet Schwartz space with the bounded approximation property and a continuous norm and for every space of continuous functions on a locally compact σ-compact space endowed with the compact open topology. In section 5 we construct new counterexamples to problems (a), (b) and (c). In particular, we show that there exist a separable (LB)-space G which is an ε-space in the sense of Hollstein and a reflexive separable Banach space X such that $G \otimes_\varepsilon X$ is not a (DF)-space. This shows that ε-spaces do not behave as well as $\mathcal{L}_\infty$-spaces.

Our notation is standard and we refer the reader to [30, 31, 33].

2. (FG)-spaces and (DFG)-spaces

In this section we introduce the classes of Fréchet spaces and locally convex spaces with a fundamental sequence of bounded sets that we will consider in this article and give some elementary properties.

DEFINITION 1. A Fréchet space E is called an *(FG)-space* if there is an increasing fundamental sequence of seminorms $(\|\cdot\|_k)_{k\in\mathbb{N}}$ such that for every sequence $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$, there is a sequence $(P_k)_{k\in\mathbb{N}} \subset L(E, E)$ satisfying

(FG1) $$x = \sum_{j\in\mathbb{N}} P_j(x) \qquad \forall x \in E,$$

(FG2) $$\|P_k(x)\|_{k-1} \leq \alpha_k \|x\|_k \qquad \forall x \in E,\ \forall k \geq 2,$$

(FG3) $$\forall j > k,\ \exists \lambda_{jk} \geq 1 \ :\ \|P_k(x)\|_j \leq \lambda_{jk} \|x\|_k \qquad \forall x \in E.$$

If we put $U_k := \{x \in E\ ;\ \|x\|_k \leq 1\}$ it is clear that we may assume that $(U_k)_{k\in\mathbb{N}}$ form a basis of 0-neighbourhoods in E. Condition (FG2) is equivalent to $P_k(U_k) \subset \alpha_k U_{k-1}$, $\forall k \geq 2$. Condition (FG3) is equivalent to any of the following facts:

(FG3') $P_k(U_k)$ is a bounded subset of E for all $k \in \mathbb{N}$,

(FG3") For every $k \in \mathbb{N}$, P_k factors continuously through the canonical Banach space $E_k := (E/\ker \|\cdot\|_k)\hat{}$ as follows

$$\begin{array}{ccc} E & \xrightarrow{\ P_k\ } & E \\ & {\scriptstyle \pi_k}\searrow \quad \nearrow{\scriptstyle \tilde{P}_k} & \\ & E & \end{array}$$

where $\pi_k\colon E \to E_k$ is the canonical mapping.

If the sequence $(P_k)_{k\in\mathbb{N}}$ in the definition can be selected to satisfy

(FG4) $$P_k P_j = \partial_{jk} P_k, \qquad \forall k, j \in \mathbb{N}$$

we will say that E is a *decomposable (FG)-space*.

It is very easy to see that every decomposable Fréchet T-space in the sense of Bonet and Díaz [18, Definition 1] is a decomposable (FG)-space.

DEFINITION 2. A locally convex space (G,t) with a fundamental sequence of bounded sets is said to be a *(dFG)-space* if there is an increasing fundamental sequence $(B_k)_{k\in\mathbb{N}}$ of absolutely convex closed bounded subsets of G and there is a locally convex topology s on G, coarser than t, such that (G,t) has a basis of s-closed absolutely convex 0-neighbourhoods and for every sequence $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$, there is a sequence $(Q_k)_{k\in\mathbb{N}} \subset L((G,t),(G,t))$ such that

(DFG1) $x = \sum_{j\in\mathbb{N}} Q_j(x)\ \forall x \in G$, where the series converges for the topology s,

(DFG2) $Q_k(B_{k-1}) \subset \alpha_k\, B_k \qquad \forall k \geq 2,$

(DFG3) $Q_k^{-1}(B_k)$ is a 0-neighbourhood in (G,t) for every $k \in \mathbb{N}$.

Condition (DFG3) is equivalent to the existence of a continuous factorization of Q_k as follows

$$\begin{array}{ccc} G & \xrightarrow{\quad Q_k \quad} & G \\ & {\scriptstyle \tilde{Q}_k}\searrow \qquad \nearrow{\scriptstyle j_k} & \\ & G_k & \end{array}$$

where $G_k = G_{B_k}$ and j_k is the canonical injection.

Condition (DFG3) implies

$$\forall k\ \forall j > k,\ \exists \lambda_{jk} > 0\ :\ B_j \subset \lambda_{jk}\, Q_j^{-1}(B_k).$$

If (G,t) is quasibarrelled this last condition is equivalent to (DFG3). If the topology s can be taken equal to t in the definition we will say that (G,t) is a *strong (dFG)-space*. If the sequence $(Q_k)_{k\in\mathbb{N}}$ can be selected to satisfy

(DFG4) $$Q_kQ_j = \partial_{kj}Q_k, \qquad \forall k,j \in \mathbb{N}$$

we say that (G,t) is a *decomposable (dFG)-space.*

If a (dFG)-space (resp. strong (dFG)-space, decomposable (dFG)-space) (G,t) is also a (DF)-space, we will say that (G,t) is a *(DFG)-space* (resp. *strong (DFG)-space, decomposable (DFG)-space*).

Proposition 3.

Let E be an (FG)-space, then $(G,t) := E'_b$ is a (DFG)-space for the topology $s = \sigma(E',E)$.

Proof. It is enough to take $B_k := U_k^\circ$, $k \in \mathbb{N}$, $Q_k := {}^tP_k$, $k \in \mathbb{N}$, to obtain the conclusion. □

Proposition 4.

(a) Let (G,t) be a reflexive (DFG)-space for a topology s finer than $\sigma(G,G')$, which is decomposable. Then $E := G'_b$ is a decomposable (FG)-space.

(b) Let E be a reflexive decomposable (FG)-space, then $(G,t) := E'_b$ is a reflexive strong decomposable (DFG)-space.

Proof. We take $U_k := B_k^\circ$, $k \in \mathbb{N}$, $P_k := {}^tQ_k$, $k \in \mathbb{N}$. Since s is finer than $\sigma(G',G)$, for every $x \in E$, $x = \sum_{j\in\mathbb{N}} P_j(x)$ for the topology $\sigma(G,G')$. Moreover E is a reflexive Fréchet space and $(P_j)_{j\in\mathbb{N}}$ is a weak Schauder decomposition in E. Consequently $x = \sum_{j\in\mathbb{N}} P_j(x)$ converges for the topology of E (see e.g. [32] or [30,14.3.2]). The proof of (b) is similar. □

The definition of a (dFG)-space (G,t) already has consequences about the locally convex structure of (G,t). We refer the reader to [30] or [33] for a study of (gDF)-spaces.

Proposition 5.

(a) Let (G,t) be a (dFG)-space which is a (gDF)-space, then (G,t) is a (DFG)-space.

(b) Let (G,t) be a strong (DFG)-space, then (G,t) is quasibarrelled.

Proof. (a) Let $(B_k)_{k\in\mathbb{N}}$ the fundamental sequence of bounded sets and let s be the locally convex topology coarser than t given by the definition of (dFG)-space (G,t). We fix a basis $\mathcal{U}$ of s-closed absolutely convex 0-neighbourhoods in (G,t). By [11, Lemma A], to show that (G,t) is (DF) it is enough to prove that for every $(W_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ and $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n > 0$,

$$T := \bigcap_{n\in\mathbb{N}} \left(W_n + \sum_{j=1}^{n} \lambda_j B_j \right)$$

is a 0-neighbourhood in (G,t). Since every (gDF)-space satisfies the countable neighbourhood property, we may find $\rho_n > 0$ such that $W := \bigcap_{n\in\mathbb{N}} \rho_n W_n$ is a 0-neighbourhood in (G,t). Now, for each $j \in \mathbb{N}$, we find $\mu_j > 0$ with $\mu_j B_j \subset W$. We put

$$\alpha_j := \min\big(\lambda_j, \min(2^{-j}\rho_n^{-1}\mu_j^{-1};\ 1 \le n \le j)\big), \quad j \in \mathbb{N}.$$

Given $(\alpha_j)_{j\in\mathbb{N}}$, we select $(Q_j)_{j\in\mathbb{N}} \subset L(G,G)$ as in the definition of (dFG)-space and we put

$$U := \bigcap_{j\in\mathbb{N}} \alpha_j Q_j^{-1}(B_j).$$

By (DFG3) U is a countable intersection of closed 0-neighbourhoods in (G,t) and by (DFG2) we have

$$\forall j, \quad \forall k > j, \qquad Q_k(B_j) \subset Q_k(B_{k-1}) \subset \alpha_k B_k,$$

hence

$$B_j \subset \bigcap_{k>j} \alpha_k Q_j^{-1}(B_k).$$

Since (G,t) is a (gDF)-space, we can apply [33, 8.3.2] to conclude that U is a 0-neighbourhood in (G,t). We prove that $U \subset W$. If $x \in U$ and $1 \leq j \leq n$, $Q_j(x) \in \alpha_j B_j \subset \lambda_j B_j$. Now

$$x - \sum_{j=1}^{n} Q_j(x) = \sum_{j=n+1}^{\infty} 2^{-j} Q_j(2^j x),$$

where the series converges for the topology s. If $j > n$,

$$Q_j(2^j x) \in 2^j \alpha_j B_j \subset 2^j \alpha_j \mu_j W \subset 2^j \alpha_j \mu_j \rho_n W_n \subset W_n.$$

Since W_n is s-closed and absolutely convex the series converges in s to an element in W_n. This implies $x \in W_n + \sum_{j=1}^{n} \lambda_j B_j$, for each $n \in \mathbb{N}$, i.e., $x \in W$.

(b) Assume now that (G,t) is a (DFG)-space for the topology $s = t$. Let T be an absolutely convex closed bornivorous subset of (G,t). For every $j \in N$ we find $\alpha_j > 0$ with $\alpha_j B_j \subset 2^{-j} T$. Given $(\alpha_j)_{j\in\mathbb{N}}$, we select $(Q_j)_{j\in\mathbb{N}}$ as in the definition of (DFG)-space. As above

$$U := \bigcap_{j\in\mathbb{N}} \alpha_j Q_j^{-1}(B_j)$$

is a 0-neighbourhood in (G,t). We check that $U \subset T$. If $x \in U$, $x = \sum_{j\in\mathbb{N}} Q_j(x)$, and the series converges for the topology t. But if $x \in U$, $Q_j(x) \in \alpha_j B_j \subset 2^{-j} T$ for each $j \in \mathbb{N}$. Since T is t-closed, $x \in T$. □

The following lemma will be used very often in the rest of the paper.

Lemma 6.

(a) Let E be an (FG)-space. Let $(P_k)_{k\in\mathbb{N}} \subset L(E,E)$ be the sequence associated with $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$. Then $\bigcup_{k\in\mathbb{N}} \alpha_k^{-1} P_k(U_k)$ is a bounded subset of E.

(b) Let (G,t) be a (DFG)-space. Let $(Q_k)_{k\in\mathbb{N}} \subset L(G,G)$ be the sequence associated with $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$. Then $\bigcap_{k\in\mathbb{N}} \alpha_k Q_k^{-1}(B_k)$ is a 0-neighbourhood in (G,t).

Proof. (a) We fix $j \in \mathbb{N}$. If $k > j$, (FG2) implies $P_k(U_k) \subset \alpha_k U_{k-1} \subset \alpha_k U_j$, hence $\bigcup_{k>j} \alpha_k^{-1} P_k(U_k) \subset U_j$. Now, by (FG3), $\bigcup_{1\leq k\leq j} \alpha_k^{-1} P_k(U_k)$ is bounded in E. The conclusion follows.

(b) Proceeding as in the proof of Proposition 5, $\bigcap_{k\in\mathbb{N}} \alpha_k Q_k^{-1}(B_k)$ is a bornivorous countable intersection of absolutely convex closed 0-neighbourhoods in (G,t) which is a (DF)-space. □

Proposition 7.

(a) The countable product of (decomposable) (FG)-spaces is again a (decomposable) (FG)-space.

(b) The countable direct sum of (decomposable, resp. strong) (DFG)-spaces is again a (decomposable, resp. strong) (DFG)-space.

Proof. (a) Let $(E_j)_{j\in\mathbb{N}}$ be a sequence of (FG)-spaces and let $(\|\cdot\|_{j,k})_{k\in\mathbb{N}}$ be the fundamental system of seminorms in E_j given by the definition. We set

$$E := \prod_{j\in\mathbb{N}} E_j, \qquad \|x\|_k := \sum_{j=1}^{k} \|x_j\|_{j,k-j+1}$$

for $x = (x_j)_{j\in\mathbb{N}} \in E$. Given $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$, we put $\beta_{j,k} := \alpha_{k+j-1}$ for all $j,k \in \mathbb{N}$, and given $(\beta_{j,k})_{k\in\mathbb{N}}$ we select $(P_{j,k})_{k\in\mathbb{N}} \subset L(E_j,E_j)$ satisfying (FG1,2,3). It is readily checked that

$$P_k(x) := (P_{1,k}(x_1), P_{2,k-1}(x_2), \ldots, P_{k,1}(x_k), 0, \ldots), \quad x \in E,$$

satisfies (FG1,2,3).

(b) Let (G_j,t_j), $j \in \mathbb{N}$, be a sequence of (DFG)-spaces. Let $(B_{j,k})_{k\in\mathbb{N}}$ be the fundamental sequence of bounded sets and s_j the locally convex topology coarser than t_j ($j \in \mathbb{N}$) be given by the definition. We put

$$(G,t) := \bigoplus_{j\in\mathbb{N}} (G_j,t_j), \quad (G,s) := \bigoplus_{j\in\mathbb{N}} (G_j,s_j),$$

$$B_k := \bigoplus_{j=1}^{k} B_{j,k-j+1}, \qquad k \in \mathbb{N},$$

then $(B_k)_{k\in\mathbb{N}}$ is a fundamental sequence of bounded sets in (G,t). Given $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$, define $\beta_{j,k} := \alpha_{k+j-1}$, $j,k \in \mathbb{N}$ and select $(Q_{j,k})_{k\in\mathbb{N}}$ associated with $(\beta_{j,k})_{k\in\mathbb{N}}$. It is easy to check that

$$Q_k(x) := (Q_{1,k}(x_1), Q_{2,k-1}(x_2), \ldots, Q_{k,1}(x_k), 0, \ldots,), \qquad x \in G,$$

have the desired properties. □

It is possible to reformulate the definition of (FG)-space (resp. strong (DFG)-space) in order to obtain the stability of the new classes by taking projective (resp. injective) tensor products. We will not include these details here. This stability is relevant in infinite dimensional holomorphy because it yields consequences about the spaces of polynomials defined on Fréchet or (DF)-spaces. We refer to [2] and [24] for details.

3. Examples of (FG)-spaces and (DFG)-spaces

In this section we will include many examples of (FG) and (DFG)-spaces. This will show that these are two wide classes and that our positive results in section 4 are applicable to many concrete spaces.

First of all, as it was already mentioned after the definition of (FG)-spaces, every decomposable Fréchet T-space in the sense of Bonet and Díaz [18] is an (FG)-space. Consequently the following are *examples* of (FG)-*spaces*:

(i) Banach spaces and countable products of Banach spaces,

(ii) Banach valued Köthe echelon spaces of order p, $1 \leq p < \infty$ or $p = 0$, $\lambda_p(A, (X_i)_{i \in \mathbb{N}})$ with X_i Banach ($i \in \mathbb{N}$), and even every X-Köthe sequence space for a Banach space X with a 1-unconditional basis (e_n) in the sense of S. Bellenot [6].

(iii) Generalized Dubinsky echelon spaces with decreasing steps wich are Montel. For instance if we take a Köthe matrix $A = (a_n)$ such that $\lambda_1(A)$ is Fréchet Montel but not Schwartz, setting

$$E := \bigcap_{n \in \mathbb{N}} l^{p+1/n}(a_n), \qquad 1 < p < \infty,$$

we obtain an (FG)-space which is Montel but which is not an X-Köthe sequence space.

(iv) The Fréchet spaces of measurable functions introduced by Reiher [34], $L_\rho(A)$ with absolutely continuous ρ. In particular the spaces $L_p((\mu_n)_{n \in \mathbb{N}})$ of Grothendieck, $1 \leq p < \infty$, where μ_i are σ-finite measures ($i \in \mathbb{N}$).

(v) Fréchet-Schwartz spaces with a finite dimensional decomposition and a continuous norm.

The positive results that will be proved in next section are valid for complemented subspaces of (FG)-spaces. Every Fréchet Schwartz space with the bounded approximation property and a continuous norm is a complemented subspace of an (FG)-space (cf. [7, 18]). Vogt has recently characterized the hilbertizable Fréchet spaces which are isomorphic to a complemented subspace of a power series space of infinite type (see [39]).

We refer to [18] for all the details and more examples. Now we include new examples. We refer the reader to [14] for the notations and the theory of Köthe echelon and co-echelon spaces of order p, $1 \leq p \leq \infty$ or $p = 0$, on an arbitrary index set I. For condition (D) in a Köthe matrix, we refer to [13] and [8]. We recall that a Köthe matrix A satisfies condition (D) if and only if $\lambda_p(I, A)$ satisfies the density condition (see [8] and [3]).

Proposition 1.

Let $A = (a_n)_{n\in\mathbb{N}}$ be a Köthe matrix on an index set I, satisfying condition (D). Then $\lambda_p(I, A)$ is a decomposable (FG)-space for $1 \leq p \leq \infty$ or $p = 0$.

Proof. By the definition of condition (D) and passing to a subsequence of the a_n's, these is an increasing sequence $J = (I_m)_{m\in\mathbb{N}}$ such that

$$(\mathrm{N,J}) \qquad \inf_{i\in I_m} \frac{a_m(i)}{a_j(i)} > 0 \quad \text{for every } j > m$$

$$(\mathrm{M,J}) \qquad \forall n \in \mathbb{N} \quad \forall I_0 \subset I \text{ with } I_0 \cap \big(I \setminus I_m\big) \neq \emptyset \text{ for each } m \in \mathbb{N},$$

$$\exists n' = n'(n, I_0) > n \text{ with } \inf_{i\in I_0} \frac{a_n(i)}{a_{n'}(i)} = 0.$$

Note that $I = \bigcup_{m\in\mathbb{N}} I_m$ by (M,J). We set $N_1 := I_1$, $N_j := I_j \setminus I_{j-1}$ $(j \geq 2)$, and denote by Q_j the canonical projection associated to N_j $(j \in \mathbb{N})$.

We only give the details if $p = \infty$, the other cases are very similar. By (N,J), for every j, $k \in \mathbb{N}$, $j < k$, there is $\gamma_{jk} > 0$ with $a_j(i)\gamma_{jk} \geq a_k(i)$ for each $i \in N_j$. Moreover, by (M,J) the series $\sum_{j\in\mathbb{N}} Q_j x$ converges to x in the sense of the topologies of the seminorms by [13, 3.7]. Now, given a sequence $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$, we put

$$\gamma_k := 2^{-1} \min\big(\alpha_k, \min(\gamma_{jk}^{-1};\ j = 1, \dots, k-1)\big), \quad k \in \mathbb{N},$$

and we define

$$J_m := \{i \in I;\ a_{m-1}(i) \leq \gamma_m a_m(i)\},$$

$$M_1 := I \setminus \Big(\bigcup_{i\geq 2} I_i\Big), \qquad M_m := J_m \setminus \Big(\bigcup_{i\geq m+1} J_i\Big) \quad (m \geq 2).$$

It is easy to see that $I_m \subset \bigcup_{j=1}^{m} M_j$ ($m \in \mathbb{N}$). We denote by P_j the canonical projection associated to M_j ($j \in \mathbb{N}$). Certainly $P_iP_j = \delta_{ij}P_j$ for each $i,j \in \mathbb{N}$. By the inclusion above we get

$$\left|x(i) - \sum_{j=1}^{n} P_j x(i)\right| \leq \left|x(i) - \sum_{j=1}^{n} Q_j x(i)\right|, \quad x \in \lambda_\infty(I,A),\ i \in I,\ n \in \mathbb{N}.$$

From this it follows that the series $\sum_{j\in\mathbb{N}} P_j x$ converges to x in λ_∞ (I,A). Moreover the following inequalities are readily checked

(a) $\|P_k x\|_{k-1} \leq \alpha_k \|x\|_k$, $k \in \mathbb{N}$

(b) $\|P_k x\|_j \leq \left(\gamma_{k+1}\,\gamma_{k+2}\,\cdots\,\gamma_j\right)^{-1}\|x\|_k$, $\quad j > k$, $j,k \in \mathbb{N}$.

This implies that $\lambda_\infty(I,A)$ is a decomposable (FG)-space. $\square$

Observe that Proposition 1 above implies that if $\lambda_\infty(A)$ satisfies the density condition, then it is an (FG)-space. By a very recent result of F. Bastin [5], for a Köthe matrix A on $\mathbb{N}$, $\lambda_\infty(A)$ is distinguished if and only if it has the density condition or equivalently if A satisfies condition (D).

By the sequence space representations of Valdivia and Vogt [39, 40] and our results above it follows that the function spaces $C^\infty(\Omega)$, $\mathcal{B}_0(\Omega)$, $\mathcal{B}(\Omega)$ and $\mathcal{D}_{L^p}$ are (FG)-spaces.

One of our main motivations to deal with (FG) and (DFG)-spaces was to include in our study the weighted inductive and projective limits of spaces of continuous functions. When one deals with these spaces it is necessary to replace the projections taken in sequence spaces by partitions of the unity, hence there is no hope to obtain decomposable (FG) or (DFG)-spaces.

First we recall some notation (also see [15]). In what follows we denote by X a locally compact σ-compact space with a fundamental sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ with $K_n \subset \overset{\circ}{K}_{n+1}$ ($n \in \mathbb{N}$). If $A = (a_n)_{n\in\mathbb{N}}$ is an increasing sequence of strictly positive continuous functions on X, we define (see [15, section 5])

$$CA(X) := \left\{f \in C(X);\ p_n(f) = \sup_{x\in X} a_n(x)|f(x)| < \infty,\ \forall n \in \mathbb{N}\right\},$$

$$CA_0(X) := \left\{f \in C(X);\ a_n f \text{ vanishes at infinity on } X,\ \forall n \in \mathbb{N}\right\},$$

both endowed with the metrizable topology defined by the seminorms $(p_n)_{n\in\mathbb{N}}$. Clearly both $CA(X)$ and $CA_0(X)$ are Fréchet spaces.

At this point it is convenient to include the definition of weighted inductive limits. Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous weights on X. We define the Banach spaces

$$C(v_n)(X) := \left\{f \in C(X);\ \sup_{x\in X} v_n(x)|f(x)| < \infty\right\},$$

$$C(v_n)_0(X) := \left\{f \in C(X);\ v_n f \text{ vanishes at infinity on } X\right\}.$$

The weighted inductive limits are defined by

$$\mathcal{V}C(X) := \operatorname{ind} C(v_n)(X) \quad \text{and}$$
$$\mathcal{V}_0C(X) := \operatorname{ind} C(v_n)_0(X).$$

The maximal Nachbin system associated with $\mathcal{V}$ is defined by

$$\bar{V} = \bar{V}(\mathcal{V}) := \Big\{\bar{v} : X \to \mathbb{R}\ ;\ \bar{v} \geq 0,\ \text{upper semicontinuous},\ \sup_{x \in X} \frac{\bar{v}(x)}{v_n(x)} < \infty\ \forall n \in \mathbb{N}\Big\}.$$

The projective hulls of the weighted inductive limits are defined by

$$C\bar{V}(X) := \big\{f \in C(X);\ p_{\bar{v}}(f) := \sup_{x \in X} \bar{v}(x)|f(x)| < \infty\ \forall \bar{v} \in \bar{V}\big\},$$
$$C\bar{V}_0(X) := \big\{f \in C(X);\ \bar{v}f \text{ vanishes at infinity on } X \quad \forall \bar{v} \in \bar{V}\big\},$$

both endowed with the locally convex topology defined by the seminorms $(p_{\bar{v}})_{\bar{v} \in \bar{V}}$. The spaces $\mathcal{V}C(X)$ and $C\bar{V}(X)$ coincide algebraically and they have the same bounded subsets

$$B_n := \big\{f \in CV(X);\ \sup_{x \in X} v_n(x)|f(x)| \leq n\big\} \quad (n \in \mathbb{N}).$$

They coincide topologically if and only if $\mathcal{V}$ satisfies condition (D) (see [4] and [12]). On the other hand $\mathcal{V}_0C(X)$ is a topological dense subspace of $C\bar{V}_0(X)$ (see [15]).

All our results will be based in the following lemma.

Lemma 2.

Let $\mathcal{V}$ be a decreasing sequence of strictly positive weights on X and $\bar{V}$ the maximal Nachbin system associated with $\mathcal{V}$. For every $(\alpha_k)_{k \in \mathbb{N}}$, $0 < \alpha_k \leq 1$, there is a sequence $(\bar{v}_k)_{k \in \mathbb{N}}$ in $\bar{V} \cap C(X)$ such that,

$$G_1 := \big\{x \in X;\ v_1(x) < 2\bar{v}_1(x)\big\},$$
$$G_k := \big\{x \in X;\ v_k(x) < \alpha_k v_{k-1}(x),\ v_k(x) < 2\bar{v}_k(x)\big\} \quad (k \geq 2)$$

form an open cover of X.

Proof. Given $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \le 1$, we put

$$\tilde{v}_m := \min\bigl(v_m, \inf_{k>m} (\alpha_{m+1}\,\alpha_{m+2}\,\dots\,\alpha_k)^{-1} v_k\bigr) \quad (m \in \mathbb{N})$$

Clearly both $\tilde{v}_m$ and $v_m \chi_{K_m}$ belong to $\bar{V}$ (here χ_{K_m} is the characteristic function of the compact set K_m). By [15, 0.2 Proposition], we can find $\bar{v}_m \in \bar{V} \cap C(X)$, strictly positive on X and such that $\max(\tilde{v}_m, v_m\chi_{K_m}) \le \bar{v}_m$ $(m \in \mathbb{N})$. We define now G_k $(k \in \mathbb{N})$ as in the statement and we prove $X = \bigcup_{k\in\mathbb{N}} G_k$. To do this we fix $x \in X$ and we select $t \in \mathbb{N}$ such that $x \in K_t$. First observe that if $k \ge t$ we have

$$(*) \qquad v_k(x) = v_k(x)\chi_{K_k}(x) \le \bar{v}_k(x) < 2\bar{v}_k(x)$$

We distinguish several cases:

(i) $\forall s \ge 2$, $v_s(x) \ge \alpha_s v_{s-1}(x)$. This implies

$$v_1(x) \le \alpha_2^{-1} v_2(x) \le (\alpha_2\,\alpha_3)^{-1} v_3(x) \le \dots ,$$

hence

$$v_1(x) \le \min\bigl(v_1(x), \inf_{k\ge2} (\alpha_2 \cdots \alpha_k)^{-1} v_k(x)\bigr) \le \bar{v}_1(x) < 2\bar{v}_1(x),$$

and $x \in G_1$.

(ii) $\exists s \ge t$ with $v_s(x) < \alpha_s v_{s-1}(x)$. From (*) for $k = s$, this readily implies $x \in G_s$.

(iii) $\exists s < t$ with $v_s(x) < \alpha_s v_{s-1}(x)$ but $v_r(x) \ge \alpha_r v_{r-1}(x)$ for all $r > s$. This yields

$$v_s(x) \le \min\bigl(v_s(x), \inf_{k>s} (\alpha_{s+1} \cdots \alpha_k)^{-1} v_k(x)\bigr) \le \bar{v}_s(x) < 2\bar{v}_s(x).$$

Consequently $x \in G_s$ and the proof is complete. □

Proposition 3.

The Fréchet space $CA_0(X)$ is an (FG)-space.

Proof. We take $p_n(f) := \sup_{x\in X} a_n(x)|f(x)|$, $f \in CA_0(X)$, $(n \in \mathbb{N})$, as the fundamental sequence of seminorms in $CA_0(X)$. We fix a sequence $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \le 1$ $(k \in \mathbb{N})$ and we denote by v_n the weight given by $v_n(x) := 1/a_n(x)$ for $x \in X$. Now we apply Lemma 2 to select $(\bar{v}_k)_{k\in\mathbb{N}} \subset \bar{V} \cap C(X)$ and the corresponding sequence $(G_k)_{k\in\mathbb{N}}$ of open subsets covering X. Since X is locally compact and σ-compact, we may find a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $C(X)$ such that $0 \le \varphi_k \le 1$, $\operatorname{supp}\varphi_k \subset G_k$ for all $k \in \mathbb{N}$, and moreover for every $m \in \mathbb{N}$ there is $k(m) \in \mathbb{N}$ with $\sum_{k=1}^{k(m)} \varphi_k(x) = 1$ for all $x \in K_m$ and $\varphi_k(x) = 0$ if $x \notin K_m$. In particular $\sum_{k \in \mathbb{N}} \varphi_k(x) = 1$ for all $x \in X$ (see e.g., [1, p. 126]). Now, for every $k \in \mathbb{N}$ we define $P_k : CA_0(X) \to CA_0(X)$ by $P_k(f) := \varphi_k f$. Since every $f \in CA_0(X)$ satisfies that $a_n f$ vanishes at infinity on X for all $n \in \mathbb{N}$, it follows that $\sum_{k \in \mathbb{N}} P_k f$ converges to f in $CA_0(X)$ for all $f \in CA_0(X)$ and (FG1) holds. Now we check (FG2). Given $f \in CA_0(X)$,

$$\begin{aligned} p_{k-1}(P_k(f)) &= \sup_{x \in X} a_{k-1}(x)\varphi_k(x)|f(x)| \leq \sup_{x \in G_k} a_{k-1}(x)|f(x)| \\ &\leq \sup_{x \in G_k} \alpha_k \, a_k(x)|f(x)| \leq \alpha_k \, p_k(f). \end{aligned}$$

To show that (FG3) is satisfied, we fix $k \in \mathbb{N}$ and first observe that, since $\bar{v}_k \in \bar{V}$, for all $j > k$ there is $\lambda_{jk} \geq 1$ with $a_j(x) \, \bar{v}_k(x) \leq \lambda_{jk} \; \forall x \in X$. Now for $j > k$ we have

$$\begin{aligned} p_j(P_k(f)) &= \sup_{x \in X} a_j(x)\varphi_k(x)|f(x)| \\ &\leq \sup_{x \in G_k} a_j(x)|f(x)| = \sup_{x \in G_k} a_j(x) \, \bar{v}_k(x) \, \frac{v_k(x)}{\bar{v}_k(x)} \, a_k(x)|f(x)| \\ &\leq 2\lambda_{jk} \, p_k(f). \qquad \square \end{aligned}$$

Remarks 4. (1) Note that the same proof given above shows that $CA(X)$ satisfies all the conditions of an (FG)-space with the exception that the convergence in (FG1) can only be obtained for the topology of pointwise convergence on X. By [13, 5.5], $CA(X) = CA_0(X)$ coincide algebraically if and only if $\forall n \; \forall Y \subset X$ non-relatively compact there is $m > n$ with

$$\inf_{y \in Y} \frac{u_n(y)}{u_m(y)} = 0.$$

(2) It is very easy to extend the former result to weighted spaces of Banach valued continuous functions.

Examples 5: Fréchet spaces which are not (FG)-spaces

According to our results in section 4 and the examples given in [36] and [19], we have the following consequences:

(a) There are Fréchet Montel spaces with finite dimensional decomposition and a continuous norm which are not (FG)-spaces. Compare with (v) above.

(b) There are quojections (i.e., separated quotients of countable products of Banach spaces) which are not (FG)-spaces. This should be compared with (i) above.

Now we provide *examples* of (DFG)-*spaces.*

(i) Normed spaces and countable direct sums of normed spaces.

(ii) Strong duals of (FG)-spaces. This follows from Proposition 2.3 and gives many examples of (DFG)-spaces simply taking strong duals of the (FG)-spaces given before.

It is our purpose now to treat the weighted inductive limits of spaces of continuous functions defined on a locally compact and σ-compact space X. In what follows $\mathcal{V} = (v_n)$ is a decreasing sequence of strictly positive continuous weights on X and $\bar{V}$ is its associated maximal Nachbin family.

Proposition 6.

The spaces $\mathcal{V}_0C(X)$, $C\bar{V}_0(X)$ and $C\bar{V}(X)$ are (DFG)-spaces for the topology of the pointwise convergence. $\mathcal{V}_0C(X)$ and $C\bar{V}_0(X)$ are even strong (DFG)-spaces. The inductive limit $\mathcal{V}C(X)$ is a (DFG)-space if $\mathcal{V}$ satisfies the condition (D) of Bierstedt and Meise (i.e. if $\mathcal{V}C(X) = C\bar{V}(X)$ holds topologically).

Proof. Without loss of generality we may assume that

$$B_n := \left\{ f \in C\bar{V}(X) \; ; \; \sup_{x \in X} v_n(x) \, |f(x)| \leq 1 \right\} \quad (n \in \mathbb{N})$$

form a fundamental sequence of bounded subsets of $C\bar{V}(X)$. In the other spaces we take as fundamental sequence the bounded sets given by the intersection of the B_n's with the corresponding space. We give the details of the proof for $C\bar{V}(X)$ and indicate the necessary changes in the other cases.

We proceed as in the proof of proposition 3, given $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$, we apply lemma 2 to find $(\bar{v}_k)_{k\in\mathbb{N}} \subset \bar{V} \cap C(X)$ and the open cover $(G_k)_{k\in\mathbb{N}}$ of X. Now we select a partition of the unity $(\varphi_k)_{k\in\mathbb{N}}$ of X subordinated to $(G_k)_{k\in\mathbb{N}}$ with the same special properties of the one selected in proposition 3. We define

$$Q_k : C\bar{V}(X) \longrightarrow C\bar{V}(X), \quad Q_k(f) := \varphi_k f \qquad (k \in \mathbb{N})$$

and we observe that Q_k maps continuously $\mathcal{V}_0C(X)$ (and $C\bar{V}_0(X)$) into itself. First of all it is clear that $C\bar{V}(X)$ has a basis of 0-neighbourhoods which are closed for the pointwise topology, namely

$$C_{\bar{v}} := \left\{ f \in C\bar{V}(X); \; \sup_{x \in X} \bar{v}(x)|f(x)| \leq 1 \right\}, \qquad \bar{v} \in \bar{V}$$

This is also the case for $\mathcal{V}_0C(X)$ and $C\bar{V}_0(X)$, but not necessarily for $\mathcal{V}C(X)$, unless $\mathcal{V}C(X) = C\bar{V}(X)$ holds topologically.

Now $f = \sum_{k\in\mathbb{N}} \varphi_k f = \sum_{k\in\mathbb{N}} Q_k f$ holds for the pointwise convergence for each $f \in C\bar{V}(X)$ and (DFG1) is satisfied. If $f \in C\bar{V}_0(X)$, the properties of $(\varphi_k)_{k\in\mathbb{N}}$ and the fact that $\bar{v}f$ vanishes at infinity on X ensure that the series converges to f for the topology of $C\bar{V}_0(X)$.

To check (DFG2) we fix $f \in B_{k-1}$ $(k \geq 2)$. We have

$$\sup_{x\in X} v_k(x)|Q_k(f)(x)| \leq \sup_{x\in G_k} v_k(x)|f(x)| \leq \alpha_k \sup_{x\in X} v_{k-1}(x)|f(x)| \leq \alpha_k,$$

hence $Q_k(B_{k-1}) \subset \alpha_k B_k$.

Finally to prove (DFG3), for each $k \in \mathbb{N}$, we show that the image by Q_k of the 0-neighbourhood

$$W_k := \left\{ f \in C\bar{V}(x);\ \sup_{x\in X} \bar{v}_k(x)|f(x)| \leq 2^{-1} \right\}$$

is contained in B_k $(k \in \mathbb{N})$. Indeed, if $f \in W_k$ we have

$$\begin{aligned} \sup_{x\in X} v_k(x)\,|f(x)| &\leq \sup_{x\in G_k} v_k(x)\,|f(x)| = \sup_{x\in G_k} \frac{v_k(x)}{\bar{v}_k(x)}\,\bar{v}_k(x)\,|f(x)| \\ &\leq 2 \sup_{x\in G_k} \bar{v}_k(x)\,|f(x)| \leq 1. \end{aligned}$$

This shows that $Q_k^{-1}(B_k)$ is a 0-neighbourhood in $C\bar{V}(X)$. The proof is now complete. □

The authors do not know if $\mathcal{V}C(X)$ is a (DFG)-space if condition (D) is not satisfied by $\mathcal{V}$. In particular, if A is a Köthe matrix not satisfying condition (D) and κ_∞ is the bornological space associated to $\lambda_1(A)'_b = k_\infty$, and G is a Banach space, is the injective tensor product $\kappa_\infty \otimes_\varepsilon G$ a (DF)-space?

Now we deal with the weighted inductive limits $K_\rho(\bar{V})$ of spaces of measurable functions introduced by Reiher [34].

Let (X, Σ, μ) a σ-finite measure space, M the set of all μ-mesurable functions and ρ a saturated funtion norm with the Fatou property. We recall that ρ is called absolutely continuous if $(\rho(f_n))_{n\in\mathbb{N}}$ tends to 0 if $(f_n)_{n\in\mathbb{N}}$ decreases to 0 in the natural lattice structure. If $a : X \to \mathbb{R}$ is a measurable function with $0 < a < \infty$ μ-a.e., we define

$$L_\rho(a) := \left\{ f \in M;\ \|f\|_a := \rho(af) < \infty \right\}.$$

This is the Köthe normed space associated to ρ and a. For examples of these spaces we refer to [34]. Given a sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$, $0 < v_{n+1} \leq v_n < \infty$ μ-a.e., of measurable functions we associate the system

$$\bar{V} := \left\{\bar{v} : X \longrightarrow \mathbb{R};\ \bar{v} \in M,\ \mu\text{-ess sup}\,\frac{\bar{v}(x)}{v_n(x)} < \infty \quad \forall n \in \mathbb{N}\right\}$$

For a function norm ρ we define

$$k_\rho(\bar{V}) := \{f \in M;\ \rho(\bar{v}f) < \infty\ \forall \bar{v} \in \bar{V}\}$$
$$\kappa_\rho(\mathcal{V}) := \operatorname{ind} L_\rho(v_n)$$

We observe that there is an increasing sequence $(X_n)_{n\in\mathbb{N}}$ of measurable subsets of X such that

(i) $\mu(X \setminus \bigcup_{n\in\mathbb{N}} X_n) = 0$ $\mu(X_n) < \infty, \quad \forall n \in \mathbb{N}$,

(ii) $\rho(\chi_{X_n}) < \infty \quad \forall n \in \mathbb{N}$,

(iii) $\exists a(m,n), b(m,n) > 0$; $a(m,n) \leq v_n \leq b(m,n)$ μ-a.e. on X_m.

This follows from [34, 1.1].

Proposition 7.

If ρ is absolutely continuous, then $k_\rho(\bar{V})$ is a strong decomposable (DFG)-space.

Proof. We use the notations established above and we fix the canonical seminorms and fundamental sequence of bounded subsets of $k_\rho(\bar{V})$ (see [34, 1.3]). Given a sequence $(\alpha_k)_{k\in\mathbb{N}}$, $0 < \alpha_k \leq 1$ $(k \in \mathbb{N})$, we define

$$\gamma_k := 2^{-1} \min\left(\alpha_{k+1}, \min\left(\frac{a(k+1,j)}{b(k,j)};\ 1 \leq j \leq k\right)\right)$$

and we put

$$J_k := \{x \in X\ ;\ v_{k+1} \leq \gamma_k v_k(x)\}$$
$$Y_1 := X \setminus \bigcup_{k\in\mathbb{N}} J_k, \qquad Y_m := J_{m-1} \setminus \bigcup_{k\geq m} J_k, \quad m \geq 2.$$

Observe that $\mu(X_i \setminus \bigcup_{k\geq i} J_k) = 0$. Indeed, if $x \in X_i \cap J_k$ $(k \geq i)$ then

$$v_{k+1}(x) \leq \gamma_k v_k(x) \leq \frac{1}{2}\,\frac{a(k+1,i)}{b(k,i)}\,b(k,i) \leq \frac{1}{2}\,a(k+1,i),$$

and this can only hold in a subset of X_i of measure zero. Consequently

$$X_1 \cup \ldots \cup X_n \subset Y_1 \cup \ldots \cup Y_n \quad \mu\text{-a.e.},$$

hence $X = \bigcup_{i \in \mathbb{N}} Y_i$ μ-a.e.

The sets Y_i are disjoint, hence the continuous projections $Q_i(f) := f\chi_{Y_i}$ $(i \in \mathbb{N})$ verify $Q_iQ_j = 0$, $i \neq j$. To check (DFG1), we fix $f \in k_\rho(\bar{V})$. Since ρ is absolutely continuous, it follows from [34, 1.4.c] that $f \in L_\rho(v_n)$ for some $n \in \mathbb{N}$. By our comments above the sequence $\left(fv_n - \sum_{i=1}^m v_n Q_i f\right)_{m \in \mathbb{N}}$ decreases to 0 in the natural lattice structure. By the absolute continuity of ρ it follows that

$$\lim_{m \to \infty} \rho\left(v_n\left(f - \sum_{i=1}^m Q_i f\right)\right) = 0,$$

hence the series $\sum_{i \in \mathbb{N}} Q_i f$ converges to f in $L_\rho(v_n)$, hence in $k_\rho(\bar{V})$.

We denote by B_n the ball of $L_\rho(v_n)$ of radius n $(n \in \mathbb{N})$. We fix $f \in B_{k-1}$ $(k \geq 2)$, i.e. $\rho(v_{k-1}f) \leq k-1$. Since $Y_k \subset J_{k-1}$, we have

$$\rho(v_k Q_k(f)) = \rho(v_k f \chi_{Y_k}) \leq \rho(\alpha_k v_{k-1} f \chi_{Y_k}) \leq \alpha_k \rho(v_{k-1} f) \leq k\alpha_k.$$

Consequently $Q_k(B_{k-1}) \subset \alpha_k B_k$, and (DFG2) is satisfied.

Finally for all $k \in \mathbb{N}$, $j > k$, the following inequality holds μ-a.e. on Y_k

$$v_k(x) < (\gamma_k \, \gamma_{k+1} \cdots \gamma_{j-1})^{-1} v_j(x).$$

Proceeding as we did before, we can find λ_{kj}, $j > k$, with $Q_k(B_j) \subset \lambda_{kj} B_k$ whence $Q_k^{-1}(B_k)$ is an absolutely convex bornivorous subset of $k_\rho(\bar{V})$ which is bornological. Thus $Q_k^{-1}(B_k)$ is a 0-neighbourhood in $k_\rho(\bar{V})$ for all $k \in \mathbb{N}$ and (DFG3) is satisfied. □

We close this section by mentioning the following example that should be compared with the examples of (DFG)-spaces (i) and (ii). By [19] and our results in section 4, there are strict inductive limits of Banach spaces which are not (DFG)-spaces and by [36] there are (DFM)-spaces with a finite dimensional decomposition and a total bounded set which are not (DFG)-spaces.

4. Main results

In this section we show that the context of (FG)- and (DFG)-spaces is an appropiate frame to give positive answer to the problems of Grothendieck. More precisely we will show the following results:

(i) If E and F are (FG)-spaces, then the problem of topologies of Grothendieck has a positive solution for $E\widehat{\otimes}_\pi F$.

(ii) If G, H are (DFG)-spaces, then $G\otimes_\varepsilon H$ is a (DF)-space.

(iii) If E is a (FG)-space and G is a (DFG)-space, then $L_b(E,G)$ is a (DF)-space.

The quasibarrelledness of $G\otimes_\varepsilon H$ and $L_b(E,G)$ as in (ii) and (iii) respectively will be considered too.

We need some preparation and some abstract results which are helpful and could be of independent interest.

First we consider two Fréchet spaces E and F. Grothendieck [26] proved that $L_b(E,F'_b)$ can be canonically identified with the dual $(E\widehat{\otimes}_\pi F)'$ of $(E\widehat{\otimes}_\pi F)$ endowed with the locally convex topology of the uniform convergence on the bounded subsets of $E\widehat{\otimes}_\pi F$ of the form $\overline{\Gamma(C\otimes D)}$, where C is bounded in E, D is bounded in F and the closure is taken in $E\widehat{\otimes}_\pi F$. *These bounded sets will be called localized in* $E\widehat{\otimes}_\pi F$. A bounded set is said to be *localizable* if it is a subset of a localized set. We denote by $\mathcal{A}$ the family of all the localizable subsets of $E\widehat{\otimes}_\pi F$. It is well-known that $\mathcal{A}$ is a saturated family of bounded sets in $E\widehat{\otimes}_\pi F$ in the sense of [31, p. 255] which covers $E\widehat{\otimes}_\pi F$ by [31, 41.4.6].

Following Taskinen [35] we say that a pair of Fréchet spaces (E,F) satisfies *property* (BB) if any bounded subset of $E\widehat{\otimes}_\pi F$ is localizable, i.e. the problem of topologies of Grothendieck has a positive answer for $E\widehat{\otimes}_\pi F$. Clearly if (E,F) has property (BB) we have $L_b(E,F'_b)=(E\widehat{\otimes}_\pi F)'_b$ holds topologically, hence $L_b(E,F'_b)$ is a (DF)-space. Taskinen [38] proved that if E and F are separable and $L_b(E,F'_b)$ is (DF), then (E,F) satisfies property (BB). We will extend this result here. In this way we show that the results about the stability of the (DF)-property for $L_b(E,G)$ if E is a Fréchet space and G is a (DF)-space, immediately yield applications for the property (BB) of pairs of Fréchet spaces.

Lemma 1.

Let E be a Fréchet space. Let $\mathcal{A}$ be a saturated family of bounded subsets of E covering E. Suppose that E' endowed with the topology of the uniform convergence on the elements of $\mathcal{A}$ is a (DF)-space. Assume that one of the following conditions is satisfied:

(i) the bounded subsets of $(E',t_{\mathcal{A}})$ are metrizable,

(ii) for every bounded subset B of E there is a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $D := \cup_{n\in\mathbb{N}} C_n$ is bounded in E and $B \subset \overline{D}$, the closure taken in E.

Then every bounded subset of E belongs to $\mathcal{A}$.

Proof. Clearly $t_{\mathcal{A}}$ is finer than $\sigma(E', E)$ and coarser than $\beta(E', E)$, consequently if $(U_n)_{n\in\mathbb{N}}$ is a basis of absolutely convex closed 0-neighbourhoods in E, then $(U_n^\circ)_{n\in\mathbb{N}}$ constitutes a fundamental sequence of bounded subsets of $(E', t_{\mathcal{A}})$. Moreover $E \subset (E', t_{\mathcal{A}})' \subset E''$ and $D \subset E''$ is $(E', t_{\mathcal{A}})$-equicontinuous if and only if there is $A \in \mathcal{A}$ with $D \subset \overline{A}$, the closure taken in $(E'', \sigma(E'', E'))$. Consequently a subset of E is $(E', t_{\mathcal{A}})$-equicontinuous if and only if $C \in \mathcal{A}$. Therefore, to conclude under the assumptions (i) or (ii), we have to show that every bounded subset of E is $(E', t_{\mathcal{A}})$-equicontinuous.

First we suppose that (i) is satisfied. By a classical result of Grothendieck (e.g. [31, 29.3.12]), $(E', t_{\mathcal{A}})$ is quasibarrelled. Now it is easy to see that every bounded subset of E is strongly bounded in $(E', t_{\mathcal{A}})'$, hence it is $(E', t_{\mathcal{A}})$-equicontinuous.

Now we assume (ii). Given a bounded subset B of E we find $(C_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ as in (ii). The set $D := \bigcup_{n\in\mathbb{N}} C_n$ is bounded in E, therefore it is strongly bounded in $(E', t_{\mathcal{A}})'$. On the other hand it is a countable union of $(E', t_{\mathcal{A}})$-equicontinuous. Since $(E, t_{\mathcal{A}})$ is a (DF)-space, D itself is also equicontinuous, i.e., $D \in \mathcal{A}$. Finally, since $\mathcal{A}$ is saturated, $\overline{D} \in \mathcal{A}$ and hence its subset B belongs to $\mathcal{A}$. □

As a consequence of this Lemma and our comments above we obtain the following result. For the density condition refer to [8].

Proposition 2.

Let E and F be Fréchet spaces such that $L_b(E, F_b')$ is a (DF)-space. Suppose that one of the following conditions is satisfied:

(i) E and F have the density condition.

(ii) Every bounded subset of $E\widehat{\otimes}_\pi F$ is contained in the closure of a bounded subset of $E\widehat{\otimes}_\pi F$ which is the union of a sequence of localizable subsets of $E\widehat{\otimes}_\pi F$.

Then (E, F) satisfies property (BB).

Proof. In case (i) use [11, 1.6] to conclude that $L_b(E, F_b')$ has metrizable bounded subsets. Case (ii) follows directly from Lemma 1. □

Remark 3. Condition (ii) in Proposition 2 is satisfied in the following cases:

(a) E and F are separable.

(b) There are $(P_j)_{j\in\mathbb{N}} \subset L(E,E)$, $(Q_j)_{j\in\mathbb{N}} \subset L(F,F)$ converging pointwisely to the identity of E and F respectively and such that for every bounded subset B of $E\widehat{\otimes}_\pi F$, $(P_j \otimes Q_j)(B)$ is localizable in $E\widehat{\otimes}_\pi F$. Here, to check that condition (ii) of Proposition 2 is satisfied one uses that $(P_j)_{j\in\mathbb{N}}$ and $(Q_j)_{j\in\mathbb{N}}$ are equicontinuous.

(c) E or F is separable and it satisfies the bounded approximation property (abreviated by b.a.p. from now on). Indeed, if E has the b.a.p., we have a sequence $(P_j)_{j\in\mathbb{N}} \subset L(E,E)$ such that $\dim P_j(E) < \infty$ for each $j \in \mathbb{N}$ and converging pointwisely to the identity. We take $Q_j = \mathrm{id}_F$ for all $j \in \mathbb{N}$. The result follows from (b), since every bounded subset of $P_j(E)\widehat{\otimes}_\pi F$ is localizable.

Lemma 4.

Let E and F be (FG)-spaces. If $L_b(E,F_b')$ is (DF), then the pair (E,F) satisfies the property (BB).

Proof. Let $(U_k)_{k\in\mathbb{N}}$, $(V_k)_{k\in\mathbb{N}}$ be the 0-basis in E and F respectively. Given $\alpha_k := 1$ ($k \in \mathbb{N}$) we find

$$(P_k)_{k\in\mathbb{N}} \subset L(E,E), \qquad (R_k)_{k\in\mathbb{N}} \subset L(F,F)$$

satisfying (FG1, 2, 3) and we set

$$\widehat{P}_i := \sum_{k=1}^{i} P_k, \qquad \widehat{R}_i := \sum_{k=1}^{i} R_k, \qquad i \in \mathbb{N}.$$

By Remark 4.3, it is enough to check that $(\widehat{P}_i \otimes \widehat{R}_i)(B)$ is contained in a localized bounded subset of $E\widehat{\otimes}_\pi F$ for every bounded subset B of $E\widehat{\otimes}_\pi F$. Moreover it is enough to check it for $(P_k \otimes R_t)(B)$, $k,t \in \mathbb{N}$. By (FG3), we have the continuous factorization

$$\begin{array}{ccc} E\widehat{\otimes}_\pi F & \xrightarrow{\;P_k\otimes R_t\;} & E\widehat{\otimes}_\pi F \\ {\scriptstyle \pi_k^E\otimes\pi_t^F}\searrow & & \nearrow{\scriptstyle \widetilde{P}_k\otimes\widetilde{R}_t} \\ & E_k\widehat{\otimes}_\pi F_t & \end{array}$$

Now $\pi_k^E \otimes \pi_t^F(B)$ is a bounded subset of the Banach space $E_k\widehat{\otimes}_\pi F_t$, hence we can find bounded subsets K, L in E_k, E_t respectively such that $\pi_k^E \otimes \pi_t^F(B) \subset \overline{\Gamma(K\otimes L)}$, the closure taken in $E_k\widehat{\otimes}_\pi F_t$. Consequently

$$P_k \otimes R_t(B) \subset \widetilde{P}_k \otimes \widetilde{R}_t\overline{(\Gamma(K\otimes L))} \subset \overline{\Gamma(\widetilde{P}_k(K)\otimes\widetilde{R}_t(L))}$$

the closure now taken in $E\widehat{\otimes}_\pi F$. □

Theorem 5.

Let E be an (FG)-space and let (G,t) be a (DFG)-space. Then $L_b(E,G)$ is a (DF)-space.

Proof. Let $(U_k)_{k\in\mathbb{N}}$ be the 0-basis in E as in the definition of (FG)-space. Let $(B_k)_{k\in\mathbb{N}}$ be the fundamental sequence of closed absolutely convex bounded subsets in (G,t) and s the locally convex topology in G coarser than t as in the definition of (DFG)-space. Let $\mathcal{U}_0$ be a basis of absolutely convex s-closed 0-neighbourhoods in (G,t). The sequence

$$\mathcal{B}_k := \Big\{f \in L_b(E,G)\ ;\ f(U_k) \subset B_k\Big\}$$

is a fundamental sequence of bounded subsets in $L_b(E,G)$ and the family of all sets of the form

$$\mathcal{W}(A,V) := \{f \in L_b(E,G)\ ;\ f(A) \subset V\}$$

for a bounded set A in E and a 0-neighbourhood V in $\mathcal{U}_0$, forms a 0-basis in $L_b(E,G)$.

According to [10, Lemma A], it is enough to check that for every sequence $(A_k)_{k\in\mathbb{N}}$ of bounded subsets of E, every sequence $(V_k)_{k\in\mathbb{N}}$ in $\mathcal{U}_0$ and every sequence $(\lambda_k)_{k\in\mathbb{N}}$, $\lambda_k > 0$, $k \in \mathbb{N}$, the set

$$\mathcal{W} := \bigcap_{n\in\mathbb{N}} \left(\mathcal{W}(A_n,V_n) + \sum_{k=1}^{n} \lambda_k \mathcal{B}_k\right)$$

is a 0-neighbourhood in $L_b(E,F)$.

Since E is metrizable we may find an absolutely convex bounded subset A of E such that for all $k \in \mathbb{N}$ there is $\rho_k > 0$ with $\rho_k A_k \subset A$. We put

$$M_k := \sup\big\{\|x\|_k\ ;\ x \in A\big\}.$$

Since (G,t) is a (DF)-space there is V in $\mathcal{U}_0$ such that for all $k \in \mathbb{N}$ there is $\mu_k > 0$ with $V \subset \mu_k V_k$. Now for every $k \in \mathbb{N}$ there is $N_k > 0$ with $N_k B_k \subset V$.

We put now for $k \in \mathbb{N}$

$$\alpha_k := \min\Big(2^{-k}, 2^{-k}\lambda_k, 2^{-k}M_k^{-1}, \min\big(2^{-2k}M_k^{-1}\mu_i^{-1}\rho_i\ ;\ 1 \le i \le k\big)\Big),$$
$$\beta_k := \min\Big(2^{-k}, 2^{-k}\lambda_k, 2^{-k}N_k, \min\big(2^{-2k}N_k\rho_i\mu_i^{-1}\ ;\ 1 \le i \le k\big)\Big).$$

Given $(\alpha_k)_{k\in\mathbb{N}}$ we select $(P_k)_{k\in\mathbb{N}} \subset L(E,E)$ satisfying (FG1,2,3) and we denote by C the closed absolutely convex hull of

$$\bigcup_{k\in\mathbb{N}} \alpha_k^{-1} P_k(U_k).$$

Given $(\beta_k)_{k\in\mathbb{N}}$ we select $(Q_k)_{k\in\mathbb{N}} \subset L(G,G)$ satisfying (DFG1,2,3) and we put

$$W := \bigcap_{k\in\mathbb{N}} \beta_k Q_k^{-1}(B_k).$$

By Lemma 2.6, C is a bounded subset of E and W is a 0-neighbourhood in (G,t). We will check that

$$\mathcal{V} := \{f \in L(E,G)\ ;\ f(C) \subset W\} \subset \mathcal{W},$$

which concludes the proof. To do this we fix $n \in \mathbb{N}$, $f \in \mathcal{V}$ and show that

$$f \in \mathcal{W}(A_n, V_n) + \sum_{k=1}^{n} \lambda_k \mathcal{B}_k.$$

If $i \in \mathbb{N}$ we define

$$f_i := \sum_{j=1}^{i} Q_j f P_i + \sum_{s=1}^{i-1} Q_i f\ P_s.$$

We first prove $\lambda_i^{-1} f_i \in \mathcal{B}_i$, $i \in \mathbb{N}$. Fix $x \in U_i$. We may write

$$\lambda_i^{-1} f_i(x) = \sum_{j=1}^{i} \frac{1}{2^{i+j}} (2^j Q_j)\ f(2^i \lambda_i^{-1} P_i(x)) + \sum_{s=1}^{i-1} \frac{1}{2^{i+s}} (2^i \lambda_i^{-1} Q_i)\ f(2^s P_s(x)).$$

For $j \leq i$ we have

$$2^i \lambda_i^{-1} P_i(x) \in 2^i \lambda_i^{-1} P_i(U_i) \subset \alpha_i^{-1} P_i(U_i) \subset C,$$

hence $f(2^i \lambda_i^{-1} P_i(x)) \in W$. From this it follows

$$(2^j Q_j)\ f(2^i \lambda_i^{-1} P_i(x)) \in 2^j \beta_j B_j \subset B_j \subset B_i.$$

On the other hand, for $s < i$ we have

$$2^s P_s(x) \in 2^s P_s(U_i) \subset 2^s P_s(U_s) \subset \alpha_s^{-1} P_s(U_s) \subset C,$$

hence $f(2^s P_s(x)) \in W$. Consequently

$$(2^i \lambda_i^{-1} Q_i)\ f(2^s P_s(x)) \in 2^i \lambda_i^{-1} \beta_i B_i \subset B_i.$$

Since

$$\sum_{j=1}^{i} \frac{1}{2^{i+j}} + \sum_{s=1}^{i-1} \frac{1}{2^{i+s}} < 1$$

and $\mathcal{B}_i$ is absolutely convex we have $\lambda_i^{-1} f_i \in \mathcal{B}_i$ as desired.

Now

$$\sum_{k=1}^{n} f_k \in \sum_{k=1}^{n} \lambda_k \mathcal{B}_k.$$

We put

$$g_n := f - \sum_{k=1}^{n} f_k.$$

Certainly $g_n \in L(E, G)$. We check $g_n \in \mathcal{W}(A_n, V_n)$. As a first step we check that for each $m > n$ and $x \in A_n$, $\sum_{i=n+1}^{m} f_i(x) \in V_n$. We set

$$\sum_{i=n+1}^{m} f_i(x) = \sum_{i=n+1}^{m} \frac{1}{2^i}(2^i f_i)(x).$$

We may write

$$2^i f_i(x) = \sum_{j=1}^{i} \frac{1}{2^{i+j}}(2^j \mu_n^{-1} Q_j)\ f(\mu_n 2^{2i} P_i(x)) + \sum_{s=1}^{i-1} \frac{1}{2^{i+j}}(2^{2i} \rho_n^{-1} Q_i)\ f(\rho_n 2^s P_s(x)).$$

Fix $j \leq i$ and recall $n < i$. Since $x \in A_n$, $\rho_n x \in A \subset M_i U_i$, this implies $x \in M_i \rho_i^{-1} U_i$, hence

$$\mu_n 2^{2i} P_i(x) \in \mu_n M_i \rho_n^{-1} 2^{2i} P_i(U_i) \subset \alpha_i^{-1} P_i(U_i) \subset C.$$

Therefore $f(\mu_n 2^{2i} P_i(x)) \in W$, consequently

$$(2^j \mu_n^{-1} Q_j)\ f(\mu_n 2^{2i} P_i(x)) \in 2^j \mu_n^{-1} \beta_j B_j \subset 2^j \mu_n^{-1} \beta_j N_j^{-1} V \subset 2^j \beta_j N_j^{-1} V_n \subset V_n.$$

Now fix $s < i$. Since $x \in A_n$,

$$\rho_n 2^s P_s(x) = 2^s P_s(\rho_n x) \in 2^s M_s P_s(U_s) \subset \alpha_s^{-1} P_s(U_s) \subset C,$$

hence $f(\rho_n 2^s P_s(x)) \in W$. This yields

$$2^{2i}\rho_n^{-1}Q_i\, f(\rho_n 2^s P_s(x)) \in 2^{2i}\rho_n^{-1}\beta_i B_i \subset 2^{2i}\rho_n^{-1}\beta_i N_i^{-1}V$$
$$\subset 2^{2i}\rho_n^{-1}\beta_i N_i^{-1}\mu_n V_n \subset V_n \quad (\text{recall } n < i).$$

This implies $\sum_{i=n+1}^{m} f_i(x) \in V_n$, $\forall m > n$, $\forall x \in A_n$. Certainly V_n is closed in $\big(G, \sigma(G,(G,s)')\big)$. If we prove that for every $x \in E$, $f(x) = \sum_{i=1}^{\infty} f_i(x)$ in the topology $\sigma(G,(G,s)')$, we get $g_n(x) = \sum_{i=n+1}^{\infty} f_i(x)$ for the topology $\sigma(G,(G,s)')$, which yield $g_n(x) \in V_n$ if $x \in A_n$ as desired. First observe that if $x \in E$, $n \in \mathbb{N}$, $\sum_{i=1}^{n} f_i(x) = (Q_1 + \cdots + Q_n)f(P_1 + \cdots + P_n)(x)$. Moreover,

$$(G,\sigma(G,(G,s)')) - \lim_{n\to\infty}(Q_1 + \ldots + Q_n)f(x) = f(x), \quad \forall x \in E.$$

Consequently we are done if we show

$$(G,\sigma(G,(G,s)')) - \lim_{n\to\infty}(Q_1 + \ldots + Q_n)\, f(x - (P_1 + \ldots + P_n)(x)) = 0, \quad x \in E.$$

Fix $v \in (G,s)' \subset (G,t)'$. It is easy to see that $\{\ {}^t(Q_1 + \ldots + Q_n)(v)\,;\, n \in \mathbb{N}\ \}$ is bounded in $(G',\sigma(G',G))$, hence $\{\,{}^tf^t(Q_1 + \ldots + Q_n)(v); n \in \mathbb{N}\}$ is bounded in $\sigma(E',E)$, hence equicontinuous. As $\lim_{n\to\infty}(P_1 + \ldots + P_n)(x) = x$ in the topology of E and

$$\langle (Q_1 + \ldots + Q_n)\, f(x - (P_1 + \ldots + P_n)(x)), v\rangle$$
$$= \langle x - (P_1 + \ldots + P_n)(x),\, {}^tf^t(Q_1 + \ldots + Q_n)(v)\rangle$$

the conclusion follows and the proof is complete. □

Corollary 6.

If E and F are (FG)-spaces, then (E,F) satisfies the property (BB).

Proof. By Proposition 2.3, F_b' is a (DFG)-space. We can apply Theorem 4.5 to conclude that $L_b(E,F_b')$ is a (DF)-space. The conclusion follows from Lemma 4. □

To treat the quasibarrelledness of the space $L_b(E,G)$ if E is an (FG)-space and G is a (DFG)-space, we need the density condition. The density condition was introduced by Heinrich [27] in the context of ultrapowers of locally convex spaces. It was studied for Fréchet spaces by Bierstedt and Bonet [8]. A Fréchet space E has the density condition if and only if the bounded subsets of E_b' are metrizable and if and only if $l_1\widehat{\otimes}_\pi E$ is distinguished and if and only if $L_b(E,l_\infty)$ is quasibarrelled. Consequently every quasinormable Fréchet space and every Fréchet

Montel space satisfies the density condition. The density condition for Köthe echelon spaces of order p, $1 \leq p < \infty$ or $p = 0$, was characterized in [8]. The dual density condition for (DF)-spaces was studied in [9,10]. A (DF)-space G has the dual density condition if and only if the bounded subsets of G are metrizable and if and only if $l_\infty(G) = L_b(l_1, G)$ is quasibarrelled. In particular every (DF)-space satisfying the strict Mackey condition (cf. [25]) and every (DF)-space with a fundamental sequence of compact sets satisfies the dual density condition. The characterization of this property for weighted inductive limits of spaces of continuous functions can be seen in [10]. More information can be seen in [9,11,12].

Proposition 7.

(a) If E is an (FG)-space with the density condition anf G is a (DFG)-space with the dual density condition, then $L_b(E,G)$ is a quasibarrelled (DF)-space.

(b) If E and F are (FG)-spaces with the density condition, then $E\widehat{\otimes}_\pi F$ satisfies the density condition, hence it is distinguished.

Proof. (a) $L_b(E,G)$ is a (DF)-space by Theorem 4.5, with metrizable bounded subsets by [11, Prop. 1.6]. The conclusion now follows from a classical result of Grothendieck (see e.g. [33, 8.3.13(ii)]).

(b) By Corollary 4.6 the pair (E,F) satisfies the property (BB), hence it is enough to apply [11, Corollary 7] to conclude. □

Remark 8. Due to the results in [9,10], Proposition 7 is optimal in the following sense: (i) If E is an (FG)-space such that $L_b(E,G)$ is quasibarrelled for every quasibarrelled (DFG)-space G, then E satisfies the density condition (take $G = l_\infty$). (ii) If G is a (DFG)-space such that $L_b(E,G)$ is quasibarrelled for every (FG)-space E, then G satisfies the dual density condition (take $E = l_1$). (iii) If E is an (FG)-space such that $E\widehat{\otimes}_\pi F$ is distinguished for every distinguished (FG)-space F, then E satisfies the density condition.

Now we study the injective tensor product of two (DFG)-spaces.

Proposition 9.

Let (G,t) be a (DFG)-space. If X is a normed space, then $G\otimes_\varepsilon X$ is a (DF)-space.

Proof. We identify $G\otimes_\varepsilon X$ with the linear subspace of $L_e(X'_{co}, G)$ of all linear continuous mappings with finite dimensional range (see e.g. [31, 44.2]). Let $(B_k)_{k\in\mathbb{N}}$ be the fundamental sequence of bounded subsets of E and let s be the locally convex topology in G coarser than t given by the definition. We denote by V the unit ball of X. A fundamental sequence of bounded subsets of $G\otimes_\varepsilon X$ is given by

$$\mathcal{B}_k := \{z \in G\otimes_\varepsilon X \ ; \ z(V^\circ) \subset B_k\}.$$

If $\mathcal{U}_0$ is the basis of s-closed absolutely convex 0-neighbourhoods in (G, t), the sets

$$\mathcal{W}(U) := \{z \in G\otimes_\varepsilon X \ ; \ z(V^\circ) \subset U\}, \qquad U \in \mathcal{U}_0,$$

form a basis of 0-neighbourhoods in $G\otimes_\varepsilon X$.

We fix a sequence $(U_k)_{k\in\mathbb{N}}$ in $\mathcal{U}_0$ and a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive numbers and we prove that

$$T := \bigcap_{n\in\mathbb{N}} \left(\mathcal{W}(U_n) + \sum_{k=1}^{n} \lambda_k \, \mathcal{B}_k \right)$$

is a 0-neighbourhood in $G\otimes_\varepsilon X$. To do this we first find $\mu_n > 0$ such that $U := \bigcap_{n\in\mathbb{N}} \mu_n U_n$ is a 0-neighbourhood in (G, t) and now $N_k > 0$ with $B_k \subset N_k U$, $k \in \mathbb{N}$. Then we put for $k \in \mathbb{N}$, $\alpha_k := \min\left(\lambda_k, \min(2^{-k} N_k^{-1} \mu_k^{-1} \ ; \ 1 \le i \le k)\right)$. Given $(\alpha_k)_{k\in\mathbb{N}}$, we select $(Q_k)_{k\in\mathbb{N}} \subset L(G, G)$ satisfying (DFG1,2,3). By Lemma 2.6, $W := \bigcap_{k\in\mathbb{N}} \alpha_k \, Q_k^{-1}(B_k)$ is a 0-neighbourhood in (G, t). We check that

$$\mathcal{W} := \{z \in G\otimes_\varepsilon X \ ; \ z(V^\circ) \subset W\} \subset T.$$

Fix $z \in \mathcal{W}$. For all $i \in \mathbb{N}$, we set $z_i := Q_i \circ z \in G\otimes_\varepsilon X$. First we show that $z_i \in \lambda_i \, \mathcal{B}_i$, $i \in \mathbb{N}$. Indeed, if $u \in V^\circ$,

$$\lambda_i^{-1} z_i(u) = \lambda_i^{-1} Q_i z(u) \in \lambda_i^{-1} Q_i(W) \subset \lambda_i^{-1} \alpha_i B_i \subset B_i.$$

Now if $n \in \mathbb{N}$, put $y_n := z - (z_1 + \cdots + z_n)$. We are done if we show that $y_n \in \mathcal{W}(U_n)$ for all $n \in \mathbb{N}$. Given $u \in V^\circ$,

$$y_n(u) = z(u) - (Q_1 + \cdots + Q_n)(z(u)) = \sum_{i=n+1}^{\infty} Q_i(z(u)),$$

where the series converges for the topology s. It is enough to check that if $i > n$, $Q_i(2^i z(u)) \in U_n$. But

$$Q_i(2^i z(u)) = 2^i Q_i(z(u)) \in 2^i \, Q_i(W) \subset 2^i \alpha_i B_i \subset 2^i \alpha_i N_i U \subset 2^i \alpha_i N_i \mu_n U_n \subset U_n.$$

This yields $z \in T$ and $G\otimes_\varepsilon X$ is a (DF)-space. □

Proceeding in the former proof we obtain

Proposition 10.

Let (G,t) be a (DFG)-space. If X is a normed space, then the ε-product of Schwartz $G\varepsilon X$ is also a (DF)-space.

Proposition 11.

Let (G,t) and (H,t') be (DFG)-spaces. Then $G\otimes_\varepsilon H$ is a (DF)-space.

Proof. This follows directly from Proposition 9 and [20, Prop. 5]. □

Proposition 12.

Let (G,t) be a strong (DFG)-space. If X is a normed space, then $G\otimes_\varepsilon X$ is quasibarrelled.

Proof. This follows with an argument similar to the proof of Proposition 9. □

Corollary 13.

(a) Let (G,t) and (H,t') be strong (DFG)-spaces. Then $G\otimes_\varepsilon H$ is quasibarrelled.
(b) Moreover, if G and H are bornological, than $G\otimes_{\tilde{\varepsilon}} H$ is bornological.

Proof. The first part follows directly from Proposition 12 and [20, 5 and 6]. Concerning the second part again by [20, 5 and 6] it is enough to prove that $G\otimes_\varepsilon X$ is bornological for every normed space X, if G is a bornological (DFG)-space. But this follows from Proposition 12 and [29, 3.4]. □

Proposition 14.

If (G,t) and (H,t') are (DFG)-spaces with the dual density condition, then $G\otimes_\varepsilon H$ is a (DF)-space with the dual density condition and hence quasibarrelled.

Proof. By Proposition 11, $G\otimes_\varepsilon H$ is a (DF)-space. On the other hand, $G\otimes_\varepsilon H$ is a topological subspace of $L_e(G'_{co},H)$, which in turn is a topological subspace of $L_b(G'_b,H)$. Indeed, first of all $L(G'_{co},H)\subset L(G'_b,H)$. Moreover, since G is quasibarrelled, every bounded subset of G'_b is (G,t)-equicontinuous. Now G'_b is a Fréchet space with the density condition and H has a fundamental sequence of bounded sets which are metrizable. By [11, Prop. 1.6] $L_b(G'_b,H)$ has metrizable bounded subsets. Consequently its topological subspace $G\otimes_\varepsilon H$ also has metrizable bounded subsets. Since it is a (DF)-space. it satisfies the dual density condition by [10, Theorem 1.5]. □

If you compare Proposition 7 and Remark 8 with the previous Corollary 13 and Proposition 14, you will immediately see that if E and F are (FG)-spaces, the spaces $L_b(E, F_b')$ and $E_b' \otimes_\varepsilon F_b'$ have a different behaviour with respect to the property of being quasibarrelled, e.g., $L_b(\lambda_2, l_\infty)$ is quasibarrelled if and only if λ_2 satisfies the density condition while $(\lambda_2)_b' \otimes_\varepsilon l_\infty$ is always quasibarrelled.

We close this section by giving a complete characterization of the quasibarrelledness of $L_b(E, G)$ when E is a Köthe echelon space of order $1 \leq p < \infty$ or $p = 0$ and G is a co-echelon space of order $1 \leq q \leq \infty$ or $q = 0$. We shall freely use the notation of [14], and we refer to this article for unexplained notation.

If $A = (a_n)_{n \in \mathbb{N}}$ is a Köthe matrix on $\mathbb{N}$, i.e. $0 < a_n(i) \leq a_{n+1}(i)$ $\forall i, n \in \mathbb{N}$, we say that A satisfies condition (M) if

$$\text{(M)} \qquad \forall n \ \forall I \subset \mathbb{N} \text{ infinite } \exists n' \in \mathbb{N} \ : \ \inf_{i \in I} \frac{a_n(i)}{a_{n'}(i)} = 0.$$

It is well-known that A satisfies condition (M) if and only if $\lambda_p(A)$ is Montel, $1 \leq p \leq \infty$ or $p = 0$ (see e.g. [14]).

We say that A satisfies condition (D) if there is an increasing sequence $J = (I_m)_{m \in \mathbb{N}}$ of subsets of $\mathbb{N}$ satisfying

$$\text{(N,J)} \qquad \forall m \ \exists n_m \ \forall k \geq n_m \ : \ \inf_{i \in I_m} \frac{a_{n_m}(i)}{a_k(i)} > 0,$$

$$\text{(M,J)} \qquad \forall n \ \forall I \subset \mathbb{N} \text{ such that } I \cap (\mathbb{N} \setminus I_m) \neq \emptyset \text{ for all } m, \exists n' > n \ : \ \inf_{i \in I} \frac{a_n(i)}{a_{n'}(i)} = 0.$$

By [8], A satisfies condition (D) if and only if $\lambda_p(A)$ has the density condition, $1 \leq p < \infty$ or $p = 0$. For $p = 1$, condition (D) is equivalent to $\lambda_1(A)$ being distinguished.

If $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ is a decreasing sequence of strictly positive weights on $\mathbb{N}$, we say that $\mathcal{V}$ satisfies condition (M) (resp. (D)) if and only if the Köthe matrix $(1/v_n)_{n \in \mathbb{N}}$ has condition (M) (resp. (D)). In what follows $\bar{V}$ will stand for the maximal Nachbin system associated to $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$, i.e. all those non-negative weights $\bar{v}$ on $\mathbb{N}$ such that $\sup_{i \in \mathbb{N}} \bar{v}(i)/v_n(i) < \infty$ for all $n \in \mathbb{N}$.

Theorem 15.

Let A be a Köthe matrix on $\mathbb{N}$ and let $\mathcal{V}$ be a decreasing sequence of strictly positive weights on $\mathbb{N}$. Let $1 \leq p < \infty$ or $p = 0$ and $1 \leq q \leq \infty$ or $q = 0$. The space $L_b(\lambda_p(A), k_q(\bar{V}))$ is always (DF). This space is quasibarrelled if and only if $\lambda_p(A)$ is distinguished, $k_q(\bar{V})$ is quasibarrelled and the conditions (1), (2), or (3) in the following diagram are satisfied

p \ q	0	1	$1<q<\infty$	∞
0	(1)	(2)	(2)	(1)
1	(1)	(1)	(1)	(2)
$1<p<\infty$	(1)	(2)	(3)	(1)

(1) : A has condition (M) or $\mathcal{V}$ has (M) or A and $\mathcal{V}$ have (D).
(2) : No condition (i.e., always quasibarrelled).
(3) : A has (M) or $\mathcal{V}$ has (M) or A and $\mathcal{V}$ have (D) or $p > q$.
In case $p \neq 1, q \neq \infty$, the result can be stated as follows, $L_b(\lambda_p(A), k_q(\bar{V}))$ is quasibarrelled if and only if every continuous linear operator from $\lambda_p(A)$ into $k_q(\bar{V})$ maps weakly convergent sequences into convergent ones or both matrices A and $\mathcal{V}$ satisfy condition (D).

Proof. If $q \neq 0$, the space $L_b(\lambda_p(A), k_q(\bar{V}))$ is the strong dual of $\lambda_p(A)\hat{\otimes}_\pi\lambda_{q'}(B)$, where $B = (b_n)_{n\in\mathbb{N}}$, $b_n = v_n^{-1}$ ($n \in \mathbb{N}$) and $q^{-1} + q'^{-1} = 1$ as usual. Consequently all the results in this case follow from [23, Theorem 15] (Observe that the case $p = 1, q \neq 0$ was already obtained in [8]).

It remains to consider the case $q = 0$. If $p = 1$ the result follows from [9, 2.5]. Hence we assume $p \neq 1, q = 0$. If A satisfies condition (M),

$$L_b(\lambda_p(A), k_0(\bar{V})) \cong \lambda_p(A)'_b\hat{\otimes}_\varepsilon k_0(\bar{V})$$

which is quasibarrelled by Proposition 4.7 above. If $\mathcal{V}$ satisfies condition (M), then $k_0(\bar{V}) = k_\infty(\bar{V})$ and we can apply the case $q = \infty$. If both A and $\mathcal{V}$ have condition (D), the result follows from Proposition 4.7.(a). To complete the proof we will show that if $L_b(\lambda_p(A), k_0(\bar{V}))$ is quasibarrelled and A and $\mathcal{V}$ do not satisfy condition (M), then A and $\mathcal{V}$ must satisfy condition (D). If A and $\mathcal{V}$ do not satisfy condition (M), then $\lambda_p(A)$ and $k_0(\bar{V})$ contain a complemented subspace isomorphic to l_p (or c_0 if $p = 0$) and c_0 respectively. Consequently both $L_b(l_p, k_0(\bar{V}))$ and $L_b(\lambda_p(A), c_0)$ must be quasibarrelled. First suppose that A does not satisfy condition (D). By [3, Prop. I.2.4 and I.2.7] there are $\lambda_j > 0$ $(j \in \mathbb{N})$, $n \in \mathbb{N}$ such that $\forall \bar{v} \in \bar{V}$ $\exists (i(m))_{m\in\mathbb{N}} \subset \mathbb{N}$ strictly increasing such that

$$\begin{gathered}\inf\left(\lambda_1 a_1(i(m))^{-1}, \ldots, \lambda_m a_m(i(m))^{-1}\right) > a_n(i(m))^{-1} \\ \bar{v}(i(m)) \le a_n(i(m))^{-1}.\end{gathered} \qquad (*)$$

We denote by

$$U_n := \left\{ x \in \lambda_p(A) \ ; \ \Big(\sum_{i\in\mathbb{N}} (a_n(i)\,|x_i|)^p\Big)^{1/p} \le 1 \right\} \qquad \text{if } p \neq 0,$$

$$U_n := \left\{ x \in \lambda_0(A) \ ; \ \sup_{i\in\mathbb{N}} a_n(i)\,|x_i| \le 1 \right\} \qquad \text{if } p = 0 \qquad n \in \mathbb{N},$$

and by V the unit ball of c_0. We put

$$\mathcal{B}_m := \left\{ f \in L(\lambda_p(A), c_0) \ ; \ f\Big(\bigcap_{n=1}^{m} \lambda_n U_n\Big) \subset V \right\}, \qquad m \in \mathbb{N}.$$

Clearly each $\mathcal{B}_m$ is closed and absolutely convex, $\mathcal{B}_m \subset \mathcal{B}_{m+1}$, $m \in \mathbb{N}$, and $\mathcal{C} := \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$ is bornivorous in $L_b(\lambda_p(A), c_0)$. Since this space is (DF), we have $\overline{\mathcal{C}} \subset 2\,\mathcal{C}$ (see e.g. [33, 8.2.27]). Consequently $\mathcal{C}$ is a 0-neighbourhood in $L_b(\lambda_p(A), c_0)$ and we can find $\bar{v} \in \bar{V}$ such that if B is the bounded subset of $\lambda_p(A)$ defined as

$$B := \left\{ x \in \lambda_p(A) \ ; \ \sum_{i\in\mathbb{N}} \left(\frac{|x_i|}{\bar{v}(i)}\right)^p \le 1 \right\} \qquad \text{if } p \neq 0$$

$$B := \left\{ x \in \lambda_0(A) \ ; \ \sup_{i\in\mathbb{N}} v(i)\,|x_i| \le 1 \right\} \qquad \text{if } p = 0,$$

then

$$\mathcal{W} := \{ f \in L_b(\lambda_p(A), c_0) \ ; \ f(B) \subset V \} \subset \mathcal{C}.$$

Given that $\bar{v} \in \bar{V}$ we use (*) to determine $(i(m))_{m\in\mathbb{N}} \subset \mathbb{N}$. Now define

$$f\colon \lambda_p(A) \longrightarrow c_0, \quad f(x) = (y_i)_{i\in\mathbb{N}}, \quad y_i = \begin{cases} a_n(i(m))x_{i(m)} & \text{if } i = i(m), \\ 0 & \text{otherwise.} \end{cases}$$

Since $x \in \lambda_p(A)$ it follows that f is well defined and continuous. Moreover if $x \in B$,

$$|a_n(i(m))\, x_{i(m)}| = a_n(i(m))\, \bar{v}(i(m))\, \frac{|x_{i(m)}|}{\bar{v}(i(m))} \le 1,$$

hence $f(x) \in V$. According to our assumption, there is some $m \in \mathbb{N}$ with $f \in \mathcal{B}_m$, i.e. $f(\bigcap_{j=1}^m \lambda_j U_j) \subset V$. But if $\mu := \inf(\lambda_1 a_1(i(m))^{-1}, \dots, \lambda_m a_m(i(m))^{-1})$, then $\mu e_{i(m)} \in \bigcap_{j=1}^m \lambda_j U_j$, but the $i(m)$-th coordinate of $f(\mu e_{i(m)})$ is $\mu a_n(i(m)) > 1$, hence $f(\mu e_{i(m)}) \notin V$. A contradiction.

Finally suppose that $L_b(l_p, k_0(\bar{V}))$ is quasibarrelled (with $l_p = c_0$ if $p = 0$) and that $\mathcal{V}$ does not satisfy condition (D). As before there are $\lambda_j > 0$ ($j \in \mathbb{N}$) and $n \in \mathbb{N}$ such that $\forall \bar{v} \in \bar{V}\ \exists\ (i(m))_{m\in\mathbb{N}} \subset \mathbb{N}$ strictly increasing with

$$\begin{gathered} \inf(\lambda_1 v_1(i(m)), \dots, \lambda_m v_m(i(m))) > v_n(i(m)) \\ \bar{v}(i(m)) \leq v_m(i(m)). \end{gathered} \tag{$**$}$$

We denote by V the unit ball of l_p and we define

$$\mathcal{B}_m := \left\{ f \in L(l_p, k_0(\bar{V}))\ ;\ \sup_{i\in\mathbb{N}} \inf\ (\lambda_1 v_1(i), \dots, \lambda_m v_m(i))\ |f(x)(i)| \leq 1 \quad \forall x \in V \right\}.$$

Clearly each $\mathcal{B}_m$ is closed and absolutely convex and $\mathcal{B}_m \subset \mathcal{B}_{m+1}$ for each $m \in \mathbb{N}$. Moreover $\mathcal{C} := \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$ is bornivorous. Again, since $L_b(l_p, k_0(\bar{V}))$ is a quasibarrelled (DF)-space, we may find $\bar{v} \in \bar{V}$ such that

$$\mathcal{W} := \left\{ f \in L(l_p, k_0(\bar{V}))\ ;\ \sup_{i\in\mathbb{N}} \bar{v}(i)\ |f(x)(i)| \leq 1 \quad \forall x \in V \right\} \subset \mathcal{C}.$$

Given $\bar{v} \in \bar{V}$ we select the sequence $(i(m))_{m\in\mathbb{N}} \subset \mathbb{N}$ by (**) and we define

$$f: l_p \to c_0(v_n) \to k_0(\bar{v})$$

for $x \in l_p$ by

$$f(x)(i) = \begin{cases} v_n(i(m))^{-1} x_{i(m)} & \text{if } i = i(m) \\ 0 & \text{otherwise} \end{cases}$$

Clearly $f: l_p \to c_0(v_n)$ is well-defined and continuous. Moreover if $x \in V$, $m \in \mathbb{N}$, we can apply (**) to get

$$\bar{v}(i(m))\ |f(x)(i(m))| = \bar{v}(i(m))\ v_n(i(m))^{-1} |x_{i(m)}| \leq 1.$$

Consequently $f \in \mathcal{W}$, and there is $m \in \mathbb{N}$ with $f \in \mathcal{B}_m$. Hence

$$\inf\ (\lambda_1 v_1(i(m)), \dots, \lambda_m v_m(i(m)))\ |f(x)(i_m)| \leq 1$$

for every $x \in V$. For $x = e_{i(n)}$ we obtain, by (**),

$$1 < \inf\ (\lambda_1 v_1(i(m)), \dots, \lambda_m v_m(i(m)))\ v_n(i_m)^{-1} \leq 1,$$

a contradiction.

This completes the proof. □

Remark 16. In the former theorem the space $k_0(\bar{V})$ may be replaced by its subspace $\kappa_0(\mathcal{V}) = \operatorname{ind} c_0(v_n)$ which is also a (DFG)-space by 3.6. Indeed, if $\mathcal{V}$ does not satisfy condition (M), even $\kappa_o(\mathcal{V})$ has a sectional subspace isomorphic to c_0. It is then enough to check that if $\mathcal{V}$ satisfies condition (M) and E is an arbitrary Fréchet space, then every bounded subset of $L_b(E, k_0(\bar{V}))$ is contained in the closure of a bounded subset of $L_b(E, \kappa_0(\mathcal{V}))$, because, in this case both spaces are simultaneously (DF) by [33, 8.3.24]. Let $(U_n)_{n\in\mathbb{N}}$ be a 0-basis of closed absolutely convex neighbourhoods in E. For $m \in \mathbb{N}$ define $\pi_m : k_0(\bar{V}) \to K_0(V)$ by $\pi_m(x) := (x_1, \ldots, x_m, 0, \ldots)$, $x \in k_0(\bar{V})$. A fundamental sequence of bounded sets in $L_b(E, k_0(\bar{V}))$ is given by

$$\mathcal{C}_n := \left\{ f \in L(E, k_0(\bar{V})) \ ; \ \sup_{i\in\mathbb{N}} v_n(i) \, |f(x)(i)| \le 1 \quad x \in U_n \right\}, \qquad n \in \mathbb{N}.$$

For each $f \in \mathcal{C}_n$,

$$(\pi_m \circ f)_{m\in\mathbb{N}} \subset L(E, K_0(V)) \qquad \text{and} \qquad \sup_{i\in\mathbb{N}} v_n(i) \, |(\pi_m \circ f)(x)(i)| \le 1 \ \forall x \in U_n.$$

It is enough to check that, if $\mathcal{V}$ has (M), then $(\pi_m \circ f)_{m\in\mathbb{N}}$ converges to f. To do this we fix a bounded subset B of E and $\bar{v} \in \bar{V}$. The bounded subset $f(B)$ of $k_0(\bar{V})$ is precompact, hence it is also precompact in the weighted c_0-space, $c_0(\bar{v})$. By a well-known characterization of the compact subsets of c_0,

$$\lim_{m\to\infty} \sup \left\{ \bar{v}(i) \, |f(x)(i)| \ ; \ x \in B, i \ge m \right\} = 0$$

and this implies

$$\lim_{m\to\infty} \sup_{x\in B} \bar{v}(i) \left| f(x)(i) - \pi_m \circ f(x)(i) \right| = 0$$

i.e. $(\pi_m \circ f)_{m\in\mathbb{N}}$ converges to f in $L_b(E, k_0(\bar{V}))$. □

Before closing this section it is important to recall that all the positive results given here remain true if the classes of (FG)-spaces and (DFG)-spaces are replaced by the classes of spaces isomorphic to complemented subspaces of (FG)-spaces and (DFG)-spaces.

We recall that every Fréchet-Schwartz space with the bounded approximation property and a continuous norm is a complemented subspace of a Fréchet-Schwartz space with finite-dimensional decomposition and a continuous norm by [7], hence a complemented subspace of an (FG)-space. Moreover by a recent result of P. Domanski, if X is a locally compact and σ-compact, then $C(X)$ endowed with the compact open topology is complemented in a countable product of Banach spaces, hence in an (FG)-space.

Vogt has characterized recently the hilbertizable Fréchet spaces which are isomorphic to a complemented subspace of a power series space $\Lambda_\infty(\alpha)$ (which is certainly an (FG)-space) in terms of conditions (DN) and (Ω) (see [41]).

5. New counterexamples to the problem of topologies of Grothendieck

In this last section we construct a separable Fréchet space E whose bounded subsets have a particular structure and a reflexive separable Banach space X such that the problem of topologies of Grothendieck is not satisfied for $E\widehat{\otimes}_\pi F$. This example in particular shows that the additional assumptions in [37, 3.2.3] are necessary.

Analizing this example we obtain some consequences about the injective tensor product of (DF)-spaces. In particular Corollary 3 shows that ε-(DF)-spaces do not behave as well as Banach spaces which are $\mathcal{L}_\infty$-spaces (see e.g. [28] or [33] for the definition of ε-space). It is important to recall that a Banach space F is an $\mathcal{L}_\infty$-space if and only if $E\otimes_\varepsilon F$ is (DF) for every (DF)-space E ([21]).

Theorem 1.

There exists a separable Fréchet space E having the property that each bounded set $B \subset E$ is contained in a bounded disk $D \subset E$ such that E_D is isomorphic to an l_1-space and there exists a separable reflexive Banach space X such that (E, X) does not satisfy the property (BB).

(Sketch of) Proof. a) Construction of E. For all $n \in \mathbb{N}$, large enough, we choose the spaces (G_n, g_n) and (M_n, g_n), $M_n \subset G_n$, as in Example 3.2.1 of [37]. We fix a continuous projection $P_n: G_n \to M_n$ and set $Q_n := id_{G_n} - P_n$. We choose an l_1-norm g'_n in the space $Q_n(G_n) =: N_n$ such that $g'_n(x) \geq g_n(x)$ for $x \in N_n$. This means that we choose some algebraic basis of N_n and form the l_1-norm with respect to this basis, and to get $g'_n \geq g_n$ we multiply this norm by a suitable constant. Since N_n is finite dimensional, there exists a number $C_n \in \mathbb{N}$ such that

$$(1) \qquad C_n g_n(x) > g'_n(x)$$

for all $x \in N_n$, and such that $(C_k)_{k=1}^\infty$ forms an increasing sequence. We define now for every $k \in \mathbb{N}$ and $x \in G_n$

$$(2) \qquad g_{nk}(x) := 2\, g_n(P_n x) + 2^{(n+k)C_{n+k}D_{n+k}}\, g'_n(Q_n x)$$

where $D_n \in \mathbb{N}$, $D_n \geq \max_{t \leq n-1}\{\dim(N_n), \|P_n\|, \|Q_n\|, D_t\}$, the operator norms taken in (G_n, g_n). Now the space E is constructed as in [35, Sect. 4], with this choice of M_n, G_n and g_n and replacing (4.4) by (2).

b) Construction of X. We choose X to be the l_2-sum of the finite dimensional Banach spaces $(M_n, g_n)'$. Then X is separable and reflexive.

c) On the bounded sets in E. Let $B \subset E$ be bounded. Then B is contained in a set $D := \{x \in E \; ; \; \sum_{m=1}^{\infty} r_m^{-1} \, \eta_m(x) \leq 1\}$, where the sequence (η_m) of norms (see [35, 4.4]) defines the topology of E, and for $m > 1$ the numbers r_m are assumed to be of the form

$$r_m = r_1 \, C_{t_m} D_{t_m} 2^{t_m C_{t_m} D_{t_m}} \tag{3}$$

for some $t_m \in \mathbb{N}$ (this is possible since the sequence $(C_t D_t \; 2^{t C_t D_t})_{t \in \mathbb{N}}$ is increasing and unbounded), and we also may assume

$$r_{m+1} > 2^{m+1} r_m \qquad \text{for all } m \in \mathbb{N}. \tag{4}$$

For $s \in \mathbb{N}$ let E_s be the s-th coordinate space of E (see [35, 4.4]). It is elementary to see from the definitions that E_D is the space

$$\Big(\bigoplus_{s \in \mathbb{N}} (E_s, E_s \cap D)\Big)_{l_1},$$

i.e. the l_1-sum of the Banach spaces E_s endowed with the Minkowski functionals of $E_s \cap D$. Hence, it suffices to show that the Banach-Mazur distance

$$d((E_s, E_s \cap D), l_1) \leq 12 \, \lambda \tag{5}$$

for all $s \in \mathbb{N}$, where λ is as in [37, Example 3.2.1].

It is easy to see (using (4) and the definition of the space E) that

$$E_s \cap D \subset \mathcal{D}_s := \{ \, x \in E_s \; ; \; r_1^{-1} h(x) + r_{s+1}^{-1} \tilde{h}(x) \leq 1 \, \} \subset 2\,(E_s \cap D) \tag{6}$$

(the meaning of h and $\tilde{h}$ is as in [35, 4.2]). Let us denote by G_{nk} the (n,k)-th coordinate space (isomorphic to G_n) of E_s. Again, it is easy to see that $(E_s, \mathcal{D}_s)$ is equal to

$$\Big(\bigoplus_{n,k} (G_{nk}, \mathcal{D}_s \cap G_{nk})\Big)_{l_1}.$$

So, to prove (5) it suffices to show that

$$d((G_{nk}, \mathcal{D}_s \cap G_{nk}), l_1^{\dim G_{nk}}) \leq 6 \, \lambda \tag{7}$$

for all n and k. Let n and k arbitrary. The Minkowski functional of $\mathcal{D}_s \cap G_{nk}$ is equal to

$$\begin{aligned} g_{nk}(x) :=& r_1^{-1} g_n(x) + r_{s+1}^{-1} g_{nk}(x) \\ =& r_1^{-1} g_n(x) + r_{s+1}^{-1}(2\, g_n(P_n x) + 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x)) \end{aligned} \tag{8}$$

where $x \in G_{nk}$. Recall that r_{s+1} is of the form (3) for some t_{s+1}. We have two cases:

(i) Assume that $t_{s+1} \geq n+k$. Then, by (3) and the choice of $(C_t)_{t\in\mathbb{N}}$, $(D_t)_{t\in\mathbb{N}}$,

$$\begin{aligned} & r_{s+1}^{-1}(2\, g_n(P_n x) + 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x)) \\ & \leq r_1^{-1} C_{t_{s+1}}^{-1} D_{t_{s+1}}^{-1} (g_n(P_n x) + g_n'(Q_n x)) \\ & \leq r_1^{-1} C_n^{-1} D_n^{-1} (g_n(P_n x) + g_n'(Q_n x)) \\ & \leq r_1^{-1} D_n^{-1} (g_n(P_n x) + g_n(Q_n x)) \\ & \leq 2\, r_1^{-1} g_n(x). \end{aligned} \tag{9}$$

So, in this case, $\forall x \in G_n$

$$r_1^{-1} g_n(x) \leq \hat{g}_{nk}(x) \leq 3\, r_1^{-1} g_n(x). \tag{10}$$

Since $d((G_n, g_n), l_1^{\dim G_n}) < \lambda$ (see [37, Example 3.2.1]), we get (7) by using (10).

(ii) Assume that $t_{s+1} \leq n+k-1$. Then

$$\begin{aligned} r_{s+1}^{-1} 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x)) &\geq r_1^{-1} C_{t_{s+1}}^{-1} D_{t_{s+1}}^{-1} 2^{C_{n+k}D_{n+k}}\, g_n'(Q_n x) \\ &\geq r_1^{-1} g_n'(Q_n x) \geq r_1^{-1} g_n(Q_n x). \end{aligned} \tag{11}$$

Now we show that

$$\begin{aligned} & \frac{1}{3}(r_1^{-1} g_n(P_n x) + r_{s+1}^{-1} 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x)) \leq \hat{g}_{nk}(x) \\ & \leq 2\, (r_1^{-1} g_n(P_n x) + r_{s+1}^{-1} 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x)). \end{aligned} \tag{12}$$

Let $x \in G_{nk}$. Case a: If $g_n(P_n x) \geq 2\, g_n(Q_n x)$, then

$$g_n(x) = g_n(P_n x + Q_n x) \geq g_n(P_n x) - g_n(Q_n x) \geq \frac{1}{2}\, g_n(P_n x). \tag{13}$$

Hence,

$$\begin{aligned} \hat{g}_{nk}(x) &= r_1^{-1} g_n(x) + r_{s+1}^{-1}(2\, g_n(P_n x) + 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x)) \\ &\geq \frac{1}{2}(r_1^{-1} g_n(P_n x) + r_{s+1}^{-1} 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x)) \end{aligned} \tag{14}$$

and the first inequality of (12) holds in this case.

Case b: If $g_n(P_n x) \leq 2\, g_n(Q_n x)$, then by (11) we have

$$r_1^{-1} g_n(P_n x) \leq 2\, r_1^{-1} g_n(Q_n x) \leq 2\, r_{s+1}^{-1} 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x)$$

and consequently the first inequality of (12) holds.

The second inequality of (12) follows in both cases a and b from (use (11))

$$\begin{aligned} r_1^{-1} g_n(x) &\leq r_1^{-1} g_n(P_n x) + r_1^{-1} g_n(Q_n x) \\ &\leq r_1^{-1} g_n(P_n x) + r_{s+1}^{-1}\, 2^{(n+k)C_{n+k}D_{n+k}} g_n'(Q_n x). \end{aligned}$$

Now (12) implies (7) also in the case (ii), by the choice of g_n' and since $d(M_n,\ l_1^{\dim M_n}) < \lambda$, by [37, 3.2.1].

d) The fact that (E, X) does not satisfy property (BB) is proved as in [35, 4.5]. □

Corollary 2.

There exist an ε-(DF)-space G and a reflexive Banach space X such that $L_b(X, G)$ is not (DF).

Proof. We take X and E as in the theorem above, and $G := E_b'$. It is clear that G is an ε-(DF)-space, moreover it is a projective limit of $\mathcal{L}_\infty$-spaces. Now, the result follows from the theorem above and Proposition 4.2 (note that E and X are separable). □

Corollary 3.

There exist an ε-(LB)-space G and a reflexive Banach space X such that $G \otimes_\varepsilon X_b'$ is not a (DF)-space.

Proof. Let E and X be as in the statement of Theorem 1. We first show that E has a predual which is an ε-(LB)-space. Indeed, E is a Fréchet space of Moscatelli type (see [17] for definition) associated with l_1 and Banach spaces H, L ($H \to L$) where H and L are l_1-sums of finite dimensional Banach spaces (see the comments after [17, 2.5]). Then, we can construct c_0-sums of finite dimensional Banach spaces, say K and Y ($Y \to K$) such that $K_b' = H$ and $Y_b' = L$. We consider the (LB)-space of Moscatelli type (see [16]) $G := \operatorname{ind} c_0((K)_{k<n}, (Y)_{k \geq n})$. According to [17, 2.6] we have that $G_b' = E$. Now E_b' is an ε-space and $E_b' = G_b'' = G_n''$ where n is the natural topology. Thus, by [28, Prop. 2.12] it follows that G is also an ε-space and our assertion is proved.

To finish we assume that $G \otimes_\varepsilon X_b'$ is a (DF). Hence it is quasibarrelled since G and X are separable (see [31, 29.3.12]). Then we apply [21, Prop. 2] (note that X has a finite dimensional decomposition so it has the approximation property) to get that (E, X) satisfies the property (BB), contradicting Theorem 1. Thus $G \otimes_\varepsilon X_b'$ is not a (DF)-space. □

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