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Rotund and uniformly rotund Banach spaces

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Abstract

In this paper we prove that the geometrical notions of Rotundity and Uniform Rotundity of the norm in a Banach space are stable for the generalized Banach products, which enclose the products that M.M. Day deals with in [2] and [3].

1. Introduction

In 1936 J.A. Clarkson [1], introduced the notion of uniform rotundity of the norm in a Banach space. He proved that for $p > 1$ the spaces L^p and l^p are uniformly rotund.

Later, in 1941, M.M. Day [2], studied the behaviour of this property in relation to the Banach products of l^p type. With this paper, we expand the Day results, and we enlarge on the family of uniformly rotund Banach spaces, establishing the stability of this property for the spaces introduced by R. Huff in [5], that generalize "Banach products" for a countable quantity of spaces. In addition, we establish analogous results for the notion of rotundity of the norm in a Banach space.

We follow the standard terminology that can be found in [6]. Let $(X, \|\cdot\|)$ be a Banach space. B_X denotes its closed unit ball, S_X the surface of B_X. $\mathbb{K}$ denotes the field of real or complex numbers.

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Now, let us consider the following definitions:

DEFINITION 1.1. A Banach space $(X, \|\cdot\|)$ is *Rotund (R)* if, given $x, y \in S_X$ with $x \neq y$, then $\left\|\frac{x+y}{2}\right\| < 1$.

DEFINITION 1.2. $(X, \|\cdot\|)$ is *Uniformly Rotund (UR)* if, given $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta \qquad \text{whenever } \|x - y\| \geq \varepsilon, \text{ and } x, y \in S_X.$$

The function $\delta : [0,2] \to [0,1]$, defined by

$$\delta(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x-y\| \geq \varepsilon\right\},$$

is called *The Modulus of Rotundity* of the space $(X, \|\cdot\|)$. S_X can be replaced by B_X in the former definition giving the same value for $\delta(\cdot)$.

It is clear, from the definition, that $(X, \|\cdot\|)$ is uniformly rotund if, and only if, for every $\varepsilon > 0$, $\delta(\varepsilon) > 0$. It is also clear that, if $\varepsilon_1 < \varepsilon_2$, $\delta(\varepsilon_1) < \delta(\varepsilon_2)$. At the same time it is obvious that $\delta(0) = 0$ and we have $0 \leq \delta(\varepsilon) \leq \varepsilon/2$, $\forall \varepsilon \in [0,2]$, therefore $\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$. Indeed, given $\varepsilon \in [0,2]$ and $x \in S_X$, we define $y = (1-\varepsilon)x$. Then $y \in B_X$ and $\|x - y\| = \varepsilon$. On the other hand

$$\left\|\frac{x+y}{2}\right\| = \left\|x - \frac{\varepsilon}{2}x\right\| = 1 - \frac{\varepsilon}{2}.$$

Therefore, $1 - \left\|\frac{x+y}{2}\right\| = \varepsilon/2$ and we have $0 \leq \delta(\varepsilon) \leq \varepsilon/2$.

It is obvious, from the definition, that

$$(UR) \longrightarrow (R).$$

DEFINITION 1.3. Let $\{X_n : n \in \mathbb{N}\}$ be a family of Banach spaces with the property *(UR)* and let δ_n be the modulus of rotundity of X_n, $n = 1, 2, \ldots$. It is said that the spaces $\{X_n : n \in \mathbb{N}\}$ have a *common modulus of rotundity* if

$$\inf\left\{\delta_n(\varepsilon) : n \in \mathbb{N}\right\} > 0 \qquad \forall \varepsilon : 0 < \varepsilon \leq 2.$$

It is clear that $\inf\{\delta_n(\varepsilon) : n \in \mathbb{N}\} > 0$ if and only if there is one function $\delta(\varepsilon) > 0$ which can be used in place of all $\delta_n(\varepsilon)$.

2. The product of rotund Banach Spaces

The following spaces have been used by R. Huff in [5].

DEFINITION 2.1. Let $(Y, \|\cdot\|)$ be a Banach space with basis $\{e_i : i \in I\}$ (unconditional if I is non-countable) and such that, for every finite subset J of I,

$$(1) \qquad 0 \leq |\alpha_j| \leq \beta_j, \quad \forall j \in J \quad \Longrightarrow \quad \Big\|\sum_{j\in J} \alpha_j e_j\Big\| \leq \Big\|\sum_{j\in J} \beta_j e_j\Big\|$$

Let $\{X_i : i \in I\}$ be a family of Banach spaces. Let us consider the space

$$Y(X_i : i \in I) = \Big\{x = (x_i)_{i\in I} \in \prod_{i\in I} X_i : \sum_{i\in I} \|x_i\| e_i \in Y\Big\},$$

endowed with the norm

$$\|x\| = \Big\|\sum_{i\in I} \|x_i\|\, e_i\Big\|_Y.$$

The space $Y(X_i : i \in I)$ with this norm is a Banach space.

In order to prove the stability of the properties *(R)* and *(UR)* for the spaces $Y(X_i : i \in I)$, as it was mentioned in the introduction, we shall use the following simple result.

Lemma 2.2

Let $(Y, \|\cdot\|)$ be a Banach space with basis $\{e_i : i \in I\}$ (unconditional if I is non-countable) satisfying (1). Let $\{X_i : i \in I\}$ be a family of Banach spaces. We define ϕ a mapping from $Z = Y(X_i : i \in I)$ into Y as

$$\phi\big((x_i)_{i\in I}\big) = \sum_{i\in I} \|x_i\| e_i, \qquad \forall (x_i) \in Z.$$

Then ϕ has the following properties:

(i) $\forall z \in Z$, $\|\phi(z)\| = \|z\|$.

(ii) $\forall z \in Z, \forall \lambda \in \mathbb{K}$, $\phi(\lambda z) = |\lambda|\, \phi(z)$.

(iii) $\forall z_1, z_2 \in Z$, $\|\phi(z_1) - \phi(z_2)\| \leq \|z_1 - z_2\|$.

(iv) $\forall z_1, z_2 \in Z$, $\|\phi(z_1) + \phi(z_2)\| \geq \|z_1 + z_2\|$.

Proof. *(i)* is just the definition of the norm in Z. *(ii)* is obvious. *(iii)* is easy: let $z_1 = (x_i^1)_{i\in I}$, $z_2 = (x_i^2)_{i\in I}$ be elements in Z. Then

$$\begin{aligned}
\|\phi(z_1) - \phi(z_2)\| &= \Big\|\sum_{i\in I}\|x_i^1\|e_i - \sum_{i\in I}\|x_i^2\|e_i\Big\| \\
&= \Big\|\sum_{i\in I}(\|x_i^1\| - \|x_i^2\|)e_i\Big\| \\
&\leq \Big\|\sum_{i\in I}\|x_i^1 - x_i^2\|\,e_i\Big\| = \|\phi(z_1 - z_2)\| = \|z_1 - z_2\|,
\end{aligned}$$

in view of $\Big|\|x_i^1\| - \|x_i^2\|\Big| \leq \|x_i^1 - x_i^2\|$, $\forall i \in I$.

(iv) is also easy: let $z_1 = (x_i^1)_{i\in I}$, $z_2 = (x_i^2)_{i\in I}$ be elements in Z. Then

$$\begin{aligned}
\|\phi(z_1) + \phi(z_2)\| &= \Big\|\sum_{i\in I}\|x_i^1\|e_i + \sum_{i\in I}\|x_i^2\|e_i\Big\| \\
&= \Big\|\sum_{i\in I}(\|x_i^1\| + \|x_i^2\|)e_i\Big\| \\
&\geq \Big\|\sum_{i\in I}\|x_i^1 + x_i^2\|e_i\Big\| = \|\phi(z_1 + z_2)\| = \|z_1 + z_2\|,
\end{aligned}$$

in view of $\|x_i^1 + x_i^2\| \leq \|x_i^1\| + \|x_i^2\|$, $\forall i \in I$. □

Theorem 2.3

Let $(Y, \|\cdot\|)$ be a Banach space with basis $\{e_i : i \in I\}$ (unconditional if I is non-countable) and such that, for every finite subset J of I,

$$0 \leq |\alpha_j| \leq \beta_j, \quad \forall j \in J \quad \Longrightarrow \quad \Big\|\sum_{j\in J}\alpha_j e_j\Big\| \leq \Big\|\sum_{j\in J}\beta_j e_j\Big\|.$$

Let $\{X_i : i \in I\}$ be a family of Banach spaces with the property (R). Then, if $(Y, \|\cdot\|)$ is a rotund space, $Z = Y(X_i : i \in I)$ has also this property.

Proof. Let us consider ϕ the mapping from Z into Y, defined in the previous lemma. Let $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I}$ be elements in Z, with $\|x\| = \|y\| = 1$ and $x \neq y$. The elements $\phi(x)$, $\phi(y)$ are in S_Y. Let us consider two cases:

(a) If $\phi(x) \neq \phi(y)$, then $\Big\|\frac{1}{2}(\phi(x) + \phi(y))\Big\| < 1$. From here, by virtue of lemma 2.2, we obtain

$$\Big\|\frac{x+y}{2}\Big\| < 1.$$

(b) If $\phi(x) = \phi(y)$, then $\|x_i\| = \|y_i\|$, $\forall i \in I$. As $x \neq y$, there will exist $i_0 \in I$ so that $x_{i_0} \neq y_{i_0}$. Due to the rotundity of the space X_{i_0}, we have

$$\Big\|\frac{1}{2}(x_{i_0} + y_{i_0})\Big\| < \frac{1}{2}\Big(\| x_{i_0}\| + \| y_{i_0}\|\Big).$$

Let us supose that $\|x + y\| = \|\phi(x + y)\| = 2$. Then

$$2 = \Big\|\sum_{i\in I}\|x_i + y_i\|\, e_i\Big\| \le \Big\|\sum_{i\in I} (\|x_i\| + \|y_i\|)\, e_i\Big\|$$
$$\le \Big\|\sum_{i\in I}\|x_i\|\, e_i\Big\| + \Big\|\sum_{i\in I}\|y_i\|\, e_i\Big\| = \|\phi(x)\| + \|\phi(y)\| = 2,$$

then

$$\Big\|\sum_{i\in I}\|x_i + y_i\|\, e_i\Big\| = \Big\|\sum_{i\in I} (\|x_i\| + \|y_i\|)\, e_i\Big\|.$$

Since Y is a rotund space, we have

$$\Big\|\sum_{i\in I} \frac{\|x_i + y_i\| + \big(\|x_i\| + \|y_i\|\big)}{2}\, e_i\Big\| < \Big\|\sum_{i\in I}\|x_i + y_i\|e_i\Big\|. \tag{2}$$

But

$$\|x_i + y_i\| \le \frac{\|x_i + y_i\| + \big(\|x_i\| + \|y_i\|\big)}{2}$$

hence

$$\Big\|\sum_{i\in I}\|x_i + y_i\|\, e_i\Big\| \le \Big\|\sum_{i\in I} \frac{\|x_i + y_i\| + \big(\|x_i\| + \|y_i\|\big)}{2}\, e_i\Big\|$$

which is in contradiction to (2). □

Bearing in mind that the space l^p with $1 < p < \infty$ is rotund Banach space [1] with basis $\{e_n : n \in \mathbb{N}\}$ that satisfies the conditions of theorem 2.3, we can obtain, as an inmediate consequence of the cited theorem, the following Day result [4]:

Corollary 2.4

Let p be with $1 < p < \infty$ and let $\{X_n : n \in \mathbb{N}\}$ be a countable family of Banach spaces. Let us consider the Banach space $l^p(X_1, X_2, \ldots)$ defined by

$$Z := l^p(X_1, X_2, \ldots) = \Big\{x = (x_n) \in \prod_{n=1}^{\infty} X_n : \|x\|_p^p = \sum_{n=1}^{\infty} \|x_n\|^p < +\infty\Big\}$$

and provided with $\| \cdot \|_p$. Then, Z is a rotund space if, and only if, the spaces X_n, $n = 1, 2, 3, \ldots$ are also rotund.

Similarly, the space L^p with $1 \leq p \leq \infty$ is rotund Banach space [1]. We choose the Haar system $\{\phi_n : n \in \mathbb{N}\}$ defined as follows. The function ϕ_1 is identically 1 on $[0,1]$; ϕ_2 is 1 on $[0,1/2)$, and -1 in $[1/2,1]$; ϕ_3 is 1 on $[0,1/4)$, -1 on $[1/4,1/2)$ and 0 on $[1/2,1]$; ϕ_4 is 1 on $[1/2,3/4)$, -1 on $[3/4,1)$ and 0 on $[0,1/2)$. For positive integers r and $k \leq 2^{r-1}$, $\phi_{2^{r-1}+k}$ is 1 on I^r_{2k-1}, -1 on I^r_{2k}, and 0 otherwise, where $[0,1]$ is partitioned into 2^r intervals $\{I^r_j : 1 \leq j \leq 2^r\}$ of equal lenghts.

$\{\phi_n : n \in \mathbb{N}\}$ is a basis that satisfies the condition of theorem 2.3. Therefore, the space

$$L^p(X) = \left\{ f : [0,1] \longrightarrow X \text{ Bochner integrable, } \|f\|^p = \int_0^1 \|f(x)\|^p \, dx < +\infty \right\}$$

is rotund if and only if $(X, \|\cdot\|)$ is also rotund. [4]

The spaces of $l^p(X_1, X_2, \ldots)$ type have been used by Day [3], in order to obtain rotund Banach spaces which cannot be renormed in a uniform rotund manner.

3. The product of uniformly rotund Banach spaces

Theorem 3.1

Let $(Y, \|\cdot\|)$ be a Banach space with basis $\{e_n : n \in \mathbb{N}\}$ and such that

$$0 \leq \alpha_n \leq \beta_n, \ \forall n \in \mathbb{N} \quad \Longrightarrow \quad \Big\|\sum_{n=1}^{\infty} \alpha_n e_n\Big\| \leq \Big\|\sum_{n=1}^{\infty} \beta_n e_n\Big\|.$$

Let $\{X_n : n \in \mathbb{N}\}$ be a family of Banach spaces with the property (UR) and a common modulus of rotundity. Let

$$Y(X_1, X_2, \ldots) = \left\{ x = (x_n) \in \prod_{n=1}^{\infty} X_n : \sum_{n=1}^{\infty} \|x_n\| e_n \in Y \right\}$$

equipped with the norm

$$\|x\| = \Big\|\sum_{n=1}^{\infty} \|x_n\| e_n\Big\|_Y$$

Then, if $(Y, \|\cdot\|)$ is uniformly rotund, $Y(X_1, X_2, \ldots)$ is uniformly rotund, too.

Proof. Let $\delta : [0,2] \longrightarrow [0,1]$ be the common modulus of rotundity of X_n, $n \in \mathbb{N}$, and $\delta_1 : [0,2] \longrightarrow [0,1]$ that of Y.

Let us consider $x = (x_n)$, $x' = (x'_n)$ elements in $Y(X_1, X_2, \ldots)$ with $\|x\| = \|x'\| = 1$. Let $\varepsilon > 0$ be given and suppose $\|x - x'\| \geq \varepsilon$. We will prove that exists $\delta_2 > 0$ such that

$$\left\|\frac{x + x'}{2}\right\| \leq 1 - \delta_2.$$

Let us consider two cases:

(a) First we assume $\|x_n\| = \|x'_n\|$, $\forall n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$

$$\|x_n + x'_n\| \leq 2\left(1 - \delta\Big(\frac{\|x_n - x'_n\|}{\|x_n\|}\Big)\right)\|x_n\|,$$

since x_n and x'_n both lie on the sphere of radius $\|x_n\|$ about the origin in X_n. Hence

$$(3)\qquad \|x + x'\| = \Big\|\sum_{n=1}^{\infty}\|x_n + x'_n\|\, e_n\Big\| \leq 2\,\Big\|\sum_{n=1}^{\infty}\Big(1 - \delta\Big(\frac{\|x_n - x'_n\|}{\|x_n\|}\Big)\Big)\|x_n\|\, e_n\Big\|$$

Consider

$$M := \left\{n \in \mathbb{N} : x_n \neq 0,\ \frac{\|x_n - x'_n\|}{\|x_n\|} > \frac{\varepsilon}{4}\right\}.$$

Therefore, if $n \in \mathbb{N} \setminus M$,

$$\|x_n\| \geq \frac{4}{\varepsilon}\|x_n - x'_n\|.$$

Then, we get

$$1 = \Big\|\sum_{n=1}^{\infty}\|x_n\|\, e_n\Big\| \geq \Big\|\sum_{n\in\mathbb{N}\setminus M}\|x_n\|\, e_n\Big\| \geq \frac{4}{\varepsilon}\Big\|\sum_{n\in\mathbb{N}\setminus M}\|x_n - x'_n\|\, e_n\Big\|,$$

that is

$$\Big\|\sum_{n\in\mathbb{N}\setminus M}\|x_n - x'_n\|\, e_n\Big\| \leq \frac{\varepsilon}{4}.$$

In addition, it can be easily proved that

$$\Big\|\sum_{n\in M}\|x_n - x'_n\|\, e_n\Big\| \geq \frac{3\varepsilon}{4}.$$

Hence

$$\Big\|\sum_{n\in M}\|x_n\|\, e_n\Big\| \geq \frac{1}{2}\Big\|\sum_{n\in M}\|x_n - x'_n\| e_n\Big\| \geq \frac{3\varepsilon}{8},$$

in view of $\|x_n - x'_n\| \leq 2\|x_n\|$, $\forall n \in \mathbb{N}$.

Now, we denote

$$y := \sum_{n\in\mathbb{N}\setminus M} \|x_n\|\, e_n, \quad y' := \sum_{n\in M} \|x_n\|\, e_n, \quad y'' := \Big(1 - 2\delta\Big(\frac{\varepsilon}{4}\Big)\Big)\, y',$$

elements in Y that satisfy

$$\|y + y''\| \leq \|y + y'\| = 1$$
$$\|y + y' - (y + y'')\| = 2\delta\Big(\frac{\varepsilon}{4}\Big)\ \|y'\| \geq 2\ \delta(\frac{\varepsilon}{4})\ \frac{3\varepsilon}{8} =: \alpha(\varepsilon).$$

By virtue of the property *(UR)* for Y

$$\frac{1}{2}\ \|y + y' + y + y''\| \leq 1 - \delta_1(\alpha(\varepsilon)),$$

that is

$$\Big\|y + \Big(1 - \delta\Big(\frac{\varepsilon}{4}\Big)\Big)\ y'\Big\| \leq 1 - \delta_1(\alpha(\varepsilon)). \tag{4}$$

From (3) and (4), we obtain

$$\begin{aligned}
\Big\|\frac{x + x'}{2}\Big\| &\leq \Big\| \sum_{\substack{n=1\\ x_n\neq 0}}^{\infty} \Big(1 - \delta\Big(\frac{\|x_n - x'_n\|}{\|x_n\|}\Big)\Big)\ \|x_n\|\, e_n\Big\| \\
&\leq \Big\| \sum_{n\in M} \Big(1 - \delta\Big(\frac{\varepsilon}{4}\Big)\Big)\ \|x_n\|\, e_n + \sum_{n\in\mathbb{N}\setminus M} \|x_n\|\, e_n\Big\| \\
&\leq \Big\|\Big(1 - \delta\Big(\frac{\varepsilon}{4}\Big)\Big)\ y' + y\Big\| \leq 1 - \delta_1(\alpha(\varepsilon)) =: 1 - \delta_0(\varepsilon),
\end{aligned}$$

where we have denoted

$$\delta_0(\varepsilon) = \delta_1(\alpha(\varepsilon)) = \delta_1\Big(\delta(\frac{\varepsilon}{4})\frac{3\varepsilon}{4}\Big).$$

(b) In the general case we suppose only that $\|x\| = \|x'\| = 1$ and that $\|x + x'\| > 2\,(1 - \delta_1(\mu))$, were $0 < \mu \leq 2$.

The lemma 2.2 shows that

$$2(1 - \delta_1(\mu)) < \|x + x'\| \leq \|\phi(x) + \phi(x')\|,$$

then

$$\|\phi(x) - \phi(x')\| = \left\|\sum_{n=1}^{\infty}(\|x_n\| - \|x'_n\|)\, e_n\right\| < \mu.$$

Now let $\varepsilon_n = \pm 1$, $n = 1,2,3,\ldots$. We will prove that, also

$$\left\|\sum_{n=1}^{\infty} \varepsilon_n \,(\|x_n\| - \|x'_n\|) e_n\right\| < \mu.$$

So, let $A = \{n \in \mathbb{N} : \varepsilon_n = 1\}$, $B = \{n \in \mathbb{N} : \varepsilon_n = -1\}$. We define

$$y_n = \begin{cases} x_n, & \text{if } n \in A \\ x'_n, & \text{if } n \in B \end{cases}, \qquad y'_n = \begin{cases} x'_n, & \text{if } n \in A \\ x_n, & \text{if } n \in B \end{cases}, \qquad n = 1,2,3,\ldots$$

Then $y = (y_n)$, $y' = (y'_n)$ are elements in $Y(X_1, X_2, \ldots)$ such that

$$\|\phi(y) + \phi(y')\| = \|\phi(x) + \phi(x')\| > 2\,(1 - \delta_1(\mu)),$$

then $\|\phi(y) - \phi(y')\| < \mu$.

But, it is easy to prove that

$$\|\phi(y) - \phi(y')\| = \left\|\sum_{n=1}^{\infty} \varepsilon_n(\|x_n\| - \|x'_n\|)\, e_n\right\|.$$

Now, we define

$$x''_n = \begin{cases} \dfrac{\|x_n\|}{\|x'_n\|}\, x'_n, & \text{if } x'_n \neq 0 \\ x_n, & \text{if } x'_n = 0 \end{cases}, \qquad n = 1,2,3,\ldots$$

We obtain $\|x''_n\| = \|x_n\|$, $\forall n \in \mathbb{N}$, then $x'' = (x''_n) \in Y(X_1, X_2, \ldots)$ and $\|x''\| = \|x\| = 1$. Also

$$\|x' - x''\| = \left\|\sum_{n=1}^{\infty} \|x'_n - x''_n\|\, e_n\right\| = \left\|\sum_{n=1}^{\infty} \varepsilon_n(\|x_n\| - \|x'_n\|)\right\| < \mu,$$

where $\varepsilon_n = \pm 1$ appropriately.

On the other hand

$$\|x + x''\| \geq \|x + x'\| - \|x'' - x'\| > 2\,(1 - \delta_1(\mu)) - \mu = 2\left(1 - \delta_1(\mu) - \frac{\mu}{2}\right).$$

Once $\varepsilon > 0$ is fixed, we take μ such that $0 < \mu < \varepsilon/2$ and $\delta_1(\mu) + \mu/2 < \delta_0(\varepsilon/2)$, where δ_0 has been defined in part (a). Then

$$\|x + x''\| > 2\left(1 - \delta_0\left(\frac{\varepsilon}{2}\right)\right).$$

By virtue of part (a) we have $\|x - x''\| < \varepsilon/2$.

Finally,

$$\|x - x'\| \leq \|x - x''\| + \|x'' - x'\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, given $0 < \varepsilon \leq 2$, μ has been defined, and therefore $\delta_2(\varepsilon) := \delta_1(\mu)$ in such a way that if $x, x' \in Y(X_1, X_2, \ldots)$ with $\|x\| = \|x'\| = 1$ and

$$\left\|\frac{x + x'}{2}\right\| > 1 - \delta_2(\varepsilon),$$

then $\|x - x'\| < \varepsilon$.

This finish the proof of the theorem. □

Since l^p with $1 < p < \infty$ is a Banach space uniformly rotund [1], with basis $\{e_n : n \in \mathbb{N}\}$ that satisfies the condition of theorem 3.2, we obtain, as an immediate consequence of the cited theorem, the result of M.M. Day [2], in which it is established that the space $l^p(X_1, X_2, \ldots)$ with the $\|\cdot\|_p$ is uniformly rotund if, and only if, the spaces X_n, $n = 1, 2, 3, \ldots$ have a common modulus of rotundity.

Analogously, $(L^p(X), \|\cdot\|_p)$ with $1 < p < \infty$ is uniformly rotund if and only if $(X, \|\cdot\|_p)$ also is. [2]

We can obtain this result as an immediate consequence of theorem 3.1 using the Haar system $\{\phi_n : n \in \mathbb{N}\}$ as a basis of the Banach space uniformly rotund L^p. It's also possible to prove it with an embedding argument similar to that used by Clarkson in analogous circumstances in L^p.

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