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Multiple periodic solutions of some forced hamiltonian systems and the generalized saddle point theorem

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Abstract

In this paper we prove the existence of geometrically distinct periodic solutions of

$$J\dot{u} + \nabla H(t,u) = 0$$

where $H(t,x)$ is periodic with respect to $t, x_1, \dots, x_p$ and goes to zero uniformly with respect to $(t, x_1, \dots, x_p)$ when $(x_{p+1}, \dots, x_{2N})$ goes to infinity.

1. Introduction

In this note we consider the following hamiltonian system

$$\text{(H)} \qquad J\dot{u} + \nabla H(t,u) = 0$$

Here, $H(t,x)\colon \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is a continuously differentiable function, periodic in t with minimal period $T > 0$. We are interested in the existence of multiple periodic solutions of (H).

We assume that H is periodic in a part of the variables x_i and resonant at infinity with respect to the other part of variables.

2. Tools

Before giving a variational formulation of (H), some preliminary materials on function spaces and norms is needed.

Let $L^2(S^1,\mathbb{R}^{2N})$ denote the set of $2N$-tuples of T-periodic functions which are square integrable. If $u \in L^2(S^1,\mathbb{R}^{2N})$, it has a Fourier expansion

$$u = \sum_{m\in\mathbb{Z}} e^{2\pi mtJ/T}\hat{u}_m,$$

where $\hat{u}_m \in \mathbb{R}^{2N}$ and $\sum_{m\in\mathbb{Z}}|\hat{u}_m|^2 < \infty$. Set

$$\|u\| = \left[\sum_{m\in\mathbb{Z}}(1+|m|)|\hat{u}_m|^2\right]^{1/2}$$

and let

$$X = W^{1/2,2}(S^1,\mathbb{R}^{2N}) = \left\{u \in L^2(S^1,\mathbb{R}^{2N}) : \|u\| < \infty\right\}.$$

For e.g. smooth $u \in X$, set

$$Q(u) := -\int_0^T \langle J\dot{u}, u\rangle\, dt.$$

Then it is easy to check that

$$Q(u) = 2\pi \sum_{m\in\mathbb{Z}} m|\hat{u}_m|^2. \tag{1}$$

Set

$$X^0 := \mathbb{R}^{2N}$$
$$X^+ = \left\{u \in X : u(t) = \sum_{m\geq 1} e^{2\pi mtJ/T}\hat{u}_m \text{ a.e.}\right\}$$
$$X^- = \left\{u \in X : u(t) = \sum_{m\leq -1} e^{2\pi mtJ/T}\hat{u}_m \text{ a.e.}\right\}.$$

Then $X = X^0 \oplus X^+ \oplus X^-$. In fact it is not difficult to verify that X^+, X^-, X^0 are respectively the subspaces of X on which Q is positive definite, negative definite, and null, and these spaces are orthogonal with respect to the bilinear form

$$B(u,v) := -\int_0^T \langle J\dot{u}, v\rangle\, dt$$

associated with Q. It is also easy to check that X^0, X^+ and X^- are mutually orthogonal in $L^2(S^1,\mathbb{R}^{2N})$.

One further analytical fact about X is needed.

Proposition I ([5])

For $s \in [1, +\infty[$, X is compactly embedded in $L^s(S^1, \mathbb{R}^{2N})$. In particular there is an $\alpha_s > 0$ such that

$$\|u\|_{L^s} \leq \alpha_s \|u\| \tag{2}$$

for all $u \in X$.

Now, we consider the operator A defined on X by

$$\langle Au, v \rangle = -\int_0^T [\langle J\dot{u}, v\rangle + \langle \bar{u}, v\rangle]\, dt \tag{3}$$

where $\bar{u} = \frac{1}{T}\int_0^T u(t)\, dt$ is the mean value of u in $[0, T]$. It is not difficult to check that A is continuous and invertible from X to X^*.

We recall the generalized saddle point theorem [2]. We assume that $X = E \times V$ where E is a Banach space and V is a complete connected Finsler manifold of class C^2. Let $E = W \oplus Z$ (topological direct sum) and $E_n = W_n \oplus Z_n$ be a sequence of closed subspaces with $Z_n \subset Z$, $W_n \subset W$, $1 \leq \dim W_n < \infty$. Define

$$X_n = E_n \times V.$$

Denoting $f_n = f|_{X_n}$ we then have $f_n \in C^1(X_n, \mathbb{R}), n \geq 1$.

Definition ([2]). Given $c \in \mathbb{R}$, we say that f satisfies the Palais-Smale condition with respect to (X_n) at level c if every sequence (x_n) satisfying

$$x_n \in X_n, \quad f(x_n) \to c, \quad \|df_n(x_n)\| \to 0$$

possesses a subsequence which converges in X to a critical point of f. The above property will be referred as the $(\mathrm{PS})_c^*$ condition with respect to (X_n).

Theorem I (Generalized Saddle Point Theorem)

Assume there exist $r > 0$ and $\alpha < \beta \leq \gamma$ such that

a) *f satisfies the $(\mathrm{PS})_c^*$ condition with respect to (X_n) for every $c \in [\beta, \gamma]$;*
b) *$f(w, v) \leq \alpha$ for every $(w, v) \in W \times V$ such that $\|w\| = r$;*
c) *$f(z, v) \geq \beta$ for every $(z, v) \in Z \times V$,*
d) *$f(w, v) \leq \gamma$ for every $(w, v) \in W \times V$ such that $\|w\| \leq r$.*

Then $f^{-1}([\beta, \gamma])$ contains at least cuplength$(V) + 1$ *critical points of f.*

3. The main result

Our main result concerning the system (H) is the following one:

Theorem 1

Assume that

(H1) $H(t,x) \neq 0$, $\forall t \in [0,T]$, $\forall x \in \mathbb{R}^{2N}$,

(H2) H *is periodic in the variables* $x_1, \ldots, x_p$,

(H3) H *and* ∇H *tend to zero uniformly in* $t, x_1, \ldots, x_p$ *as* $(x_{p+1}, \ldots, x_{2N})$ *tends to infinity in* $\mathbb{R}^{2N-p}$.

Then the system (H) has at least $(p+1)$ T*-periodic solutions.*

Proof. We can assume that

$$H(t,x) < 0, \qquad \forall t \in [0,T],\ \forall x \in \mathbb{R}^{2N}$$

and we consider the continuously differentiable functional

$$\varphi(u) = -\int_0^T \left[\tfrac{1}{2}\langle J\dot{u}, u\rangle + H(t,u)\right] dt$$

defined on the space X introduced above. One has

$$\varphi'(u)v = -\int_0^T \langle J\dot{u} + \nabla H(t,u), v\rangle\, dt,$$

and it is well known that the critical points of the functional φ correspond to the T-periodic solutions of the system (H).

To find critical points of φ we will apply the generalized Saddle Point Theorem to φ. Let

$$e_i = (0,\ldots,0,\underset{i\text{-th}}{1},0,\ldots,0)$$

$$Y_0 = \langle e_1, \ldots, e_p\rangle$$
$$Y_1 = \langle e_{p+1}, \ldots, e_{2N}\rangle$$

Let $W = Y_1 \oplus X^-$, $Z = X^+$ and V be the quotient space $Y_0/\{x + e_i \sim x,\ i = 1,\ldots p\}$ which is nothing but the torus T^p. Now regard the function f as defined on $E = (W \oplus Z) \times V$ and apply theorem 1. We have

$$\forall (z,v) \in Z \times V, \qquad \varphi(z+v) = -\int_0^T \left[\tfrac{1}{2}\langle J\dot{z}, z\rangle + H(t, z+v)\right] dt.$$

Let (z_n, v_n) be a minimizing sequence: $\varphi(z_n, v_n) \to \inf_{Z\times V} \varphi$ then by (H3) and the formula (1), (z_n) is bounded in X. Therefore, up to a subsequence, there exists $(z_0, v_0) \in Z \times V$ such that (z_n) (resp. (v_n)) is weakly convergent to z_0 (resp. v_0). Moreover, the embedding map $X \to L^2$, $u \to u$ is compact, so $z_n \to z_0$ in L^2. Then, by taking a subsequence if it is necessary, we can assume that $z_n(t) \to z_0(t)$ a.e. and by Lebesgue Theorem, we have

$$\int_0^T H(t, z_n + v_n)\, dt \longrightarrow \int_0^T H(t, z_0 + v_0)\, dt.$$

Now, it is not difficult to see that $\varphi(z_n + v_n) \to \varphi(z_0 + v_0)$ and then φ attains its minimum on $Z \times V$ at (z_0, v_0). We then have

$$\beta = \inf_{Z\times V} \varphi \geq -\int_0^T H\big(t, z_0 + v_0\big)\, dt.$$

Let $0 < \alpha < \beta$, we have for all $(u, v) \in W \times V$

$$\varphi(u + v) = -\int_0^T \Big[\tfrac{1}{2}\langle J\dot{\tilde{u}}, \tilde{u}\rangle + H(t, u_1 + \tilde{u} + v)\Big]\, dt$$

where u_1 is the mean value of u and $\tilde{u} = u - u_1$. By (H2) and formula (1), it is easy to see that

$$\lim_{\|\tilde{u}\|\to\infty} \varphi(u + v) = -\infty, \qquad \lim_{|\tilde{u}_1|\to\infty} \varphi(u + v) \leq 0 \qquad \text{uniformly in } v.$$

So there exists $r > 0$ such that

$$\forall (u, v) \in W \times V, \qquad \|u\| = r \Rightarrow \varphi(u + v) \leq \alpha.$$

φ is also bounded from above on $B_r \times V$ by a constant $\gamma \geq \beta$, where B_r is the closed disc in W centered in zero, with radius r.

Now, we will prove that for all $c \in [\beta, \gamma]$, φ satisfies the Palais-Smale condition at level c with respect to

$$E_n = \Big[Y_1 \oplus \Big\{u \in X : u(t) = \sum_{1\leq|m|\leq n} e^{2\pi m t J/T}\hat{u}_m\Big\}\Big] \times V, \quad n \in \mathbb{N}.$$

Let (u_n) be a sequence such that

$$u_n \in E_n,\ \forall n \in \mathbb{N};\ \varphi(u_n) \to c;\ \big\|d\varphi_n(u_n)\big\| \to 0$$

where $\varphi_n = \varphi|_{E_n}$. Set

$$u_n = \bar{u}_n + \widetilde{u}_n + v_n, \qquad \text{with } \bar{u}_n \in Y_1.$$

By the formula (1), (4) we have

$$(5) \qquad \varphi'(u_n)\cdot\left(\widetilde{u}_n^+ - \widetilde{u}_n^-\right) = 2\pi \sum_{1\le|m|\le n} |m|\left|\widetilde{u}_m\right|^2 - \int_0^T \left\langle \nabla H(t,u_n), \widetilde{u}_n^+ - \widetilde{u}_n^- \right\rangle dt$$

and we deduce by the assumption (H3) the inequality

$$\left\|\widetilde{u}_n\right\|^2 \le \text{const}\left\|\widetilde{u}_n\right\|$$

so $(\widetilde{u}_n)$ is bounded in X and we can assume that $\widetilde{u}_n(t) \to \widetilde{u}(t)$ a.e. We claim that $(\bar{u}_n)$ is bounded. Otherwise, by Lebesgue theorem, we have

$$(6) \qquad \lim_{n\to\infty}\int_0^T H\left(t, \bar{u}_n + \widetilde{u}_n + v_n\right) dt = 0$$

and

$$(7) \qquad \lim_{n\to\infty}\int_0^T \left|\nabla H\left(t, \bar{u}_n + \widetilde{u}_n + v_n\right)\right|^2 dt = 0$$

By Hölder inequality

$$\left|\int_0^T \left\langle \nabla H(t,u_n), \widetilde{u}_n^+ - \widetilde{u}_n^- \right\rangle dt\right| \le \left|\nabla H(t,u_n)\right|_{L^2} \left|u_n\right|_{L^2}$$
$$\le \left|\nabla H(t,u_n)\right|_{L_2} \left\|\widetilde{u}_n\right\| \le M\left|\nabla H(t,u_n)\right|_{L_2},$$

we deduce from (7) that

$$\int_0^T \left\langle \nabla H(t,u_n), \widetilde{u}_n^+ - \widetilde{u}_n^- \right\rangle dt \longrightarrow 0, \qquad \text{if } n\to\infty.$$

Elsewhere, we have

$$\left|\varphi_n'(u_n)\cdot\left(\widetilde{u}_n^+ - \widetilde{u}_n^-\right)\right| \le \left\|\varphi_n'(u_n)\right\|_{X^*}\left\|\widetilde{u}_n\right\|_X \le M\left\|\varphi_n'(u_n)\right\|_{X^*}$$

and so

$$\varphi_n'(u_n)\cdot\left(\widetilde{u}_n^+ - \widetilde{u}_n^-\right) \longrightarrow 0, \qquad \text{if } n\to\infty.$$

Consequently, we deduce from (5) that $(\widetilde{u}_n)$ goes to zero in X and therefore by (6)

$$\varphi(u_n) = \pi \sum_{1\leq|m|\leq n} m|\hat{u}_m|^2 - \int_0^T H(t,u_n)\,dt \longrightarrow 0, \qquad n\to\infty$$

in contradiction with $\varphi(u_n)\to c>0$. Then (u_n) is bounded in E and we can assume that $u_n \rightharpoonup u$ in E.

Now, let P_n (resp. Q_n) be the projector from E to E_n (resp. $E_n^{\perp}$), we have $E=\overline{\bigcup_{n\geq 0} E_n}$, so for all $u\in E$,

$$u = P_n u + Q_n u; \quad P_n u \to u, \quad Q_n u \to 0 \quad \text{when } n\to\infty.$$

Elsewhere, we have

$$\varphi'(u)\cdot v = \langle Au, v\rangle + \langle B(u), v\rangle$$

where A is the linear operator introduced in the paragraph 2 and

$$\langle B(u), v\rangle = \int_0^T \Big[\langle -\nabla H(t,u), v\rangle + \langle \bar{u}, \bar{v}\rangle\Big]\,dt.$$

The operator B is compact, therefore $B(u_n)\to B(u)$. Let f_n be a representative of $\varphi'(u_n)$ in X given by the Riesz Theorem, then

$$Au_n + B(u_n) = f_n + Q_n B(u_n).$$

Since $Q_n B(u_n) \to 0$ and A is a continuously invertible operator, then $u_n \to A^{-1}B(u)$, which proves that φ satisfies the $(\mathrm{PS})_c^*$.

φ verifies all the generalized Saddle Point Theorem assumptions, so φ has at least cuplength$(V)+1$ critical points, and since V is the torus T^p, then cuplength$(V)=p$ and the theorem is proved. □

Remark. Writing $x=(x_1,\ldots,x_{2N})$ and taking $p\in\{1,\ldots,2N-1\}$, we can replace (H2) and (H3) respectively by

(H2') H is periodic in the variables $x_{\sigma(1)},\ldots,x_{\sigma(p)}$,

(H3') H and ∇H tend to zero uniformly in $t, x_{\sigma(1)},\ldots,x_{\sigma(p)}$ as $(x_{\sigma(p+1)},\ldots,x_{\sigma(2N)})$ tends to infinity in $\mathbb{R}^{2N-p}$, where σ is a permutation of the set $\{1,\ldots,2N\}$.

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