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A weighted Plancherel formula III. The case of the hyperbolic matrix ball

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Abstract

The group $\mathrm{SU}(2,2)$ acts naturally on an L^2-space on a hyperbolic matrix ball (type one bounded symmetric domain) with respect to the usual weighted measure. We will find the corresponding invariant Laplace operator and study its spectral resolution. The spherical functions (K-invariant eigenfunctions) can be expressed using hypergeometric functions. It turns out that, besides the weighted Bergman space, some discrete parts enter into the decomposition. The number of the discrete parts equals to the number of the orbits of the Weyl group action on the zeros (in the "lower half plane") of the generalized Harish-Chandra **c**-function. We calculate their reproducing kernels in a special case. As an application, we obtain decompositions of the tensor products of holomorphic discrete series representations. This improves an earlier result by J. Repka.

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0. Introduction

In the papers [21] and [27], we studied the problem of the explicit decomposition of the action of the Moebius group on weighted Hilbert spaces of functions on the rank one symmetric domains, which contain the usual weighted Bergman spaces as irreducible components. In particular, we found all the other irreducible discrete components in this decomposition and described these spaces using certain invariant Cauchy-Riemann operators.

In this paper. we study the same problem in the case of a hyperbolic matrix ball (a bounded symmetric domain of type one). We will find the invariant Laplacian and study its spectral resolution via the Harish-Chandra c-function. In our case, the c-function has zeros in a cone (lower half plane) in the (complexification of the) dual of a Cartan subalgebra, while the Weyl group acts on the zeros via permutation. Each orbit of the action gives us a discrete part in the decomposition. The invariant Laplacian is of the form Laplace-Beltrami operator plus a first order differential operator. We find exactly all of the discrete and the continuous spectrum.

As an application of our result, we find also the discrete parts of tensor products of holomorphic discrete series. This refines an earlier result obtained by Repka [23]. In the case of the unit disk, the irreducible decomposition of the tensor products of holomorphic discrete series was studied in [22], [15] and [28]. Our considerations also lead us to orthogonality relations of the continuous dual Hahn polynomials previously studied by Wilson [26]. In particular, we find that these polynomials are actually the Clebsch-Gordan coefficients for the tensor products of holomorphic discrete series of $SU(1,1)$ and that the weight in the orthogonality relations is a product of the Harish-Chandra c-function and the symbol function of a certain Berezin transform.

We note that a similar problem of decompostion of a Hilbert space of functions on the Shilov boundary of a bounded symmetric domain was studied in [16]. If we lift to the universal covering group of the automorphism group, the restriction of the representation to the isotropy group gives us a character of this group, while the spaces that we are studying consist of functions on the universal covering group which are right invariant with respect to the character.

We will restrict ourself on the 2×2 hyperbolic matrix ball. Much of the calculation can be carried out on any type I domain, and some for general symmetric domains as well; it is our belief that we eventually will be able to do the whole theory on that level of generality.

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1. The invariant Laplacian

Let D be the rank two type I tube type domain, that is, $D = D(I_{2,2})$ consists of all complex 2×2 matrices

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

with $Z^*Z < I$. Its automorphism group is the matrix group $SU(2,2)$ of all 4×4 matrices (written in block form)

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $\det g = 1$, which are unitary with respect to the indefinite metric

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 .$$

The action on D is

$$gZ = (AZ + B)(CZ + D)^{-1} .$$

Let $K(Z, W)$ be the reproducing kernel for the Bergman space on D. It is well-known that

$$K(Z, W) = \det (I - ZW^*)^{-4},$$

see [14], p. 84. The Bergman metric is then defined by $\partial\bar{\partial} \log K$. The corresponding Laplace-Beltrami operator is (see [14], pp. 117-118)

$$L = \operatorname{tr} \left[(I - ZZ^*)\bar{\partial}_Z \cdot (I - Z^*Z) \cdot \partial_Z'\right], \tag{1.1}$$

where ∂_Z is the differential operator

$$\partial_Z = \begin{pmatrix} \dfrac{\partial}{\partial z_{11}} & \dfrac{\partial}{\partial z_{12}} \\ \dfrac{\partial}{\partial z_{21}} & \dfrac{\partial}{\partial z_{21}} \end{pmatrix},$$

and the dots here indicate that the factor $(I - Z^*Z)$ is not differentiated. This operator commutes with the weight zero group action, that is

$$L(f(gZ)) = Lf(gZ),\ f \in C^\infty(D).$$

Now let us consider the Hilbert space $L^2(D, d\mu_\alpha)$, where $\alpha > -1$ and is not an odd integer, while $d\mu_\alpha(z) = C_\alpha \det(1 - Z^*Z)^\alpha\, dm(z)$, with

$$C_\alpha = \pi^4 \frac{\Gamma(\alpha+1)^2\Gamma(\alpha+2)^2}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(\alpha+4)},$$

dm being the Lebesgue measure, and $\mu_\alpha(D) = 1$. We let

$$\boxed{\nu = \alpha + 4}\,.$$

(The number 4 is the genus of D.) The group SU(2,2) admits the following action on $L^2(D, d\mu_\alpha)$:

$$U_g^{(\nu)}\colon f(Z) \to f(gZ)(J_g(Z))^{\frac{\nu}{4}},$$

where $J_g(Z)$ is the complex Jacobian of the transformation g. (If α is not an integer, we have to pass to the universal cover.) For simplicity, we drop the index ν in the notation writing $U_g = U_g^{(\nu)}$. We will find an invariant Laplacian intertwining the group representation. Let

$$l_\nu = \det(I - Z^*Z)^{-\nu} \operatorname{tr}\left[(I - ZZ^*)\bar\partial_Z \cdot (I - Z^*Z) \cdot \partial'_Z(\det(I - Z^*Z)^\nu)\right],$$

and set

$$L_\nu = L + l_\nu.$$

Lemma 1

The operator L_ν is invariant under $U = U^{(\nu)}$, that is, we have

$$L_\nu U_g = U_g L_\nu \quad (g \in \mathrm{SU}(2,2)) \tag{1.2}$$

on the space $C^\infty(D)$.

Proof. We let

$$E(Z) = I - ZZ^*,\ F(Z) = I - Z^*Z.$$

In this notation,

$$L = \operatorname{tr}\left[E\bar{\partial}_Z \cdot F\partial'_Z\right],$$
$$l_\nu = \det E^{-\nu}\operatorname{tr}\left[E\bar{\partial}_Z \cdot F\partial'_Z(\det E^\nu)\right].$$

We have $\det E = \det F$ and

$$\det E(g(Z)) = J_g(Z)\det E(Z)\overline{J_g(Z)}. \tag{1.3}$$

If $f \in C^\infty(D)$, then

$$LU_gf(Z) = L\left(f(gZ)\right)\left(J_g(Z)\right)^{\frac{\nu}{4}} + \operatorname{tr}\left[E\bar{\partial}_Z\left(f(gZ)\right)F\partial'_Z\left(J_g^{\frac{\nu}{4}}(Z)\right)\right].$$

In view of (1.1) the first term to the right equals $U_gLf(Z)$. In order to prove the lemma, we therefore need only to establish the identity

$$U_gl_\nu f = l_\nu U_gf + \operatorname{tr}\left[E\bar{\partial}_Z(f(gZ))F\partial'_ZJ_g^{\frac{\nu}{4}}\right]. \tag{1.4}$$

The left hand side of this formula is

$$\text{LHS} = l_\nu f(gZ)(J_g(Z))^{\frac{\nu}{4}}.$$

Moreover,

$$l_\nu f(Z) = (\det E(Z))^{-\nu}\operatorname{tr}\left[E(Z)\bar{\partial}_Zf(Z)F(Z)\partial'_Z(\det E)^\nu(Z)\right].$$

Now we have the following transformation formulae (see [14], p. 117)

$$\begin{aligned}
\partial_Zf(gZ) &= (BZ^* + A)\bar{\partial}_Z\left(f(gZ)\right)(Z^*C^* + D^*),\\
E(gZ) &= (ZB^* + A^*)^{-1}E(Z)(BZ^* + A)^{-1},\\
F(gZ) &= (Z^*C^* + D^*)^{-1}F(Z)(CZ + D)^{-1},\\
\left(\partial'_Z(\det E)^\nu\right)(gZ) &= (CZ + D)\partial'_Z\left((\det E(gZ))^\nu\right)(ZB^* + A^*).
\end{aligned}$$

Multiplying these equalities together and taking trace, we get

$$\begin{aligned}
&(l_\nu f)(gZ)J_g^{\frac{\nu}{4}}(Z)\\
&= \det E^{-\nu}(gZ)\operatorname{tr}\left[E(Z)\partial_Z\left(f(gZ)\right)F(Z)\partial'_Z\left(\det E^\nu(gZ)\right)J_g^{-\frac{\nu}{4}}(Z)\right].
\end{aligned} \tag{1.5}$$

It follows from (1.3) that

$$\begin{aligned}\partial'_Z(\det{}^\nu E(gZ)) &= \partial'_Z\left(J_g^{\frac{\nu}{4}}(Z)\det{}^\nu E(Z)J_g^{\frac{\nu}{4}}(Z)\right)\\ &= J_g^{\frac{\nu}{4}}(Z)\partial'_Z(\det{}^\nu E(Z))\bar{J}_g^{\frac{\nu}{4}}(Z)+\partial'_Z\left(J_g^{\frac{\nu}{4}}(Z)\right)\det{}^\nu E(Z)J_g^{\frac{\nu}{4}}(Z).\end{aligned}$$

Substituting this into (1.5) and cancelling factors, we get

$$\begin{aligned}\mathrm{LHS} &= \det E^{-\nu}(Z)\,\mathrm{tr}\left[E\bar{\partial}_Z\left((f\circ g)J_g^{\frac{\nu}{4}}\right)\partial_Z(\det E^\nu)\right](Z)+\\ &\quad+\mathrm{tr}\left[E\partial_Z(f\circ g)F\partial'_Z J_g^{\frac{\nu}{4}}\right](Z)\\ &= L_\nu U_g f(Z)+\mathrm{tr}\left[E\partial(f\circ g)F\partial' J_g^{\frac{\nu}{4}}\right](Z) = \mathrm{RHS}.\end{aligned}$$

This proves (1.4). □

Next we calculate the radial part of the differential operator L_ν. Let us first introduce the following notation; see [13]. We fix the Cartan subalgebra $\mathbf{a}$ of $\mathbf{su}(2,2)$, consisting of all matrices of the type

$$T = \begin{pmatrix} 0 & \mathrm{diag}(t_1,t_2)\\ \mathrm{diag}(t_1,t_2) & 0\end{pmatrix}.$$

We will identify it with $\mathbb{R}^2$. The corresponding subgroup $\mathbf{A}$ consists of all matrices of the type

$$a_T = \begin{pmatrix}\mathrm{diag}(\mathrm{ch}\, t_1,\mathrm{ch}\, t_2) & \mathrm{diag}(\mathrm{sh}\, t_1,\mathrm{sh}\, t_2)\\ \mathrm{diag}(\mathrm{sh}\, t_1,\mathrm{sh}\, t_2) & \mathrm{diag}(\mathrm{ch}\, t_1,\mathrm{ch}\, t_2)\end{pmatrix}.$$

Let α_1, α_2 be the linear functionals on $\mathbf{a}$ defined by $\alpha_1(T) = t_1$, $\alpha_2(T) = t_2$. The roots system is then $\{\pm 2\alpha_1, \pm 2\alpha_2, \pm(\alpha_1+\alpha_2), \pm(\alpha_1-\alpha_2)\}$. We choose the Weyl chamber defined by $\mathbf{A}^+ = \{(t_1,t_2) : t_1 > t_2 > 0\}$. The positive roots are then $2\alpha_1$, $2\alpha_2$, $\alpha_1+\alpha_2$, $\alpha_1-\alpha_2$. We also define $\rho = \alpha_1+\alpha_2+(\alpha_1+\alpha_2)+(\alpha_1-\alpha_2) = 3\alpha_1+\alpha_2$. The Weyl group W consists of all transformations of the type

$$(t_1,t_2)\mapsto(\epsilon_1 t_{\sigma(1)},\epsilon_2 t_{\sigma(2)})$$

where $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$ and σ stands for permutation of the indices 1 and 2.

Let

$$\omega(T) = 2(\mathrm{ch}\, 2t_1 - \mathrm{ch}\, 2t_2).$$

The radial part of the Laplace-Beltrami operator is (see [13])

$$\frac{1}{4}\omega^{-1}(L_1 + L_2)\omega,$$

where

$$L_i = \frac{\partial^2}{\partial t_i^2} + 2\coth 2t_i\frac{\partial}{\partial t_i}.$$

It is not difficult to find the radial part of first order differential operator l_ν:

$$-\frac{\nu}{2}\left(\tanh t_1\frac{\partial}{\partial t_1} + \tanh t_2\frac{\partial}{\partial t_2}\right).$$

Since

$$\omega^{-1}\left(\tanh t_1\frac{\partial}{\partial t_1} + \tanh t_2\frac{\partial}{\partial t_2}\right)\omega = \tanh t_1\frac{\partial}{\partial t_1} + \tanh t_2\frac{\partial}{\partial t_2} + \frac{1}{2},$$

we see thus that the radial part of the invariant Laplace operator is, apart from a constant term,

$$\omega^{-1}\left(\sum_{i=1}^{2}\frac{1}{4}L_i - \frac{\nu}{2}\tanh t_i\frac{\partial}{\partial t_i}\right)\omega.$$

2. The spherical functions and the Harish-Chandra c-function

We will study the eigenvalue problem for the radial part of the invariant Laplace operator:

$$\omega^{-1}\sum_{i=1}^{2}\left(\frac{1}{4}L_i - \frac{\nu}{2}\tanh t_i\frac{\partial}{\partial t_i}\right)\omega\phi = -\frac{1}{4}(2(\nu-1)^2 + \lambda_1^2 + \lambda_2^2)\phi, \tag{2.1}$$

for $\Lambda = (\lambda_1, \lambda_2)$, using the method in [13].

First we look at the rank one case. The eigenfunction problem then becomes

$$\frac{d^2\Phi_\lambda}{d^2r} + 2(\coth 2r - \nu\tanh r)\frac{d\Phi_\lambda}{dr} = -((\nu-1)^2 + \lambda^2)\Phi_\lambda.$$

This is the classical hypergeometric differential equation transformed. The function

$$\varphi_\lambda = F\left(\frac{1-\nu+i\lambda}{2}, \frac{1-\nu-i\lambda}{2}; 1; -\mathrm{sh}^2 t\right)$$

is a solution, which is regular at the point $t = 0$ (see [27]). The equation has two other solutions Φ_λ and $\Phi_{-\lambda}$, where

$$\Phi_\lambda(t) = (e^t - e^{-t})^{\nu-1+i\lambda} F\left(\frac{1-\nu+i\lambda}{2}, \frac{1-\nu+i\lambda}{2}; 1+i\lambda; -\mathrm{sh}^{-2} t\right).$$

By Kummer's formula ([7], 2.10 (2)) we have

$$\phi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda},$$

where the function

$$c(\lambda) = \frac{2^{-\nu+1-i\lambda}\Gamma(i\lambda)}{\Gamma(\frac{-\nu+1+i\lambda}{2})\Gamma(\frac{\nu+1+i\lambda}{2})}$$

is the generalized Harish-Chandra c-function (see [27]).

Now let us define a function on the Weyl chamber $\mathbf{A}^+$ by the product formula

$$\Phi_\Lambda(a_T) = \frac{\Phi_{\lambda_1}(t_1)\Phi_{\lambda_2}(t_2)}{\omega(a_T)}, \quad \Lambda = (\lambda_1, \lambda_2).$$

It is obvious that $\Phi_\Lambda(a_T)$ is a solution of (2.1). Next we define the function

$$\phi_\Lambda(a_T) = \frac{-2^4}{(\lambda_1^2 - \lambda_2^2)\omega(a_T)} \begin{vmatrix} \phi_{\lambda_1}(t_1) & \phi_{\lambda_1}(t_2) \\ \phi_{\lambda_2}(t_1) & \phi_{\lambda_2}(t_2) \end{vmatrix}.$$

We can use the same method as in [13] to prove that

$$\phi_\Lambda(0) = 1$$

and that

$$\phi_\Lambda(a_T) = \sum_{s\in W} \mathbf{C}(s\Lambda)\Phi_{s\Lambda}(a_T), \tag{2.2}$$

with

$$\mathbf{C}(\Lambda) = -2^4 \frac{c(\lambda_1)c(\lambda_2)}{\lambda_1^2 - \lambda_2^2}. \tag{2.3}$$

Therefore $\Phi_\Lambda(a_T)$ is also a solution of the differential equation (2.1).

Now we study the zero set of our generalized Harish-Chandra function $\mathbf{C}(\lambda)$. Since $c(\lambda)$ has zeros on the lower half plane in the set

$$P = \{p_l = -i(\nu - 1 - 2l), l = 0, 1, 2, \ldots, k\}$$

where

$$k = [\frac{\nu - 1}{2}] \quad (= [\frac{\alpha + 3}{2}]),$$

the $\mathbf{C}$-function has zeros in the generalized lower half plane (cone)

$$\{(\lambda_1, \lambda_2) \in \mathbb{C}^2, \Im\lambda_1, \Im\lambda_2 > 0\}$$

in the set

$$P^2 = \{(\lambda_1, \lambda_2). \lambda_i \in P\}.$$

We now establish the corresponding Harish-Chandra expansion of the function ϕ_λ. First we note that the function $\Phi_\lambda(t)$ has the following series expansion

$$\Phi_\lambda(t) = e^{(\nu-1+i\lambda)t} \sum_{n=0}^{\infty} \Gamma_n(\lambda) e^{-nt}, \tag{2.4}$$

while the function $\omega(a_T)$ has the expansion

$$\begin{aligned}
\omega^{-1}(a_T) &= \frac{1}{2}(\operatorname{ch} 2t_1 - \operatorname{ch} 2t_2)^{-1} = (e^{2t_1} + e^{-2t_1} - e^{2t_2} - e^{-2t_2})^{-1} \\
&= e^{-2t_1}(1 - e^{-2(t_1-t_2)})^{-1}(1 - e^{-2(t_1+t_2)})^{-1} \\
&= e^{-2t_1} \sum_{p,q=0}^{\infty} e^{-2p(t_1-t_2)-2q(t_1+t_2)}.
\end{aligned}$$

Therefore we find that Φ_Λ has the series expansion

(2.5)

$$\begin{aligned}
\Phi_\Lambda(a_T) &= \frac{\Phi_{\lambda_1}(t_1)\Phi_{\lambda_2}(t_2)}{\omega(a_T)} \\
&= e^{(\nu-2+i\lambda_1)t_1} e^{(\nu-1+i\lambda_2)t_2} \sum_{p,q=0}^{\infty} e^{-2p(t_1-t_2)-2q(t_1+t_2)} \prod_{i=1,2} \sum_{s=0}^{\infty} \Gamma_s(\lambda_i) e^{-st_i} \\
&= e^{(\nu-\rho+i\Lambda)T} \sum_{\mu \in \mathbf{L}} \Gamma_\mu(\Lambda) e^{-\mu(T)},
\end{aligned}$$

where $\mathbf{L}$ is the semigroup in $\mathbb{C}^n$ generated by the linear functionals α_1, $\alpha_1 - \alpha_2$, and where we have used the notation ν for the linear functional $\nu(T) = (\nu t_1, \nu t_2)$.

Next we need an estimate for the coefficients of this expansion.

Lemma 2

For any $H \in \mathfrak{a}^+$, there is a constant K_H such that

$$|\Gamma_\mu(\Lambda)| \leq K_H e^{\mu H}.$$

Proof. Since $\Gamma_\mu(\lambda)$ is product of the coefficients in the expansion (2.4), we can use the corresponding result in the rank one case [27] to obtain the desired estimate. □

Lemma 3

If $\Lambda = (\lambda_1, \lambda_2)$, $\Im\lambda_1 \leq 0$, $\Im\lambda_2 \leq 0$ then we have the following estimate

$$\mathbf{C}^{-1}(\Lambda)| \leq K(1 + |\Lambda|)^3,$$

where K is a constant.

Proof. By (2.3) we find

$$|\mathbf{C}^{-1}(\Lambda)| - |2^4(\lambda_1^2 - \lambda_2^2)c^{-1}(\lambda_1)c^{-1}(\lambda_1)|.$$

It follows from the Lemma 2.3 in [27] that if $\Im\lambda < 0$,

$$|c^{-1}(\lambda)| \leq C(1 + |\lambda|)^{\frac{1}{2}}.$$

Therefore we find that if $\Lambda = (\lambda_1, \lambda_1)$, $\Im\lambda_1 \leq 0$, $\Im\lambda_2 \leq 0$, then

$$|\mathbf{C}^{-1}(\Lambda)| \leq C(1 + |\Lambda|)^3.$$

If $\Im\Lambda = 0$, we can pass to the limit to produce the same estimate. □

3. The residue calculation

Let $F(\Lambda)$ be an entire function on $\mathbb{C}^2$ and of exponential type $R > 0$. In this section we will calculate the following integral.

Lemma 4

If $T = (t_1, t_2)$ and $t_1 > t_2 > R$. Then we have

$$\int_{\mathbb{R}^2} F(\Lambda)\phi_\Lambda(a_T)|\mathbf{C}(\Lambda)|^{-2}d\Lambda = -\sum_{l_1 < l_2} c_{l_1,l_2} F(p_{l_2}, p_{l_1})\phi_{(p_{l_1}, p_{l_2})},$$

for some positive constants c_{l_1,l_2}.

Proof. We have by (2.2)

$$\int_{\mathbb{R}} F(\Lambda)\phi_\Lambda(a_T)|\mathbf{C}(\Lambda)|^{-2}d\Lambda = \int_{\mathbb{R}^2} F(\Lambda)\sum_{s\in W} C(s\Lambda)\Phi_{s\Lambda}(a_T)|\mathbf{C}(\Lambda)|^{-2}d\Lambda .$$

However, it is also easy to see from (2.3) that we have, for any $\Lambda \in \mathbb{R}^2$ and $s \in W$,

$$|\mathbf{C}(\Lambda)|^2 = \mathbf{C}(s\Lambda)\mathbf{C}(-s\Lambda).$$

Therefore the above expression becomes

$$\int_{\mathbb{R}^2}\sum_{s\in W} F(\Lambda)\Phi_{s\Lambda}(a_T)\mathbf{C}^{-1}(-s\Lambda)d\Lambda = |W|\int_{\mathbb{R}^2} F(\Lambda)\Phi_{-\Lambda}(a_T)\mathbf{C}^{-1}(\Lambda)d\lambda$$

where $|W| = 8$ is the number of elements of the Weyl group, and the equality is obtained by change of variables and the fact that F is invariant under W. Using the expansion (2.5) and Lemma 2 and Lemma 3, we see that the above is

$$\sum_{\mu\in\mathbf{L}} e^{(\nu-\rho)(T)}e^{-\mu(T)}|W|\int_{\mathbb{R}^2} F(\Lambda)e^{-i\Lambda(T)}\Gamma_\mu(\Lambda)\mathbf{C}^{-1}(\Lambda)d\Lambda.$$

Let us calculate the integral in the above sum. From formula (2.3), we see that if $T = (t_1, t_2), t_1 > t_2 > R$, then

(3.1)

$$\begin{aligned}&|W|\int_{\mathbb{R}^2} F(\Lambda)e^{-i\Lambda(T)}\Gamma_\mu(\Lambda)\mathbf{C}^{-1}(\Lambda)d\Lambda\\ &= 2^{-4}|W|\int_{\mathbb{R}} c^{-1}(\lambda_2)\left(\int_{\mathbb{R}} F(\lambda_1,\lambda_2)e^{-i\Lambda(T)}\Gamma_\mu(-\Lambda)(\lambda_1^2-\lambda_2^2)c^{-1}(\lambda_1)d\lambda_1\right)d\lambda_2 .\end{aligned}$$

Since $\Gamma_\mu(-\Lambda)$ is a rational function of Λ (see [11], p. 453, for the case $\nu = 0$), so we can perform the residue calculation as in [27]. By the computation there, we see that the inner λ_1-integral is

$$-\sum_{l_1}^{k} c'_{l_1}(p_{l_1}^2-\lambda_2^2)F(p_{l_1},\lambda_2)e^{-i(p_{l_1},\lambda_2)(T)}\Gamma_\mu(-p_{l_1},\lambda_2),$$

where

$$c'_{l_1} = 2^{-4}|W|2\pi i\lim_{\lambda_1\to p_{l_1}}(\lambda_1-p_{l_1})c^{-1}(\lambda_1).$$

Thus the integral (3.1) is

$$\sum_{l_1}^{k} c'_{l_1} \int_{\mathbb{R}} (p_{l_1}^2 - \lambda_2^2) F(p_{l_1}, \lambda_2) e^{-i(p_{l_1},\lambda_2)(T)} \Gamma_\mu(-p_{l_1}, \lambda_2) c^{-1}(\lambda_2)\, d\lambda_2 .$$

Similarly, we can calculate this integral using the same residue method. We find that it equals

$$\sum_{l_1,l_2=0, l_1 \neq l_2}^{k} c_{l_1,l_2} F(p_{l_1}, p_{l_2}) e^{-i(p_{l_1},p_{l_2})(T)} \Gamma_\mu(-p_{l_1}, -p_{l_2}),$$

where

$$c'_{l_1,l_2} = c'_{l_1}(p_{l_1}^2 - p_{l_2}^2) 2\pi i \lim_{\lambda_2 \to p_{l_2}} (\lambda_2 - p_{l_2}) c^{-1}(\lambda_2). \tag{3.3}$$

Now we take summation over the lattice $\mathbf{L}$. We find that the integral in (3.1) is

(3.4)

$$\begin{aligned}
&\sum_{l_1,l_2=0, l_1 \neq l_2}^{k} c'_{l_1,l_2} F(p_{l_1}, p_{l_2}) \sum_{\mu} e^{(\nu-\rho)(T)} e^{\mu(T)} e^{-i(p_{l_1},p_{l_2})(T)} \Gamma_\mu(-p_{l_1}, -p_{l_2}) \\
&= \sum_{l_1,l_2=0, l_1 \neq l_2}^{k} c'_{l_1,l_2} F(p_{l_1}, p_{l_2}) \Phi_{(-p_{l_1}, -p_{l_2})} \\
&= \sum_{l_1,l_2=0, l_1 < l_2}^{k} F(p_{l_1}, p_{l_2}) (c'_{l_1,l_2} \Phi_{(-p_{l_1},-p_{l_2})} + c'_{l_2,l_1} \Phi_{(-p_{l_2},-p_{l_1})}).
\end{aligned}$$

On the other hand, we have already seen that

$$\mathbf{C}(\lambda_1, \lambda_2) = 0 \text{ if } \lambda_1 \text{ or } \lambda_2 \in P.$$

Hence the expansion (2.2) for the function Φ_Λ then reduces to

$$\varphi_{(-p_{l_1},-p_{l_2})}(a_T) = \mathbf{C}(-p_{l_1}, -p_{l_2}) \Phi_{(-p_{l_1},-p_{l_2})} + \mathbf{C}(-p_{l_2}, -p_{l_1}) \Phi_{(-p_{l_2},-p_{l_1})}.$$

However, we can write

$$\begin{aligned}
c'_{l_1,l_2} \Phi_{(-p_{l_1},-p_{l_2})} + c'_{l_2,l_1} \Phi_{(-p_{l_2},-p_{l_1})} \\
= c'_{l_1,l_2} \mathbf{C}^{-1}(-p_{l_1}, -p_{l_2}) \mathbf{C}(-p_{l_1}, -p_{l_2}) \Phi_{(-p_{l_1},-p_{l_2})} \\
+ \text{the same item with } (l_1, l_2) \text{ replaced by } (l_2, l_1).
\end{aligned}$$

Let us put

$$c_{l_1,l_2} = -\frac{1}{2}\mathbf{C}^{-1}(p_{l_1},p_{l_2})c'_{l_1,l_2}.$$

We can compute these constants:

$$c_{l_1,l_2} = 2^{-5}\pi^2|W|(l_1 - l - 2)^2(\nu - 1 - l_1 - l_2)^2\frac{2^{-4(l_1+l_2)}(\nu - 2l_1)_{l_1}(\nu - 2l_2)_{l_2}}{l_1!l_2!\Gamma(\nu - 1 - 2l_1)\Gamma(\nu - 1 - 2l_2)},$$

which are positive and invariant under the permutation $l_1 \mapsto l_2$. Therefore we find the the sum (3.4) is

$$\sum_{l_1,l_2=0,l_1<l_2}^{k} -c_{l_1,l_2}F(p_{l_2},p_{l_1})\phi_{p_{l_1},p_{l_2}}(a_T).$$

Hence we have proved that

$$\int_{\mathbb{R}^2} F(\Lambda)\Phi_\Lambda(a_T)|\mathbf{C}(\Lambda)|^{-2}d\Lambda + \sum_{l_1<l_2} c_{l_1,l_2}F(p_{l_2},p_{l_1})\phi_{(p_{l_1},p_{l_2})} = 0.$$

This finishes the proof. □

Let $f \in C_0^\infty(D)$ be invariant under the group $K = S(U(2) \times U(2))$, that is, $f(UzV) = f(z)$ for every pair of unitary operators U, V with $\det U \cdot \det V = 1$. Define its generalized Fourier transform by

$$\tilde{F}(\Lambda) = \int_D f(Z)\overline{\phi_\Lambda(Z)}d\mu_\alpha,$$

where $\phi_\Lambda(Z)$ is the K-invariant extension on D of $\phi_\Lambda(a_T)$ on $\mathbf{A}^+\cdot 0$.

From Lemma 4 and using the same argument as in [27], we can then prove the following:

Theorem 1

Let f be a K-invariant function $C_0^\infty(D)$. Then we have the following inversion formula

$$cf(0) = \int_{\mathbb{R}^2} \tilde{F}(\Lambda)|C(\Lambda)|^{-2}d\Lambda + \sum_{\substack{l_1,l_2=0 \\ l_1<l_2}}^{k} c_{l_1,l_2}\tilde{F}(p_{l_1},p_{l_1}),$$

for some constant c.

4. The discrete parts and the reproducing kernels

In this section we calculate the reproducing kernels for the spaces obtained from the decomposition in a special case.

By Theorem 1 in §3, the invariant Laplacian L_ν has the continuous spectrum

$$\left(-\infty, -\frac{1}{2}(\nu-1)^2\right],$$

while the discrete spectrum consists of the $\binom{k+1}{2}$ points

$$\frac{1}{4}\left(2(\nu-1)^2-(\nu-1-2l_1)^2-(\nu-1-2l_1)^2\right), \quad l_1, l_2 = 0, 1 \dots, k,\ l_1 < l_2.$$

Let $A_{i,j}$ be the eigenspace of L_ν correponding to l_i, l_j. This space is generated by the spherical function $\phi_{p_{l_1}, p_{l_2}}$. We first calculate ϕ_{p_0,p_0}. Since $p_0 = -i(\nu-1)$, $p_1 = -i(\nu-3)$, we see that

$$\frac{1-\nu+ip_0}{2} = 0, \quad \frac{1-\nu+ip_1}{2} = -1, \quad \frac{1-\nu-ip_1}{2} = 2-\nu.$$

Hence $\phi_{p_0} = F(0, 2-\nu; 1, -\mathrm{sh}^2 t) = 1$ and $\phi_{p_1}(t) = F(-1, 2-\nu; 1, -\mathrm{sh}^2 t) = 1 + (2-\nu)\,\mathrm{sh}^2 t$. From this we find that

$$\begin{aligned}
\phi_{p_0,p_1}(a_T) &= \frac{-2^4}{(p_0^2-p_1^2)\omega(a_T)} \times \\
&\quad \times \det\begin{pmatrix} F(0,2-\nu;1,-\mathrm{sh}^2 t) & F(-1,2-\nu;1,-\mathrm{sh}^2 t) \\ F(-1,2-\nu;1;-\mathrm{sh}^2 t_1) & F(1,2-\nu;1;-\mathrm{sh}^2 t_2) \end{pmatrix} = \\
&= \frac{-2^4}{(\nu-3)^2-(\nu-1)^2} \cdot \frac{1}{2(\mathrm{ch}\, 2t_1 - \mathrm{ch}\, 2t_2)} \times \\
&\quad \times \det\begin{pmatrix} 1 & 1 \\ 1+(2-\nu)\mathrm{sh}^2 t_1 & 1+(2-\nu)\mathrm{sh}^2 t_2 \end{pmatrix} \equiv 1.
\end{aligned}$$

In view of the Moebius invariance, we see that $A_{0,1}$ is just the weighted Bergman space with the reproducing kernel

$$K_{0,1}(Z, W) = K(Z, W) \cdot \det(1 - ZW^*)^{-\alpha-4}.$$

This is of course trivial and well-known.

Using this fact we can evaluate the constant in Theorem 1 in §3. By checking the equality for the constant function 1, we find $c = c_{0,1}$.

Remark. The constant $\frac{c_{l_1,l_2}}{c_{0,1}}$ is the norm of $\phi_{p_{l_1},p_{l_2}}$ in $L^2(D, d\mu_\alpha)$. Therefore we can obtain it by calculating the residue of the Harish-Chandra function. Other calculations of L^2 norms of hypergeometric functions are in [5].

Now if f is a C^∞-function in A_{l_1,l_2}, so $g(Z) = \int_K f(kZ)dk$ is K-invariant and in A_{l_1,l_2}. By Theorem 1 in §3, we see that

$$f(0) = g(0) = \frac{c_{l_1,l_2}}{c_{0,1}}\left(g, \phi_{p_{l_1},p_{l_2}}\right) = \frac{c_{l_1,l_2}}{c_{0,1}}\left(f, \phi_{p_{l_1},p_{l_2}}\right),$$

where the inner product is one in $L^2(D, d\mu_\alpha)$.

Thus the the reproducing kernel of the space A_{l_1,l_2} is

$$\begin{aligned} K_{l_1,l_2}(Z,W) &= \frac{c_{l_1,l_2}}{c_{0,1}}(J_{\psi_W}(Z))^{\frac{\nu}{4}}(\overline{J_{\psi_W}(Z)})^{\frac{\nu}{4}}\phi_{p_{l_1},p_{l_2}}(\psi_W(Z)) \\ &= \frac{c_{l_1,l_2}}{c_{0,1}}K(Z,W)\phi_{p_{l_1},p_{l_2}}(\psi_W(Z)), \end{aligned}$$

where ψ_W is any Moebius transformation sending W to 0.

We let $\alpha = 2$. Then $\nu = \alpha + 4 = 6$ and $k = [\frac{\nu-1}{2}] = 2$. So there are $k+1 = 3$ discrete parts. Moreover, $p_0 = -5i$, $p_1 = -3i$, $p_2 = -i$ in this case.

As we just calculated $\phi_{p_0,p_1} = 1$. The spherical function ϕ_{p_0,p_2} is,

$$\begin{aligned} \phi_{p_0,p_2}(a_T) &= \frac{-2^4}{p_0^2 - p_2^2}\frac{1}{\omega(a_T)}\det\begin{pmatrix} 1 & 1 \\ F(-2,-3;1;-\mathrm{sh}^2 t_1) & F(-2,-3;1;-\mathrm{sh}^2 t_2) \end{pmatrix} \\ &= 1 - \frac{1}{2}(\mathrm{sh}t_1^2 + \mathrm{sh}t_2^2) \\ &= 1 - \frac{1}{2}\mathrm{tr}(a_T a_T^*)(I - a_T a_T^*)^{-1}. \end{aligned}$$

Therefore by the above, the reproducing kernel of the corresponding discrete space is

$$K_{0,2}(Z,W) = c_{0,2}K(Z,W)\left[1 - \frac{1}{2}\mathrm{tr}\left((\psi_W(Z)\psi_W(Z)^*(I - \psi_W(Z)\psi_W(Z)^*)^{-1}\right)\right],$$

where ψ_W is a Moebius transformation which sends W to the origin O.

Similarly we find that

$$
\begin{aligned}
\phi_{p_1,p_2}(a_T) &= \frac{-2^4}{p_1^2-p_2^2}\,\frac{1}{\omega(T)}\times \\
&\times \det\begin{pmatrix} F(\frac{1-\nu+ip_1}{2},\frac{1-\nu-ip_1}{2};1;-\mathrm{sh}^2t_1) & F(\frac{1-\nu+ip_1}{2},\frac{1-\nu-ip_1}{2};1;-\mathrm{sh}^2t_2) \\ F(\frac{1-\nu+ip_2}{2},\frac{1-\nu-ip_2}{2};1;-\mathrm{sh}^2t_1) & F(\frac{1-\nu+ip_2}{2},\frac{1-\nu-ip_2}{2};1;-\mathrm{sh}^2t_2) \end{pmatrix} \\
&= \frac{1}{\mathrm{sh}^2t_1-\mathrm{sh}^2t_2}\det\begin{pmatrix} F(-1,-4;1;-\mathrm{sh}^2t_1) & F(-1,-4;1;-\mathrm{sh}^2t_2) \\ F(-2,-3;1;-\mathrm{sh}^2t_1) & F(-2,-3;1;-\mathrm{sh}^2t_2) \end{pmatrix} \\
&= 1-\frac{3}{2}(\mathrm{sh}^2t_1+\mathrm{sh}^2t_2)+6\,\mathrm{sh}^2t_1\mathrm{sh}^2t_2 \\
&= 1-\frac{3}{2}(\mathrm{sh}^2t_1+\mathrm{sh}^2t_2)+3(\mathrm{sh}^2t_1+\mathrm{sh}^2t_2)^2-3(\mathrm{sh}^4t_1+\mathrm{sh}^4t_2) \\
&= 1-\frac{3}{2}\mathrm{tr}(a_Ta_T^*)(I-a_Ta_T^*)+3\,|\mathrm{tr}(a_Ta_T^*)(I-a_Ta_T^*)|^2 \\
&\quad -3\,\mathrm{tr}\left[(a_Ta_T^*)(I-a_Ta_T^*)\right]^2 .
\end{aligned}
$$

From this we see that the reproducing kernel of the space corresponding to (p_1,p_2) is

$$
\begin{aligned}
K_{1,2}(Z,W) = c_{1,2}K(Z,W)\Big\{1 &- \frac{3}{2}\mathrm{tr}\big(\psi_W(Z)\psi_W(Z)^*(I-\psi_W(Z)\psi_W(Z)^*)^{-1}\big) \\
&+ 3\big[\mathrm{tr}\big(\psi_W(Z)\psi_W(Z)^*(I-\psi_W(Z)\psi_W(Z)^*)^{-1}\big)\big]^2 \\
&- 3\,\mathrm{tr}\big(\psi_W(Z)\psi_W(Z)^*(I-\psi_W(Z)\psi_W(Z)^*)^{-1}\big)^2\Big\}.
\end{aligned}
$$

5. Tensor products of weighted Bergman spaces

In this section, we give an application of our results to tensor products of weighted Bergman spaces. We will write down explicitly the intertwining operators between tensor products of weighted Bergman spaces and weighted L^2 spaces.

The following considerations have a general nature, so for a while we let D be an arbitrary bounded symmetric domain in $\mathbb{C}^n$.[1] Points of D will be denoted by lower case letters (such as z or w).

Let p denote the genus of D. (Thus $p = 4$ if $D = D(\mathrm{II}_{2,2})$ is our hyperbolic matrix ball.) Consider, for each real number $\alpha > -1$ the Hilbert space

[1] Necessary background on symmetric domains can be found in e.g. [1].

$L^2(D, d\mu_\alpha)$ of square integrable functions with respect to the measure $d\mu_\alpha(z) = C_\alpha K^{-\frac{\alpha}{p}}(z,z)dm(z)$ on D, where $dm(z)$ is the Euclidean measure and $K(z,w)$ is the Bergman kernel of D, and $\mu_\alpha(D) = 1$.

Let $G = \operatorname{Aut} D$ be the group of all biholomorphic transformations of D into itself. If ν and κ are positive real parameters such that $\nu, \kappa > p-1$, there is a unitary representation of G on $L^2(D, d\mu_{\nu+\kappa-2p})$ given by the formula

$$U_g^{(\nu,\kappa)} : f(z) \mapsto f(g(z))(J_g(z))^{\frac{\nu}{p}}(\overline{J_g(z)})^{\frac{\kappa}{p}} \qquad (g \in G)$$

where J_g stands for the Jacobian of the transformation g. Therefore $U^{(\nu,0)} = U^\nu$.

Lemma 5

The representation $U^{(\nu,\kappa)}$ is unitarily equivalent to the representation $U^{(\nu-\kappa,0)}$.

Proof. Define an operator T form $L^2(D, d\mu_{\nu+\kappa-2p})$ to $L^2(D, d\mu_{\nu-\kappa-p})$ by

$$T: f(z) \mapsto f(z)K(z,z)^{\frac{-\kappa}{p}}.$$

Then T is a bounded operator. From the transformation formula

$$K(gz, gz) = (J_g(z))^{-1}(\overline{J_g(z)})^{-1}K(z,z),$$

we see that T is an intertwining operator

$$TU^{(\nu,\kappa)} = U^{(\nu-\kappa,0)}T.$$

The Lemma is proved. □

Corollary

Two representations $U^{(\nu,\kappa)}$ and $U^{(\nu',\kappa')}$ unitarily equivalent provided $\nu - \kappa = \nu' - \kappa'$.

The representation $U^{(\nu)}$, where $\nu = \alpha + p$, has, provided $\alpha > -1$, an important subrepresentation on the weighted Bergman space $A^{\alpha,2}(D)$ of analytic functions in $L^2(D, d\mu_\alpha)$. We write $\bar{U}^{(\nu)}$ for the corresponding representation on the space $\overline{A^{\alpha,2}(D)}$ of conjugate analytic functions in $L^2(D, d\mu_\alpha)$.

Lemma 6

The representation $U^{(\nu)} \otimes \bar{U}^{(\kappa)}$ restricted to tensor product $A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$, where $\nu = \alpha + p$, $\kappa = \beta + p$, is equivalent to the representation $U^{(\nu,\kappa)}$ on the space $L^2(D, d\mu_{\alpha+\beta+p})$.

Before proving the Lemma, we invoke the following

Lemma 7

Let H and H_1 be positive self-adjoint 1-1 operators in a Hilbert space and let U and U_1 be unitary operators there. Assume that $UH = H_1U_1$. Then $U = U_1$ and $H_1 = UHU^{-1}$.

Proof. We can write the relation $UH = H_1U_1$ as $UHU^{-1} = HU_1U^{-1}$. Thus, upon replacing H by UHU^{-1}, we may reduce to the special case $U = 1$. So we have to show that if $UH = H_1$ then $U = 1$ and $H_1 = H$.

For simplicity assume first that H is onto. We have then $U = H_1H^{-1}$. Thus $1 = U^*U = H^{-1}H_1H_1H^{-1} = H^{-1}H_1^2H^{-1}$ or $H^2 = H_1^2$. This again implies $H = H_1$ and $U = 1$.

It is now easy to fix up the proof, even if H is not assumed to be onto, only 1-1. □

Proof of Lemma 6. Let $f \in A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$. Then by the reproducing property we have for each $z \in D$

$$f(z,z) = \int_D f(z,w)K^{\frac{\kappa}{p}}(w,z)d\mu_\beta(w).$$

Hence by Schwarz's inequality

$$\begin{aligned}|f(z,z)|^2 &\leq \int_D |K^{\frac{\kappa}{p}}(w,z)|^2 d\mu_\beta(w) \int_D |f(z,w)|^2 d\mu_\beta(w) = \\ &= K^{\frac{\kappa}{p}}(z,z)\int_D |f(z,w)|^2 d\mu_\beta(w).\end{aligned}$$

Integrating this yields

$$\begin{aligned}&\int_D |f(z,z)|^2 d\mu_{\alpha+\beta+p}(z) = \\ &= \frac{C_{\alpha+\beta+p}}{C_\alpha}\int_D |f(z,z)|^2 K^{-\frac{\kappa}{p}}(z,z)d\mu_\alpha(z) \leq \\ &\leq C\iint_{D\times D} |f(z,w)|^2 d\mu_\alpha(z)d\mu_\beta(w) = C\|f\|_{A^{\alpha,2}(D)\otimes\overline{A^{\beta,2}(D)}}.\end{aligned}$$

So if R denotes the operation of taking the restriction to the diagonal, $R : f(z,w) \mapsto f(z,z)$, it follows that R is a bounded operator from $A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$

into $L^2(D, d\mu_{\alpha+\beta})$. In addition, it is clear that R has the following intertwining property:

$$R(U_g^{(\nu)} \otimes \bar{U}_g^{(\kappa)}) = U_g^{(\nu,\kappa)} R \qquad (g \in G). \tag{5.1}$$

Since a function $f(z, w)$ holomorphic in z and anti-holomorphic in w is uniqulely determined by its restriction on the diagonal, $f(z, z)$ it is easy to see that R is 1-1 and has a dense range. So we have the polar decomposition

$$R = (RR^*)^{\frac{1}{2}} V = |R| V\,,$$

where $V : A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)} \mapsto L^2(D, d\mu_{\alpha+\beta+p})$ is a unitary operator. Thus (5.1) can be rewritten as

$$|R| V(U_g^{(\nu)} \otimes U_g^{(\kappa)}) V^{-1} = U_g^{(\nu,\kappa)} |R| \qquad (g \in G). \tag{5.1$'$}$$

Applying Lemma 6 with $H = H_1 = |R|$, $U = V(U_\xi^{(\lambda)} \otimes U_\xi^{(\kappa)}) V^{-1}$, $U_1 = U_\xi^{(\lambda,\kappa)}$, it follows that

$$V(U_\xi^{(\nu)} \otimes \bar{U}_\xi^{(\kappa)}) V^{-1} = U_\xi^{(\nu,\kappa)} \qquad (\xi \in G)\,.$$

Thus V gives a unitary equivalence between the representations $U^{(\nu)} \otimes U^{(\kappa)}$ and $U^{(\nu,\kappa)}$. □

From our Theorem 1 in §3 and Lemma 5 and 6 we immediately get the following

Theorem 2

Assume $\nu, \kappa > 3$, and $\nu - \kappa > 3$. Then the representation $U^{(\nu)} \otimes \overline{U^{(\kappa)}}$ of $SU(2,2)$ on $A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$ is unitarily equivalent to $U^{(\nu-\kappa)}$ on $L^2(D, \mu_{\alpha-\beta-4})$ and it has at least

$$\binom{[\frac{\nu-\kappa-1}{2}] + 1}{2}$$

discrete parts in its irreducible decomposition.

Remark. A similar result for the group $SU(1,1)$ is due to Repka [22]. For general groups, the first half of our theorem was obtained by him in [23] using Schur's lemma. Our theorem refines the result in [23].

Appendix 1. The Berezin type transform

In order to get a better feel for what we are doing in §5 we shall (*infra*) consider an example. Now let thus $D = D(\mathrm{I}_{1,1})$ be the unit disk ($p = 2$, $G = \mathrm{SU}(1,1)$). Assume that $\alpha \geq \beta > -1$ and $\alpha - \beta$ is not an odd integer and in this case $\nu = \alpha + 2$, $\kappa = \beta + 2$.

First we find an explicit expression for the operator RR^*, whose square root we are interested in. (This reasoning is valid for general symmetric domains.) We consider the inner product

$$\begin{aligned}(Rf,\phi) &= \int_D f(\zeta,\zeta)\overline{\phi(\zeta)}d\mu_{\alpha+\beta+p}(\zeta)\\ &= \int_D \iint_{D\times D}\left(K^{\frac{\nu}{p}}(\zeta,z)K^{\frac{\kappa}{p}}(\zeta,w)f(z,w)d\mu_\alpha(z)d\mu_\beta(w)\right)\overline{\phi(\zeta)}d\mu_{\alpha+\beta+p}(\zeta) =\\ &\iint_{D\times D}\overline{\int_D K^{\frac{\nu}{p}}(\zeta,z)K^{\frac{\kappa}{p}}(\zeta,w)\phi(\zeta)d\mu_{\alpha+\beta+p}(\zeta)}f(z,w)d\mu_\alpha(z)d\mu_\beta(w)\end{aligned}$$

for any $f \in A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$ and $\phi \in L^2(D, d\mu_{\alpha+\beta+p})$. As $(Rf,\phi) = (f, R^*\phi)$, it follows that

$$R^*\phi(z,w) = \int_D K^{\frac{\nu}{p}}(z,\zeta)\bar{K}^{\frac{\kappa}{p}}(w,\zeta)\phi(\zeta)d\mu_{\alpha+\beta+p}(\zeta),$$

so restricting to the diagonal ($z = w$) yields

$$\text{(a.1)}\qquad RR^*\phi(z) = \int_D K^{\frac{\nu}{p}}(z,\zeta)\bar{K}^{\frac{\kappa}{p}}(z,\zeta)\phi(\zeta)d\mu_{\alpha+\beta+p}(\zeta).$$

This is an integral operator and so is its square root. *Thus finding the unitary transformation V is reduced to solving an integral equation.*

If $\alpha = \beta$, we see that we have essentially the *Berezin transform* (cf. [4], [2], [19]). This suggests that the operator RR^* might be a function of an invariant operator. Indeed, this is so at least in the rank one case. We now calculate this function for the disk $D = D(\mathrm{I}_{1,1})$.

Then (a.1) spelled out reads

$$\text{(a.1}'\text{)}\qquad RR^*\phi(z) = (\nu+\kappa-1)\int_D (1-z\bar{\zeta})^{\frac{\nu}{2}}(1-z\zeta)^{\frac{\kappa}{2}}\phi(\zeta)(1-|\zeta|^2)^{\alpha+\beta+2}dm(\zeta).$$

The corresponding invariant Laplace operator $\Delta = \Delta_{\alpha,\beta}$ has the following eigenfunctions (see [21]) [2]:

$$(a.2) \qquad e_{\lambda,\omega}(z) = \frac{1}{(1-|z|^2)^{\frac{\nu+\kappa}{2}-\frac{1}{2}-i\frac{\lambda}{2}}(1-z\bar{\omega})^{\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2}}(1-\bar{z}\omega)^{-\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2}}},$$

where $\omega \in \partial D = \mathbb{T}$ and λ is a complex numbers.

Note that λ is not quite an eigenvalue of Δ, the relation to the eigenvalues δ reads as follows (cf. [21]):

$$\frac{\lambda}{2} = \sqrt{-\delta - \left(\frac{\nu-\kappa+1}{2}\right)^2}.$$

This suggests introducing the self-adjoint operator

$$\nabla = \sqrt{4\Delta - (\nu-\kappa+1)^2}.$$

One checks easily the following transformation formula

$$e_{\lambda,g(\omega)}(g(z))(g'(z))^{\frac{\nu}{2}}(\overline{g'(z)})^{\frac{\kappa}{2}}(g'(\omega))^{\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2}}(\overline{g'(\omega)})^{-\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2}} = e_{\lambda,z}(z) \quad (g \in G)$$

or in abreviated notation

$$\left(U_g^{(\nu,\kappa)} \otimes U_g^{(\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2},\, -\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2})}\right) e_{\lambda,z} = e_{\lambda,z}\,.$$

(We have a group action on functions on the product $D \times \partial D$). Using this again one sees that

$$\left(U_g^{(\nu,\kappa)} \otimes U_g^{(\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2},\, -\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2})}\right) RR^* e_{\lambda,z} = RR^* e_{\lambda,z}\,.$$

As the group G acts transitively on the set $D \times \partial D$, it follows that we must have

$$(a.3) \qquad RR^* e_{\lambda,z} = \mathrm{const} \cdot e_{\lambda,z}\,.$$

To find the constant in (a.3) it suffices to evaluate the integral implicitly enterring in the left hand side of this formula for $z = 0$ and $\omega = 1$. It is question (see (a.1′), and notice that $\alpha + \beta + 2 = \nu + \kappa - 2$)

$$\int_D \frac{1}{(1-\zeta)^{\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2}}(1-\bar{\zeta})^{-\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2}}}(1-|\zeta|^2)^{\frac{\nu+\kappa}{2}-\frac{3}{2}+i\frac{\lambda}{2}} dm(\zeta)\,.$$

[2] There only the case $\beta = 0$ was considered; rewriting the formula there we get the present more symmetric formula.

Using the binomial expansion

$$(1-\zeta)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\zeta^n$$

and remembering that the monomials z^n ($n=0,1,2,\ldots$) form an orthogonal set with respect to any radial weight, in particular, one of the type $(1-|\zeta|^2)^c$, we end up with the series

$$\sum_{n=0}^{\infty}\frac{(\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2})_n}{n!}\frac{(-\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2})_n}{n!}\int_D|\zeta|^{2n}(1-|\zeta|^2)^{\frac{\nu+\kappa}{2}-\frac{3}{2}+i\frac{\lambda}{2}}dm(\zeta)\,.$$

Now, quite generally,

$$\int_D|\zeta|^{2n}(1-|\zeta|^2)^c dm(\zeta)=\frac{n!}{(c+1)(c+2)_n}\,.$$

Recall also a famous formula for the hypergeometric function (cf. e.g. [17], p. 8):

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,.$$

In our case this yields the expression

$$\frac{1}{\frac{\nu+\kappa}{2}-\frac{1}{2}+i\frac{\lambda}{2}}\sum_{n=0}^{\infty}\frac{(\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2})_n}{n!}\frac{(-\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2})_n}{n!}\frac{n!}{(\frac{\nu+\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2})_n}=$$

$$=\frac{1}{\frac{\nu+\kappa}{2}-\frac{1}{2}+i\frac{\lambda}{2}}F\left(\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2},-\frac{\nu-\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2};\frac{\nu+\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2};1\right)=$$

$$=\frac{1}{\frac{\nu+\kappa}{2}-\frac{1}{2}+i\frac{\lambda}{2}}\frac{\Gamma(\frac{\nu+\kappa}{2}+\frac{1}{2}+i\frac{\lambda}{2})\Gamma(\frac{\nu+\kappa}{2}-\frac{1}{2}-i\frac{\lambda}{2})}{\Gamma(\nu)\Gamma(\kappa)}=$$

$$\frac{|\Gamma(\frac{\nu+\kappa}{2}-\frac{1}{2}+i\frac{\lambda}{2})|^2}{\Gamma(\nu)\Gamma(\kappa)}\,.$$

This shows that

$$RR^*=(\nu+\kappa-1)\frac{1}{\Gamma(\nu)\Gamma(\kappa)}\left|\Gamma\left(\frac{\nu+\kappa}{2}-\frac{1}{2}+i\frac{\nabla}{2}\right)\right|^2$$

and so, finally,

$$(RR^*)^{\frac{1}{2}}=(\nu+\kappa-1)^{\frac{1}{2}}\left(\frac{1}{\Gamma(\nu)\Gamma(\kappa)}\right)^{\frac{1}{2}}\left|\Gamma\left(\frac{\nu+\kappa}{2}-\frac{1}{2}+i\frac{\nabla}{2}\right)\right|.$$

If $\alpha=\beta$ (and therefore $\nu=\kappa$) this is full agreement with what is in [4], [2], [19]. Note that in view of Lemma 5 one can in this case even reduce to the case $\alpha=\beta=0$.

Appendix 2. A class of hypergeometric orthogonal polynomials

In this appendix we will give an application of our consideration in §5 and Appendix 1 to hypergeometric orthogonal polynomials.

First we restate the Plancherel formula in [21] as follows. Let D be the unit disk and we keep the notation in Appendix 1. We invoked there the invariant Laplacian $\Delta = \Delta_{\alpha,\beta}$ on $L^2(D, d\mu_{\alpha+\beta+2})$, which is unitarily equivalent to the operator $\square_{\nu-\kappa}$ on $L^2(D, d\mu_{\alpha-\beta-2})$ (see [21]),

$$\square_{\nu-\kappa} = (1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} - (\nu-\kappa)\bar{z}(1-|z|^2)\frac{\partial}{\partial \bar{z}}.$$

From [21], we know that the operator Δ has the eigenfunction (a.2). We define the generalized Fourier-Helgason transform on radial functions as follows:

$$\tilde{f}(\lambda) = \int_D f(z) e_{\lambda,1}(z) d\mu_{\alpha+\beta+2}(z).$$

Then we have the Plancherel formula

$$\int_D |f(z)|^2 d\mu_{\alpha+\beta+2}(z) = \int_{\mathbb{R}^+ \cup \{\lambda_1,\dots,\lambda_k\}} |\tilde{f}(\lambda)|^2 d\rho(\lambda),$$

where

$$d\rho(\lambda) = \frac{(\nu-1)^2}{(\nu-\kappa-1)(\nu+\kappa-1)^2}\, 2^{-2(\nu-\kappa)+1}\pi^{-2}|c(\lambda)|^{-2}$$

on $\mathbb{R}^+$, while

$$\rho(\{\lambda_l\}) = \frac{(\nu-1)^2}{(\nu-\kappa-1)(\nu+\kappa-1)^2}\,(\nu-\kappa-1-2l),$$

with $\lambda_l = -i(\nu-\kappa-1-2l)$ $(l = 0, 1, \dots, k)$, $k = [\frac{\nu-\kappa-1}{2}]$, C is a constant (depending only on ν and κ); here $c(\lambda)$ is the generalized Harish-Chandra c-function

$$c(\lambda) = \frac{2^{-(\nu-\kappa)+1-i\lambda}\Gamma(i\lambda)}{\Gamma(\frac{-(\nu-\kappa)+1+i\lambda}{2})\Gamma(\frac{\nu-\kappa+1+i\lambda}{2})}.$$

Now we consider the tensor product $A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$. As shown in §5, it is unitarily equivalent to $L^2(D, d\mu_{\alpha+\beta+2})$. Now we are going to find the Clebsch-Gordan coefficients. Let R be the restriction operator with the polar decomposition $R = |R|V = (RR^*)^{\frac{1}{2}}V$, where

$$V : A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)} \to L^2(D, d\mu_{\alpha+\beta+2})$$

is a unitary operator and

$$V(U^{(\nu)} \otimes U^{(\kappa)}) = U^{(\nu,\kappa)} V .$$

From these relations we get

$$VC = (\Delta + \kappa)V, \quad |R|\Delta = \Delta|R| ,$$

where C is the Casimir operator in [28],

$$C = (1 - z\bar{w})^2 \frac{\partial^2}{\partial z \partial \bar{w}} - \nu w(1 - zw)\frac{\partial}{\partial w} - \kappa z(1 - z\bar{w})\frac{\partial}{\partial z} + \nu\kappa z\bar{w} .$$

We have a natural orthonormal vectors $\{e_n\}$ in $A^{\alpha,2}(D) \otimes \overline{A^{\beta,2}(D)}$, where

$$e_n(z, w) = \frac{(z\bar{w})^n}{\Gamma^{\frac{1}{2}}_{n,\nu}\Gamma^{\frac{1}{2}}_{n,\kappa}},$$

with

$$\Gamma_{n,\nu} = \frac{\Gamma(n+1)\Gamma(\nu)}{\Gamma(n+\nu)} = \|z^n\|^2_{A^{\alpha,2}(D)} \quad \text{and similarly for } \Gamma_{n,\kappa} .$$

A simple computation reveals that the operator C has the matrix form on $\{e_n\}$:

$$Ce_n = a_n e_{n+1} + b_n e_n + c_n e_{n-1}, \tag{a.4}$$

with

$$a_n = (n+\nu)(n+\kappa)\frac{\Gamma^{\frac{1}{2}}_{n+1,\nu}\Gamma^{\frac{1}{2}}_{n+1,\kappa}}{\Gamma^{\frac{1}{2}}_{n,\nu}\Gamma^{\frac{1}{2}}_{n,\nu}}, \; b_n = -n(2n+\nu+\kappa), c_n = n^2\frac{\Gamma^{\frac{1}{2}}_{n-1,\nu}\Gamma^{\frac{1}{2}}_{n-1,\kappa}}{\Gamma^{\frac{1}{2}}_{n,\nu}\Gamma^{\frac{1}{2}}_{n,\kappa}};$$

note that $a_n = c_{n+1}$.

Now we consider the Fourier-Helgason transforms of Ve_n defined above. It is clear that Ve_n are radial functions. So we can use our result in the beginning of Appendix 1.

Taking the Fourier-Helgason transform using (a.4), we then get

$$\Lambda\widetilde{Ve_n}(\Lambda) = a_n\widetilde{Ve_{n+1}}(\Lambda) + b_n\widetilde{Ve_n}(\Lambda) + c_n\widetilde{Ve_{n-1}}(\Lambda) ,$$

where

$$\Lambda = \kappa - \frac{1}{4}\Big((\nu - \kappa - 1)^2 + \lambda^2\Big).$$

Set

$$q_n(\Lambda) = \frac{\widetilde{Ve_n}(\Lambda)}{\Gamma_{n,\nu}^{\frac{1}{2}}\Gamma_{n,\kappa}^{\frac{1}{2}}}. \tag{a.5}$$

With these notations, we have

$$\Lambda q_n(\Lambda) = (n+1)^2 q_{n+1}(\Lambda) - n(2n+\nu+\kappa)q_n(\Lambda) + (n-1+\nu)(n-1+\kappa)q_{n-1}(\Lambda).$$

In order to identify the functions $q_n(\lambda)$, we need to determine

$$q_0(\Lambda) = \frac{\widetilde{Ve_0}(\Lambda)}{\gamma_{0,\nu}^{\frac{1}{2}}\gamma_{0,\kappa}^{\frac{1}{2}}}.$$

However, as $R = |R|V$, we have $V = |R|^{-1}R = (RR^*)^{-\frac{1}{2}}R$. Hence, by the selfadjointness of $(RR^*)^{-\frac{1}{2}}$, we have that

$$\begin{aligned}
\widetilde{Ve_0}(\lambda) &= \int_D (RR^*)^{-\frac{1}{2}}R1(z)e_{\lambda,1}(z)d\mu_{\alpha+\beta+2}(z)\\
&= \int_D (RR^*)^{-\frac{1}{2}}1(z)e_{\lambda,1}(z)d\mu_{\alpha+\beta+2}(z)\\
&= \int_D (RR^*)^{-\frac{1}{2}}e_{\lambda,1}(z)d\mu_{\alpha+\beta+2}(z)\\
&= (\nu+\kappa-1)^{-\frac{1}{2}}\Gamma(\nu)^{\frac{1}{2}}\Gamma(\kappa)^{\frac{1}{2}}\left|\Gamma(\frac{\nu+\kappa-1+i\lambda}{2})\right|^{-1}\int_D e_{\lambda,1}(z)d\mu_{\alpha+\beta+2}(z)\\
&= (\nu+\kappa-1)^{-\frac{1}{2}}\Gamma(\nu)^{\frac{1}{2}}\Gamma(\kappa)^{\frac{1}{2}}\left|\Gamma(\frac{\nu+\kappa-1+i\lambda}{2})\right|^{-1}\times\\
&\quad\times(\nu+\kappa-1)\Gamma(\nu)^{-1}\Gamma(\kappa)^{-1}\left|\Gamma(\frac{\nu+\kappa-1+i\lambda}{2})\right|^{2}\\
&= (\nu+\kappa-1)^{\frac{1}{2}}\Gamma(\nu)^{-\frac{1}{2}}\Gamma(\kappa)^{-\frac{1}{2}}\left|\Gamma(\frac{\nu+\kappa-1+i\lambda}{2})\right|.
\end{aligned}$$

Therefore we have

$$q_0(\Lambda) = (\nu+\kappa-1)^{\frac{1}{2}}\Gamma(\nu)^{-\frac{1}{2}}\Gamma(\kappa)^{-\frac{1}{2}}\left|\Gamma(\frac{\nu+\kappa-1+i\lambda}{2})\right|.$$

Recall now the hypergeometric polynomial p_n in [28],

$$\begin{aligned}
&p_n(\Lambda)\\
&= \frac{(\frac{\nu+\kappa-1+i\lambda}{2})_n}{n!}\times\\
&\quad\times {}_3F_2\Big(-n,\frac{\nu-\kappa+1-i\lambda}{2},\frac{-\nu+\kappa+1-i\lambda}{2};1,\frac{-\nu-\kappa-2n+3-i\lambda}{2};1\Big).
\end{aligned}$$

It follows from the result in [28] that p_n satisfy the same recursion formula as q_n and $p_0 = 1$. We therefore have

$$
\begin{aligned}
\text{(a.6)} \qquad q_n(\Lambda) &= q_0(\Lambda)p_n(\Lambda) \\
&= (\nu+\kappa-1)^{\frac{1}{2}}\Gamma(\nu)^{-\frac{1}{2}}\Gamma(\kappa)^{-\frac{1}{2}} \left|\Gamma\left(\frac{\nu+\kappa-1+i\lambda}{2}\right)\right| p_n(\Lambda).
\end{aligned}
$$

Since

$$\int_D e_n(z)\overline{e_m(z)}d\mu_{\alpha+\beta+2}(z) = \delta_{n,m},$$

we have

$$\int_{\mathbb{R}^+\cup\{\lambda_1,\dots,\lambda_k\}} \widetilde{Ve_n}(\Lambda)\overline{\widetilde{Ve_m}(\Lambda)}d\rho(\lambda) = \delta_{n,m}.$$

Using (a.5) and (a.6), we get

$$
\begin{aligned}
\widetilde{Ve_n}(\Lambda) &= \Gamma_{n,\nu}^{\frac{1}{2}}\Gamma_{n,\kappa}^{\frac{1}{2}}q_n(\Lambda) \\
&= \Gamma_{n,\nu}^{\frac{1}{2}}\Gamma_{n,\kappa}^{\frac{1}{2}}(\nu+\kappa-1)^{\frac{1}{2}}\Gamma(\nu)^{-\frac{1}{2}}\Gamma(\kappa)^{-\frac{1}{2}} \left|\Gamma\left(\frac{\nu+\kappa-1+i\lambda}{2}\right)\right| p_n(\Lambda).
\end{aligned}
$$

Therefore, we have

$$\frac{\nu+\kappa-1}{\Gamma(\nu)\Gamma(\kappa)}\int_{\mathbb{R}^+\cup\{\lambda_1,\dots,\lambda_k\}} p_n(\Lambda)\overline{p_m(\Lambda)}\left|\Gamma\left(\frac{\nu+\kappa-1+i\lambda}{2}\right)\right|^2 d\rho(\lambda) = \frac{\delta_{n,m}}{\Gamma_{n,\nu}\Gamma_{n,\kappa}}.$$

This is the orthogonality relation we have looked for. It is also one of the orthogonality relations for the hypergeometric polynomials in [26]. We now explain this in detail.

First we recall Thomae's transformation formula (see [9], p. 59):

$${}_3F_2(-n,a,b;c,d;1) = \frac{(d-b)_n}{(d)_n}{}_3F_2(-n,c-a,b;c,1+b-d-n;1).$$

It follows that

$$
\begin{aligned}
&p_n(\Lambda)\\
&= \frac{(\frac{\nu+\kappa-1+i\lambda}{2})_n}{n!}\times\\
&\quad\times {}_3F_2\Big(-n, \frac{\nu-\kappa+1-i\lambda}{2}, \frac{-\nu+\kappa+1-i\lambda}{2}; 1, \frac{-\nu-\kappa-2n+3-i\lambda}{2}; 1\Big)\\
&= \frac{(\frac{\nu+\kappa-1+i\lambda}{2})_n}{n!}\,\frac{(1-n-\kappa)_n}{(\frac{-\nu-\kappa-2n+3-i\lambda}{2})_n}\times\\
&\quad\times {}_3F_2\Big(-n, \frac{-\nu+\kappa+1+i\lambda}{2}, \frac{-\nu+\kappa+1-i\lambda}{2}; 1, \kappa, 1\Big)\\
&= \frac{(\kappa)_n}{n!}\,{}_3F_2\Big(-n, \frac{-\nu+\kappa+1+i\lambda}{2}, \frac{-\nu+\kappa+1-i\lambda}{2}; 1, \kappa; 1\Big)\\
&= \frac{1}{(n!)^2}(1)_n(\kappa)_n\,{}_3F_2\Big(-n, \frac{-\nu+\kappa+1+i\lambda}{2}, \frac{-\nu+\kappa+1-i\lambda}{2}; 1, \kappa; 1\Big)\\
&= \frac{1}{(n!)^2}\,S_n\Big(-(\frac{\lambda}{2})^2; \frac{\kappa-\nu+1}{2}, \frac{\nu-\kappa+1}{2}, \frac{\nu+\kappa-1}{2}\Big),
\end{aligned}
$$

where

$$S_n(x^2; a, b, c) = (a+b)_n(a+c)_n\,{}_3F_2(-n, a+ix, a-ix; a+b, a+c; 1)$$

are the continuous dual Hahn polynomials (see [3], [26]). In particular, as a consequence of our result, we get a new proof of the Wilson's orthogonality relation and, further of the fact, that the weight function is a product of the Harish-Chandra c-function and the symbol function of a Berezin transform.

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