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A projection property and weak sequential completeness of α-duals

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Abstract

In [7] the Hahn-Schur Theorem from summability was used to give a simple proof of a result of Bennett on the weak sequential completeness of the α-dual of a monotone sequence space ([2]). In this note we point out that the same method of proof employed in [7] can be used to give a generalization of Bennett's result to a wider class of sequence spaces than the monotone spaces. As was the case in [7], the methods are also applicable to vector-valued sequence spaces.

1. Introduction

Let E be a vector space of real-valued sequences. The α-dual of E, E^α, is defined to be the space of all real-valued sequences $y = (y_i)$ such that $\sum_{i=1}^\infty y_i x_i$ is absolutely convergent for every $x = (x_i) \in E$; we write $y \cdot x = \sum_{i=1}^\infty y_i x_i$ when $y \in E^\alpha$, $x \in E$. If E contains the vector space Φ of all sequences which are eventually 0, then (E, E^α) form a dual pair under the bilinear map $y \cdot x$. We denote the weak topology on E^α from this pairing by $\sigma(E^\alpha, E)$. The space E is said to be monotone if the sequence $tx = (t_i x_i) \in E$ for every $x \in E$ and sequence $t = (t_i)$ in m_0, the space of all sequences with finite range. Bennett showed that $(E^\alpha, \sigma(E^\alpha, E))$ is sequentially complete whenever E is a monotone space ([2]). Bennett's proof used some deep results from functional analysis, but a simple proof based on the Hahn-Schur Theorem was given in [7]. We show the methods of [7] can be used to give an extension of Bennett's Theorem.

2. The scalar case

DEFINITION 1. A family $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$, which contains the finite subsets of $\mathbb{N}$ is called a Hahn-Schur family (HS-family) if whenever $t^i \in \ell^1$ and $\lim_i \sum_{j\in\mathcal{A}} t^i_j$ exists for all $\mathcal{A} \in \mathcal{F}$, then $t = (t_j) \in \ell^1$, when $t_j = \lim_i t^i_j$, and $\lim_i \|t^i - t\|_1 = 0$.

If $\mathcal{F} = \mathcal{P}(\mathbb{N})$, then $\mathcal{F}$ is an HS-family by the classical Hahn-Schur Theorem ([1], Corollary 15, p. 41); however, proper subsets of $\mathcal{P}(\mathbb{N})$ can be HS-families. For example, we can obtain a class of HS-families using the following results of J. Sember and R. Samaratunga ([6]).

DEFINITION 2. A family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a finitely quasi-σ-family (or an FQσ-family) if $\mathcal{F} \supset F_0 = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ and for each disjoint sequence (A_n) of members of F_0, there exists a subsequence (A_{n_k}) such that $\bigcup_k A_{n_k} \in \mathcal{F}$.

If $A \subset \mathbb{N}$, let C_A be the characteristic function of A. We denote by $\overline{\mathcal{F}}$ the linear span of $\{C_A : A \in \mathcal{F}\}$.

The following theorem shows that an FQσ-family is a Hahn-Schur family.

Theorem 3 ([6], Theorem 2.5)

Let $F \subseteq \mathcal{P}(\mathbb{N})$ be any FQσ-family and let $(t^i) \subset \ell^1$. The following are equivalent:

(i) $t^i \to 0$ in $\sigma(\ell^1, \overline{\mathcal{F}})$
(ii) $t^i \to 0$ in $\|\cdot\|_1$.

Let E be a sequence space containing Φ and if x is a sequence, let $C_A x$ denote the pointwise product of C_A and x. If $y \in E^\alpha$, note $C_A y \in E^\alpha$ for every $y \in E^\alpha$ since E^α is monotone. We now define the property of the sequence spaces which we will consider.

DEFINITION 4. Let

$$\mathcal{A} = \{A \subseteq \mathbb{N} : C_A : E^\alpha \to E^\alpha \text{ is } \sigma(E^\alpha, E) \text{ sequentially continuous}\}.$$

If $\mathcal{A}$ is an HS-family, then E will be said to have the Hahn-Schur property (HS-property).

Note that any monotone space has the HS-property since $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and $C_A y \cdot x = y \cdot C_A x$ for every $A \subset \mathbb{N}$, $y \in E^\alpha$, $x \in E$. We now give our generalization of Bennett's Theorem.

Theorem 5

If E has the HS-property, then E^α is $\sigma(E^\alpha, E)$ sequentially complete.

Proof. Let (y^i) be a $\sigma(E^\alpha, E)$-Cauchy sequence and let $y_j = \lim_i y_j^i$. We first show that $y = (y_j) \in E^\alpha$. For any $x \in E$, and any $A \in \mathcal{A}$, $\lim_i C_A y^i \cdot x = \lim_i \sum_{j \in A} y_j^i x_j$ exists since C_A is $\sigma(E^\alpha, E)$ sequentially continuous. Since $\mathcal{A}$ is an HS family,

$$(y_j x_j) \in \ell^1.$$

This implies that $y \in E^\alpha$. Because

$$\lim_i \sum_{j=1}^{\infty} \left| y_j^i x_j - y_j x_j \right| = 0,$$

$y^i \to y$ in $\sigma(E^\alpha, E)$, which proves the result. □

Remark 6. Note that if E has the HS-property, then $C_A : E^\alpha \to E^\alpha$ is $\sigma(E^\alpha, E)$-sequentially continuous for all $A \subseteq \mathbb{N}$. Let $M(E) = \{(y_i)_{i=1}^\infty : (x_i y_i)_{i=1}^\infty \in E \; \forall (x_i)_{i=1}^\infty \in E\}$, that is, the multiplier space of E.

Corollary 7

If $\mathcal{M} = \{A : C_A \in M(E)\}$ is an HS-family, then E^α is $\sigma(E^\alpha, E)$ sequentially complete.

Proof. Let $(y^i) \subseteq E^\alpha$ and assume $y^i \to 0$ in $\sigma(E^\alpha, E)$. For any $A \in \mathcal{M}$, $C_A x \in E$ for all $x \in E$. Therefore

$$C_A y^i \cdot x = y^i \cdot C_A x \longrightarrow 0,$$

so $C_A : E^\alpha \to E^\alpha$ is $\sigma(E^\alpha, E)$ sequentially continuous. $\mathcal{M}$ is an HS-family so the result follows from the theorem. □

If E is a monotone space, $\mathcal{M} = \{A : A \in M(E)\} = \mathcal{P}(\mathbb{N})$, so Theorem 5, in principle, generalizes the result of Bennett that E monotone implies E^α is $\sigma(E^\alpha, E)$ sequentially complete ([2]).

We present an example of a nonmonotone space E with the HS-property, using the following result of Richard Haydon. This will show that Theorem 5 gives a generalization of Bennett's Theorem. Haydon's terminology and notation have been changed for consistency.

Proposition 8 ([3], Proposition 1E)

There is an algebra $U \subset \mathcal{P}(\mathbb{N})$ which is an FQσ-family, but for no infinite $A \subseteq \mathbb{N}$ do we have $\mathcal{P}(A) = \{A \cap B : B \in U\}$.

A technical lemma also is needed:

Lemma 9

Let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ be a set algebra. Then $\forall x \in \overline{\mathcal{A}}$, the support of x

$$\operatorname{supp}(x) = \{n \in N : x_n \neq 0\} \in \mathcal{A}.$$

Proof. Each $x \in \overline{\mathcal{A}}$ is a simple function and so can be written as $\sum_{i=1}^{n} \alpha_i C_{S_i}$, $S_i \in \mathcal{A}$. Without loss of generality we can assume that the S_i's are pairwise disjoint and that $\alpha_i \neq 0$ $\forall i$ (if not, a standard disjointification procedure can be used). So

$$\operatorname{supp}(x) = \bigcup_{i=1}^{n} S_i \in \mathcal{A}$$

since $\mathcal{A}$ is an algebra. □

We can now present

EXAMPLE 10. A nonmonotone sequence space E containing Φ with the HS-property.

Let U be the algebra in Proposition 8 and $E = \overline{U}$. $\Phi \subset E$ since U contains the finite subsets of $\mathbb{N}$. E is nonmonotone because for any infinite $A \in U$ $\exists$ $B \subset A$ such that $B \notin U$. By Lemma 9, $C_B \notin E$ so E is not monotone.

Let (y^n) be a $\sigma(E^\alpha, E)$-null sequence. Then, by Theorem 3, (y^n) is $\|\cdot\|_1$-null. This implies that $C_A y^n \to 0$ in $\|\cdot\|_1$ $\forall A \subseteq \mathbb{N}$ and therefore C_A is $\sigma(E^\alpha, E)$ sequentially continuous.

We next show that the HS-property is not necessary for E^α to be $\sigma(E^\alpha, E)$ sequentially complete. For this we need the following definition and theorem of Noll.

Let

$$I_0 = \big\{\{n, n+1, \ldots, n+k\} : n, k \in \mathbb{N}\big\},$$

that is, the set of all finite subintervals of $\mathbb{N}$. A sequence $(I_n) \subset I_0$ is increasing if $\min(I_{n+1}) > \max(I_n)$ $\forall n$.

DEFINITION 11 ([5]). Let E be a sequence space containing Φ. E is said to have the weak gliding hump property (WGHP) if given any $x \in E$ and any increasing $(I_n) \subset I_0$, there exists a subsequence (I_{n_k}) such that $C_{\cup_k I_{n_k}} x \in E$.

Theorem 12 ([5], Theorem 6)

Let E be a sequence space containing Φ and having the WGHP. Then E^β is $\sigma(E^\beta, E)$ sequentially complete, where $E^\beta = \{y : \sum_{i=1}^\infty y_i x_i$ converges for every $x \in E\}$.

Remark 13. Every monotone space has the WGHP. It is not difficult to confirm that $cs = \{(x_i) : \sum_{i=1}^\infty x_i$ converges$\}$ is a nonmonotone space with the WGHP.

We now construct the example.

EXAMPLE 14. Choose $z \in cs \backslash \ell^1$ such that $\sup_i\{|z_i|\} = 1$, and let $\mathcal{F} = \{A \subseteq \mathbb{N} : \sum_{i \in A} z_i$ converges$\}$. Then $\overline{\mathcal{F}}$ is a sequence space containing Φ that is nonmonotone and has the WGHP. This follows from the remark above.

Let

$$E = c_0 + \overline{\mathcal{F}} = \{x + y : x \in c_0, y \in \overline{\mathcal{F}}\}.$$

E is nonmonotone and has the WGHP. Since $c_0^\alpha = c_0^\beta = \ell^1$ and $\overline{\mathcal{F}} \subset m_0$, $E^\alpha = E^\beta = \ell^1$. Thus, by Theorem 12, E^α is $\sigma(E^\alpha, E)$ sequentially complete. However, E^α does not have the HS-property, as we now show.

We construct an increasing sequence $(I_n) \subset I_0$ such that $\bigcup_n I_n = \mathbb{N}$ and $1 \leq \sum_{i \in I_n} |z_i| \leq 2$ for all n. Let n_1 be the smallest integer such that $\sum_{i=1}^{n_1} |z_i| \geq 1$; n_1 exists because $(z_i) \notin \ell^1$. Since

$$\sup_i\{|z_i|\} = 1, \quad \sum_{i=1}^{n_1} |z_i| \leq 2.$$

Set $I_1 = \{1, \ldots, n_1\}$. Choose n_2 to be the smallest integer such that

$$\sum_{i=n_1+1}^{n_2} |z_i| \geq 1,$$

and set $I_2 = \{n_1 + 1, \ldots, n_2\}$. Continue inductively.

For notational convenience let $z^n = C_{I_n} z$. Since $z^n \in \Phi$, we can consider the sequence (z^n) to be contained in $E^\alpha = \ell^1$, and show that $z^n \to 0$ in $\sigma(E^\alpha, E)$.

If $w \in E$ then $w = u + v$, $u \in c_0$, $v \in \overline{\mathcal{F}}$. First assume $w \in c_0$. Then

$$\begin{aligned} |z^n \cdot w| = \left|\sum_{i=1}^\infty z_i^n w_i\right| &\leq \sup_{i \geq \min(I_n)} |w_i| \sum_{i=1}^\infty |z_i^n| \\ &\leq \sup_{i \geq \min(I_n)} 2|w_i| \longrightarrow 0 \end{aligned}$$

as $n \to \infty$. Next if $w \in \overline{\mathcal{F}}$, $w = \alpha_1 C_{A_1} + \alpha_2 C_{A_2} + \cdots + \alpha_m C_{A_m}$, $A_i \in \mathcal{F}\ \forall i$. Therefore

$$z^n \cdot w = z^n \cdot (\alpha_1 C_{A_1} + \cdots + \alpha_m C_{A_m}) = \sum_{i=1}^{m} \alpha_i (C_{A_i} \cdot z^n).$$

Since $\sum_{j \in A_i} z_j$ exists, $C_{A_i} \cdot z^n \to 0$ as $n \to \infty$. So

$$\sum_{i=1}^{m} \alpha_i (C_{A_i} \cdot z^n) \longrightarrow 0$$

as $n \to \infty$, and we can conclude that (z^n) is $\sigma(E^\alpha, E)$-null.

However, since $1 \le \sum_{i=1}^{\infty} |z_i^n| \le 2$ for all n, the sum of the positive or negative terms in the sequence $(z_i^n)_{i=1}^{\infty}$ must equal or exceed $1/2$ in absolute value for each n. Without loss of generality assume that there exists an increasing sequence (n_k) such that the sum of the positive terms in the sequence z^{n_k} is at least $1/2$. Let $A = \bigcup_{k=1}^{\infty} \{i : z_i^{n_k} > 0\}$. Since $\mathbb{N} \in \mathcal{F}$, $z^{n_k} \cdot C_{\mathbb{N}} \to 0$, but $|C_A z^{n_k} \cdot C_{\mathbb{N}}| \ge 1/2$ for all k. So C_A is not $\sigma(E^\alpha, E)$-sequentially continuous.

By the remark following Theorem 5, this completes the proof.

3. The vector case

Let X be a topological vector space. Denote by $E(X)$ an X-valued sequence space. All X-valued sequence spaces will be assumed to contain $\Phi(X)$, the vector space of all sequences with finite support.

For topological vector spaces X and Y, let $L(X,Y)$ denote the continuous linear operators from X into Y. Following Maddox ([4]) we can define the β-dual of $E(X) = E$ (with respect to Y) by $E^{\beta Y} = \{(A_k) \subset L(X,Y) : \sum_{k=1}^{\infty} A_k x_k$ converges $\forall (x_k) \in E\}$. If Y is a normed space, the α-dual of E is defined by $E^{\alpha Y} = \{(A_k) \subset L(X,Y) : \sum_{k=1}^{\infty} \|A_k x_k\|$ converges $\forall (x_k) \in E\}$. In contrast to the scalar case, even when E is monotone, the α and β duals may be different.

Proposition 15

Let $c_0(X) = \{(x_i)_{i=1}^{\infty} : \lim_i \|x_i\| = 0\}$, and $\ell^\infty(X) = \{(x_i)_{i=1}^{\infty} : (\|x_i\|)_{i=1}^{\infty}$ is bounded$\}$. If X and Y are infinite-dimensional Banach spaces, then $E(X)^{\alpha Y} \neq E(X)^{\beta Y}$ for $E(X) = c_0(X)$ or $\ell^\infty(X)$.

Proof. The Dvoretsky-Rogers Theorem states that if Y is an infinite-dimensional Banach space and (λ_n) a sequence of positive numbers satisfying $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$, then $\exists (y_n) \subset Y$ such that $\|y_n\| = \lambda_n$ and $\sum_{n=1}^{\infty} y_n$ is unconditionally (and therefore subseries) convergent.

For $E(X) = \ell^\infty(X)$ or $c_0(X)$, choose $(y_n) \subset Y$ such that $\|y_n\| = 1/n^{2/3}$ and $\sum_{n=1}^{\infty} y_n$ is subseries convergent. For any sequence $(z_n) \subset c_0(X)$ satisfying $\|z_n\| = 1/n^{1/3}$ we can find, by the Hahn-Banach Theorem, a sequence $(x_n') \subset X'$ such that $\|x_n'\| = 1$ and $< x_n', z_n >= \|z_n\|$. Define $A_n : X \to Y$ by

$$A_n(x) = < x_n', x > y_n, \; n = 1, 2, 3, \ldots$$

and note that

$$\|A_n(x)\| \leq \|x_n'\| \, \|x\| \, \|y_n\| \leq \left(\frac{1}{n}\right)^{2/3} \|x\|,$$

so $A_n \in L(X, Y)$.

For any $(x_n) \in \ell^\infty(X)$, $\sum_{n=1}^{\infty} A_n x_n = \sum_{n=1}^{\infty} \langle x_n', x_n \rangle \, y_n$ converges because

$$\left| \langle x_n', x_n \rangle \right| \leq \|x_n'\| \, \|x_n\| \leq \sup_n \|x_n\| < \infty,$$

and unconditional convergence is equivalent to bounded multiplier convergence in Banach spaces ([8], Thm. 5, p. 417). Since (x_n) is arbitrary, this implies that

$$(A_n) \in \ell^\infty(X)^{\beta Y} \subset c_0(X)^{\beta Y}.$$

However $(z_n) \in c_0(X)$, and

$$\begin{aligned} \sum_{n=1}^{\infty} \|A_n z_n\| &= \sum_{n=1}^{\infty} \left| \langle x_n', z_n \rangle \right| \|y_n\| \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/3} \left(\frac{1}{n}\right)^{2/3} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty, \end{aligned}$$

so

$$(A_n) \notin c_o(X)^{\alpha Y} \supset \ell^\infty(X)^{\alpha Y}.$$

See also [4] p. 22. □

We introduce another dual space which seems to be more appropriate than the α-dual for a vector valued generalization of Theorem 5. We define the σ-dual of E to be $E^{\sigma Y} = \left\{ (A_k) \subset L(X, Y) : \sum_{k=1}^{\infty} A_k x_k \text{ is subseries convergent } \forall (x_k) \in E \right\}$. If E is monotone, $E^{\beta Y} = E^{\sigma Y}$, but Proposition 15 shows that, in general, $E^{\alpha Y} \neq E^{\sigma Y}$ even when E is monotone. We show that the dual space $E^{\sigma Y}$ allows a very straightforward generalization of Theorem 5. For this generalization we require the following vector form of the Hahn-Schur Theorem.

Theorem 16 ([1], Theorem 1, p. 75)

Let G be a normed group and $x_{ij} \in G \ \forall i,j \in \mathbb{N}$. Assume that the rows of the matrix (x_{ij}) are subseries convergent and $\lim_i x_{ij} = x_j$ exists for each j. If $(\sum_{j\in A} x_{ij})$ is convergent in G for each $A \subseteq \mathbb{N}$, then
(i) the series $\sum x_j$ is subseries convergent and
(ii) $\lim_i \sum_{j\in A} x_{ij} = \sum_{j\in A} x_j$ uniformly for $A \subseteq \mathbb{N}$.

As in the scalar case, a vector-valued sequence space $E(X)$ is monotone if and only if for any $x \in E(X)$ and $A \subseteq \mathbb{N}$, $C_A x \in E(X)$. Since $E^{\sigma Y}$ is monotone, $C_A : E^{\sigma Y} \to E^{\sigma Y}$. We denote the weak topology on $E^{\sigma Y}$ by $w(E^{\sigma Y}, E)$; $T^n \to T$ in $w(E^{\sigma Y}, E)$ if and only if $T^n \cdot x = \sum_{i=1}^{\infty} T_i^n x_i$ converges in $Y \ \forall (x_i) \in E(X)$.

We have the following generalization of Definition 4.

DEFINITION 17. Let X and Y be normed vector spaces. An X-valued sequence space $E(X) \supseteq \Phi(X)$ will be said to have the Hahn-Schur Property (HS-property) if $C_A : E^{\sigma Y} \to E^{\sigma Y}$ is $w(E^{\sigma Y}, E)$ sequentially continuous $\forall A \subseteq \mathbb{N}$. As in the scalar case, if $E(X)$ is monotone, then it has the HS-property.

We say that the pair (X, Y) has the Banach-Steinhaus property if $(T_j) \subset L(X, Y)$ and $\lim T_j x = Tx$ exists for each $x \in X$ implies that $T \in L(X, Y)$, i.e., if the conclusion of the classical Banach-Steinhaus Theorem holds. If X is an F-space or if X is barrelled and Y is a locally convex space, the pair (X, Y) has the Banach-Steinhaus property ([8]).

We have a vector generalization of Theorem 5.

Theorem 18

Let (X, Y) have the Banach-Steinhaus property and let $E(X)$ be an X-valued sequence space with the HS-property. Then $E^{\sigma Y}$ is $w(E^{\sigma Y}, E)$ sequentially complete.

Proof. Let $(T^i) \subset E^{\sigma Y}$ be a $w(E^{\sigma Y}, E)$-Cauchy sequence. Since $T_j x = \lim_i T_j^i x$ exists for all $x \in X$, $T_j \in L(X, Y)$ by the Banach-Steinhaus property. We want to show that $T = (T_j) \in E^{\sigma Y}$ and $T^i \to T$ in $w(E^{\sigma Y}, E)$. Let $x = (x_j) \in E(X)$. By hypothesis,

$$\lim_i C_A T^i \cdot x = \lim_i \sum_{j\in A} T_j^i x_j$$

exists for every $A \subset \mathbb{N}$.

Therefore, Theorem 16 implies $\sum T_j x_j$ is subseries convergent $\forall (x_j) \in E(X)$, so $T \in E^{\sigma Y}$ and $\lim_i \sum_{j\in A} T_j^i x_j = \sum_{j\in A} T_j x_j$ uniformly for $A \subseteq \mathbb{N}$. This means $T^i \to T$ in $w(E^{\sigma Y}, E)$. □

Remark 19. If $E(X)$ is monotone, then $E(X)$ has the IIS-property, so $E^{\sigma Y} = E^{\beta Y}$ is $w(E^{\sigma Y}, E)$ sequentially complete (see Theorem 8 of [7]).

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