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Some remarks on interpolation of bilinear operators

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ABSTRACT

In the present paper the problem of interpolation of bilinear operators is considered. It is shown that under certain conditions the theorem of bilinearity holds.

1. Introduction

The aim of this paper is to prove some results on interpolation of bilinear operators in the framework of Banach spaces.

Before introducing the main results we recall some notation from interpolation theory (cf. [2], [3]).

A pair $\overline{X} = (X_0, X_1)$ of Banach spaces is called a Banach couple if X_0 and X_1 are both continuously embedded in some Hausdorff topological vector space V.

For a Banach couple $\overline{X} = (X_0, X_1)$ we can form the intersection $\Delta(\overline{X}) = X_0 \cap X_1$ and the sum $\Sigma(\overline{X}) = X_0 + X_1$. They are both Banach spaces with the natural norms

$$\|x\|_\Delta = \max\{\|x\|_{X_0}, \|x\|_{X_1}\}$$

and

$$\|x\|_\Sigma = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1\}.$$

DEFINITION 1. Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1)$ and $\overline{Z} = (Z_0, Z_1)$ be Banach couples. T is said to be a bilinear operator from $\overline{X} \times \overline{Y}$ into $\overline{Z}$ ($T \in \mathcal{B}(\overline{X} \times \overline{Y}, \overline{Z})$) if $T \in \mathcal{B}(X_i \times Y_i, Z_i)$, where $\mathcal{B}(X_i \times Y_i, Z_i)$ is the space of bilinear bounded operators from $X_i \times Y_i$ into Z_i ($i = 0, 1$).

DEFINITION 2. F is an interpolation functor on the category $\overline{\mathcal{C}}$ of interpolation couples of Banach spaces into the category of Banach spaces, if for any two Banach couples $\overline{X}$ and $\overline{Y}$, $F(\overline{X})$ and $F(\overline{Y})$ are interpolation spaces with respect to $\overline{X}$ and $\overline{Y}$, i.e., for any operator T from $\overline{X}$ into $\overline{Y}$ we have $T(F(\overline{X})) \subset F(\overline{Y})$.

For a given Banach couple $\overline{X} = (X_0, X_1)$ and a quasi-concave function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ (where $\psi(s) \le \max(1, s/t)\psi(t)$ for all $s, t > 0$) we denote by $\Lambda_\psi(\overline{X})$ the space of all $x \in \Sigma(\overline{X})$ which can be represented in the form

$$x = \sum_{\nu=-\infty}^{\infty} x_\nu, \quad x_\nu \in \Delta(\overline{X}) \qquad (\text{convergence in } \Sigma(\overline{X}))$$

with

$$\sum_{\nu=-\infty}^{\infty} \psi(2^\nu)^{-1} \max(\|x_\nu\|_{X_0}, 2^\nu \|x_\nu\|_{X_1}) < \infty.$$

It is well known (cf. [9] for example) that $\Lambda_\psi(\overline{X})$ is a Banach space with the norm

$$\|x\|_\psi = \inf\left\{ \sum_{\nu=-\infty}^{\infty} \psi(2^\nu)^{-1} \max(\|x_\nu\|_{X_0}, 2^\nu \|x_\nu\|_{X_1}) : x = \sum_{\nu=-\infty}^{\infty} x_\nu,\ x_\nu \in \Delta(\overline{X}) \right\}.$$

Furthermore Λ_ψ is an interpolation functor.

In the present paper we would like to consider the following problem concerning interpolation of bilinear operators. What conditions must be satisfied in order to make the following statement come true: if $T \in \mathcal{B}(\Sigma(\overline{X}) \times \Sigma(\overline{Y}), \Sigma(\overline{Z}))$ then $T \in \mathcal{B}(F_0(\overline{X}) \times F_1(\overline{Y}), F_2(\overline{Z}))$ where F_i ($i = 0, 1, 2$) are some interpolation functors.

It seems unlikely that the statement holds for any linear interpolation method. Theorems concerning our problem are well known for the real and complex methods (see [2], [4], [5], [7]).

The problem of interpolation of bilinear operators is interesting because of important applications in a general theory of Banach space. The widest application of this problem can be found when $F_i = F$ ($i = 0, 1, 2$).

Using the technique developed in [5], we show in the present work that the statement holds in two cases:

1° $F_0 = \Lambda_\psi$, $F_1 = F_2 = F$, where ψ stands for a special function generated by an interpolation functor F.

2° $F_2 = M_\varphi$, F_0, F_1 are any interpolation functors, which satisfy certain conditions concerning duality, where φ is a specially constructed function.

Here and throughout the paper M_ϕ is an interpolation functor generated by any function $\phi \in \Phi$ in the following way:

$$M_\phi(\overline{X}) = \left\{ x \in \Sigma(\overline{X}) : \|x\|_{\phi,\infty} = \sup_{s,t>0} \frac{K(s,t,x;\overline{X})}{\phi(s,t)} < \infty \right\},$$

where ϕ denotes the class of non-vanishing functions $\phi: \mathbb{R}^2_+ \to \mathbb{R}_+$, which are non-decreasing in each argument and positive homogeneous of degree one, and

$$K(s,t,x;\overline{X}) = \inf\left\{ s\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1 \right\}$$

for $s, t > 0$.

Note that for any function $\phi \in \Phi$ the following holds:

$$\phi(s,t) \leq \max(s/u, t/v)\, \phi(u,v)$$

for any $s, t, u, v > 0$. This implies that

$$(*) \qquad \phi(s,t) = \inf_{u,v>0} \max(s/u, t/v)\, \phi(u,v)$$

for any $s, t > 0$.

2. Results

It is well known that results on interpolation of bilinear operators obtained by complex as well as real methods have interesting applications in the theory of Banach spaces. It seems natural and important to research and define new interpolation functors for which the theorem of bilinearity holds. Some contribution in this research may be Theorem 1 and Theorem 2 generalizing some results obtained by A. Favini [5].

Lemma 1

Let F be an interpolation functor and let

$$\varphi_F(s,t) = \sup\left\{ \|T\|_{F(\overline{X})\to F(\overline{Y})} : \|T\|_{X_0 \to Y_0} \leq s,\ \|T\|_{X_1 \to Y_1} \leq t,\ \overline{X}, \overline{Y} \in \overline{\mathcal{C}} \right\}$$

for all $s, t \geq 0$. If $\overline{X} = (X_0, X_1)$ is any Banach couple, then

$$\|x\|_{F(\overline{X})} \leq \varphi_F(s^{-1}, t^{-1}) \max\{ s\|x\|_{X_0}, t\|x\|_{X_1} \}$$

for $x \in \Delta(\overline{X})$ and $s, t > 0$.

Proof. Let $\overline{X} = (X_0, X_1)$, $\overline{Y} = (Y_0, Y_1) = (sX_0 \cap tX_1, sX_0 \cap tX_1)$, $s, t > 0$, and let $T = \mathrm{Id}: \overline{Y} \to \overline{X}$, where the norm in space tA is defined by formula $\|a\|_{tA} = t\|a\|_A$. For all $x \in \Delta(\overline{X})$, we have

$$\|Tx\|_{X_0} \le s^{-1} \max\{\|x\|_{sX_0}, \|x\|_{tX_1}\} = s^{-1}\|x\|_{Y_0},$$
$$\|Tx\|_{X_1} \le t^{-1} \max\{\|x\|_{sX_0}, \|x\|_{tX_1}\} = t^{-1}\|x\|_{Y_1}.$$

From definition $\varphi_F(s,t)$ we have $\|T\|_{F(\overline{Y}) \to F(\overline{X})} \le \varphi_F(s^{-1}, t^{-1})$.

Now by the simple calculation we obtain

$$\begin{aligned}\|Tx\|_{F(\overline{X})} &= \|x\|_{F(\overline{X})} \le \|T\| \, \|x\|_{F(\overline{Y})} \le \varphi_F(s^{-1}, t^{-1}) \, \|x\|_{F(\overline{Y})} \\ &= \varphi_F(s^{-1}, t^{-1}) \max\{s\|x\|_{X_0}, t\|x\|_{X_1}\} \qquad \text{for } x \in \Delta(\overline{X}). \ \square\end{aligned}$$

Remark. It is easy to show that $\varphi_F(s,t) = s^{1-\theta}t^{\theta}$ for the complex interpolation functor of Calderón $F = [\cdot]_\theta$, $0 < \theta < 1$, and that $\varphi_F(s,t) \approx s^{1-\theta}t^{\theta}$ in the case of Lions-Peetre functor $F = (\cdot)_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$.

The proof of the following lemma is similar to the proof of Theorem 3.5.2(b) in [2] (for the sake of completeness we give a proof).

Lemma 2

Let F be an interpolation functor. Then

$$\Lambda_{\varphi_*}(\overline{X}) \subset F(\overline{X})$$

for any Banach couple $\overline{X}$, where $\varphi_*(t) = 1/\varphi_F(1, t^{-1})$ for $t > 0$.

Proof. By Lemma 1 it follows that

$$\|x\|_{F(\overline{X})} \le \varphi_*(t)^{-1} J(t, x; \overline{X}) \tag{1}$$

for every $x \in \Delta(\overline{X})$ and all $t > 0$, where $J(t, x; \overline{X}) = \max\{\|x\|_{X_0}, t\|x\|_{X_1}\}$. Now fix $\varepsilon > 0$ and take $x \in \Lambda_{\varphi_*}(\overline{X})$. Then $x = \sum_{\nu=-\infty}^{\infty} x_\nu$ (convergence in $\Sigma(\overline{X})$) with $x_\nu \in \Delta(\overline{X})$ and

$$\sum_{\nu=-\infty}^{\infty} \varphi_*(2^\nu)^{-1} J(2^\nu, x_\nu; \overline{X}) \le \|x\|_{\varphi_*} + \varepsilon.$$

This implies that

$$\sum_{\nu=-\infty}^{\infty} \|x_\nu\|_{F(\overline{X})} \le \|x\|_{\varphi_*} + \varepsilon \tag{2}$$

by (1). In consequence Σx_ν is convergent in $F(\overline{X})$, thus in $\Sigma(\overline{X})$. So $x = \sum_{\nu=-\infty}^{\infty} x_\nu$ and thus, by (2)

$$\|x\|_{F(\overline{X})} \le \|x\|_{\varphi_*}. \ \square$$

Theorem 1

Let F be an interpolation functor and let $\overline{X}$, $\overline{Y}$, $\overline{Z}$ be Banach couples. If $T \in \mathcal{B}(\Sigma(\overline{X}) \times \Sigma(\overline{Y}), \Sigma(\overline{Z}))$, then T is a bounded bilinear operator from $\Lambda_{\varphi_*}(\overline{X}) \times F(\overline{Y})$ into $F(\overline{Z})$, where $\varphi = \varphi_F$.

Proof (cf. [5]). For a fixed $x \in \Delta(\overline{X})$ put $S_x(y) = T(x,y)$ for $y \in \Sigma(\overline{Y})$. By assumption it follows that $S_x: \overline{Y} \to \overline{Z}$ and

$$\|S_x(y)\|_{Z_i} \le M_i \|x\|_{X_i} \|y\|_{Y_i}, \qquad i = 0,1$$

for some constants $M_i > 0$.

Since F is an interpolation functor, $S_x: F(\overline{Y}) \to F(\overline{Z})$ and

$$\|S_x\|_{F(\overline{Y}) \to F(\overline{Z})} \le \varphi\big(M_0\|x\|_{X_0}, M_1\|x\|_{X_1}\big).$$

Hence

$$\|S_x\|_{F(\overline{Y}) \to F(\overline{Z})} \le M\, \varphi\big(\|x\|_{X_0}, \|x\|_{X_1}\big), \tag{3}$$

where $M = \max(M_0, M_1)$. On $\Delta(\overline{X})$ we put the norm

$$\begin{aligned}\|x\| &= \max\left\{ \sup_{y \in F(\overline{Y})} \frac{\|S_x y\|_{F(\overline{Z})}}{M\|y\|_{F(\overline{Y})}},\ \|x\|_{\varphi_*} \right\} \\ &= \max\left\{ M^{-1}\|S_x\|_{F(\overline{Y}) \to F(\overline{Z})},\ \|x\|_{\varphi_*} \right\}.\end{aligned}$$

From (3), we have

$$\|x\| \le \max\left\{ \varphi(\|x\|_{X_0}, \|x\|_{X_1}),\ \|x\|_{\varphi_*} \right\} \qquad \text{for } x \in \Delta(\overline{X}).$$

Note that if $x \in \Delta(\overline{X})$, then $x = \sum_{\nu=-\infty}^{\infty} x_\nu$, where $x_n = x$ and $x_\nu = 0$ for $\nu \ne n$. This yields

$$\|x\|_{\varphi_*} \le \varphi_*(2^n)^{-1} J(2^n, x; \overline{X}) = \varphi(1, 2^{-n}) \max\{\|x\|_{X_0}, 2^n\|x\|_{X_1}\}.$$

Since $\varphi \in \Phi$, we obtain

$$\varphi\big(\|x\|_{X_0}, \|x\|_{X_1}\big) \le \varphi(1, 2^{-n}) \max\big(\|x\|_{X_0}, 2^n\|x\|_{X_1}\big). \tag{4}$$

From (4) and the definition of the norm in $\Lambda_{\varphi_*}(\overline{X})$ we get

$$\|x\| \le \varphi_*(2^n)^{-1} J(2^n, x; \overline{X})$$

for all $x \in \Delta(\overline{X})$. Thus it follows that if we show, that the completion $\tilde{\Delta} = (\Delta, \|\cdot\|)$ is contained in $\Sigma(\overline{X})$, then in the same way as in the proof of Lemma 2 we obtain that $\Lambda_{\varphi_*}(\overline{X}) \subset \tilde{\Delta}$. Now we will show that $\tilde{\Delta} \subset \Sigma(\overline{X})$. The necessary and sufficient condition for this is (see [1]): if Cauchy sequence $\{x_n\}$ in $(\Delta, \|\cdot\|)$ converges in $\Sigma(\overline{X})$ to 0, then $\|x_n\| \to 0$ in $(\Delta, \|\cdot\|)$. We have

$$\|x\| = \max\{M^{-1}p(x), \|x\|_{\varphi_*}\}$$

for $x \in \Delta(\overline{X})$, where $p(x) = \|S_x\|_{F(\overline{Y}) \to F(\overline{Z})}$.

Now suppose $\{x_n\}$ is a Cauchy sequence in $(\Delta, \|\cdot\|)$ which converges to 0 in $\Sigma(\overline{X})$. Then $\{x_n\}$ is a Cauchy sequence in $\Lambda_{\varphi_*}(\overline{X})$, so $x_n \to 0$ in $\Lambda_{\varphi_*}(\overline{X})$, by the continuity of embedding $\Lambda_{\varphi_*}(\overline{X})$ into $\Sigma(\overline{X})$. Furthermore $\{S_n\} = \{S_{x_n}\}$ is a Cauchy sequence in $\mathcal{B}(F(\overline{Y}), F(\overline{Z}))$.

Thus $\|S_n - S\|_{F(\overline{Y}) \to F(\overline{Z})} \to 0$ for some S, so

$$\|T(x_n, y) - Sy\|_{F(\overline{Z})} \longrightarrow 0$$

for all $y \in F(\overline{Y})$. This yields $T(x_n, y) \to Sy$ in $\Sigma(\overline{Z})$. So $S = 0$. In consequence $\|x_n\| \to 0$.

For all $x \in \Delta(\overline{X})$, $\|x\|_{\tilde{\Delta}} \le \|x\|_{\varphi_*}$. On the other hand, from the definition of $\|\cdot\|$, we get $\|x\|_{\tilde{\Delta}} \ge \|x\|_{\varphi_*}$ for all $x \in \Delta(\overline{X})$. Since $\Delta(\overline{X})$ is dense in $\tilde{\Delta}$ as well as in $\Lambda_{\varphi_*}(\overline{X})$, we have

$$\Lambda_{\varphi_*}(\overline{X}) = \tilde{\Delta}$$

isometrically. Thus, by the definition of $\|\cdot\|$, it follows that

$$\sup_{y \in F(\overline{Y})} \frac{\|S_x(y)\|_{F(\overline{Z})}}{M\|y\|_{F(\overline{Y})}} \le \|x\|_{\varphi_*}.$$

In consequence

$$\|T(x, y)\|_{F(\overline{Z})} \le M\,\|x\|_{\varphi_*}\,\|y\|_{F(\overline{Y})}$$

holds for all $x \in \Delta(\overline{X})$ and $y \in F(\overline{Y})$. Since $\Delta(\overline{X})$ is dense in $\Lambda_{\varphi_*}(\overline{X})$ the proof is finished. □

Before showing the second theorem we introduce one more definition (cf. [3]).

DEFINITION 3. The space X' dual to the intermediate space X of Banach couple $\overline{X}$ is defined by $X' = \left(\Delta(\overline{X}), \|\cdot\|_X\right)^*$, where Y^* stands for the topological dual space of Y.

By $\overline{X}'$ we denote a Banach couple (X_0', X_1'). The following formula holds (cf. [2], Theorem 2.7.1 or [3], Proposition 2.4.6)

$$(**) \qquad \Sigma(\overline{X})' = \Delta(\overline{X}')$$

with equality of norms.

For a Banach space X containing $X_0 \cap X_1$ the closure of $X_0 \cap X_1$ in X will be denoted by X^0. A Banach couple (X_0^0, X_1^0) is denoted by $\overline{X}^0$.

If F is an interpolation functor, then F^0 is an interpolation functor defined by $F^0(\overline{X}) = F(\overline{X})^0$ for any Banach couple $\overline{X}$.

Theorem 2

Let $\overline{X}$, $\overline{Y}$, $\overline{Z}$ be Banach couples and let F_0, F_1 be interpolation functors. If $T \in \mathcal{B}(\Sigma(\overline{X}) \times \Sigma(\overline{Y}), \Sigma(\overline{Z}))$, $F_0(\overline{X}^0) = F_0(\overline{X})$ and $F_0(\overline{Y}') \subset F_1(\overline{Y})'$, then T is a bounded bilinear operator from $F_0^0(\overline{X}) \times F_1^0(\overline{Y})$ into $M_\varphi^0(\overline{Z})$, where $\varphi = \varphi_{F_0}$.

Proof. Let $f \in Z_0' \cap Z_1'$, $f \neq 0$ be fixed. For $x \in \Delta(\overline{X})$, we define the functional f_x on $\Delta(\overline{Y})$ by

$$f_x(y) = f(T(x,y))$$

for $y \in \Delta(\overline{Y})$. We have

$$|f_x(y)| \leq \|f\|_{Z_i'} \|T(x,y)\|_{Z_i} \leq \|f\|_{Z_i'} \|T\|_i \|x\|_{X_i} \|y\|_{Y_i} \quad (i = 0, 1).$$

Thus, it follows that $f_x \in Y_0' \cap Y_1'$ and

$$\|f_x\|_{Y_i'} \leq \|f\|_{Z_i'} \|T\|_i \|x\|_{X_i} \quad (i = 0, 1).$$

Now we define a linear operator S from $X_0 \cap X_1$ into $Y_0' \cap Y_1'$ by $Sx = f_x$. Then

$$\|Sx\|_{Y_i'} \leq \|f\|_{Z_i'} \|T\|_i \|x\|_{X_i} \qquad \text{for } x \in \Delta(\overline{X}) \text{ (i=0,1)}.$$

Hence there exists an extension $S_i : X_i^0 \to Y_i'$ of S satisfying

$$(5) \qquad \|S_i\|_{X_i^0 \to Y_i'} \leq \|f\|_{Z_i'} \|T\|_i \quad (i = 0, 1).$$

Since $S_0 = S_1 = S$ on $\Delta(\overline{X})$, we extend S to $\overline{S}$ from $X_0^0 + X_1^0$ into $Y_0' + Y_1'$ in a natural way:

$$\overline{S}x = S_0x_0 + S_1x_1$$

for $x \in \Sigma(\overline{X})$, where $x = x_0 + x_1$ is any representation of x with $x_i \in X_i$, i=0,1. Then by (5) it follows that $\overline{S}: X^0 \to \overline{Y}'$ with

$$\|\overline{S}\|_{X_i^0 \to Y_i'} \le \|f\|_{Z_i} \|T\|_i \quad (i = 0,1). \tag{6}$$

Hence, by assumption $F_0(\overline{X}^0) = F_0(\overline{X})$ and interpolation, we have

$$\overline{S}: F_0(\overline{X}) \to F_0(\overline{Y}') \subset F_1(\overline{Y})'.$$

The closed graph theorem yields

$$\|\overline{S}x\|_{F_1(\overline{Y})'} \le C\|\overline{S}x\|_{F_0(\overline{Y}')} \tag{7}$$

for some $C > 0$ and any $x \in F_0(\overline{X})$. Thus from the definition of the function φ, (6) and (7) it follows that

$$\|\overline{S}x\|_{F_1(\overline{Y})'} \le CM\varphi(\|f\|_{Z_0'}, \|f\|_{Z_1'})\|x\|_{F_0(\overline{X})} \tag{8}$$

for any $x \in F_0(\overline{X})$, where $M = \max\|T\|_i$ $(i = 0,1)$.

Since $\overline{S} = S$ on $\Delta(\overline{X})$ and

$$\|Sx\|_{F_1(\overline{Y})'} = \sup_{y\in\Delta(\overline{Y})} \frac{|f_x(y)|}{\|y\|_{F_1(\overline{Y})}} = \sup_{y\in\Delta(\overline{Y})} \frac{|f(T(x,y))|}{\|y\|_{F_1(\overline{Y})}}$$

for $x \in \Delta(\overline{X})$, we conclude ($f \in Z_0' \cap Z_1'$ is arbitrary) by (8) that

$$\sup_{f\in Z_0'\cap Z_1'} \frac{|f(T(x,y))|}{\varphi(\|f\|_{Z_0'}, \|f\|_{Z_1'})} \le CM\|x\|_{F_0(\overline{X})} \|y\|_{F_1(\overline{Y})} \tag{9}$$

for any $x \in \Delta(\overline{X})$ and $y \in \Delta(\overline{Y})$.

Now observe that by (*) and (**) it easily follows that

$$\|z\|_{\varphi,\infty} = \sup_{s,t>0} \frac{K(s,t,z;\overline{Z})}{\varphi(s,t)} = \sup_{f\in Z_0'\cap Z_1'} \frac{|f(z)|}{\varphi(\|f\|_{Z_0'}, \|f\|_{Z_1'})}$$

for any $z \in \Delta(\overline{Z})$. This yields by (9) that

$$\|T(x,y)\|_{M_\varphi^0(\overline{Z})} \le CM\|x\|_{F_0^0(\overline{X})} \|y\|_{F_1^0(\overline{Y})}$$

and the proof is finished. □

Remark. In [3] (cf. also [6] and [9]) some results on duality of interpolation functors are given. In particular it is shown that, under the general condition, $F_1(\overline{X})' = F_0(\overline{X}')$ for any Banach couple $\overline{X}$, where F_1 (resp. F_0) stands for the minimal or maximal method of interpolation. Note also that there is quite a large class of interpolation functors satisfying $F(\overline{X}^0) = F(\overline{X})$ for any Banach couple $\overline{X}$ (cf. [6], Lemma 2).

3. Further remarks

It is equally important as researching new interpolation functors to study the form functors which are already known. The next theorem shows that a very strong condition must be imposed on generations functions so that the bilinear theorem will hold. It will be shown on example of Ovchinnikov's interpolation functors.

Recall that Ovchinnikov in [8] constructed the lower (the upper) method of interpolation φ_ℓ (resp. φ_u) depending on parameter φ, $\varphi \in \Phi$. The method φ_ℓ may be defined as follows:

$$\varphi_\ell(\overline{X}) = \left\{Ta_\varphi; T\colon (\ell_\infty, \ell_\infty(2^{-n})) \to \overline{X}\right\}, \quad \text{where } a_\varphi = \{\varphi(1, 2^n)\}_{n=-\infty}^{\infty}.$$

The space $\varphi_\ell(\overline{X})$ is equipped with the norm

$$\|x\| = \inf\left\{\max(\|T\|_{\ell_\infty \to X_0}, \|T\|_{\ell_\infty(2^{-n}) \to X_1})\colon Ta_\varphi = x\right\}.$$

From the result proved in [8] (cf. also [9]), it follows that if $\overline{E} = (E_0, E_1)$ is a couple of Banach lattices with the Fatou property, then

$$\varphi_\ell(\overline{E}) = \varphi_u(\overline{E}) = \varphi(E_0, E_1), \tag{10}$$

where $\varphi(E_0, E_1)$ is a Calderón-Lozanovskiĭ space of all functions x such that $|x| = \varphi(|x_0|, |x_1|)$, where $x_0 \in E_0$, $x_1 \in E_1$. A norm on $\varphi(E_1, E_0)$ may be defined by $\|x\|_\varphi = \inf\{\max(\|x_0\|_{E_0}, \|x_1\|_{E_1}) : |x| = \varphi(|x_0|, |x_1|)\}$.

Proposition 1

Suppose the bilinear theorem holds for the interpolation functor φ_ℓ, with φ strictly concave and $\varphi(t, 1) \to 0$ as $t \to 0$. Then there exist $C > 0$ such that

$$\varphi(st, 1) \geq C\varphi(s, 1)\varphi(t, 1).$$

Proof. From (10) it is easy to show that if we put

$$\varphi(s,t) = tM^{-1}(s/t) \tag{+}$$

if $t > 0$ and $\varphi(s,t) = 0$ if $t = 0$ then for any measure space (Ω, μ)

$$\varphi_\ell(L_1(\mu), L_\infty(\mu)) = L_M(\mu) \tag{11}$$

where $L_M(\mu) = \{x \in L_0 : \int_\Omega M(|x|/\lambda)\, d\mu < \infty \text{ for some } \lambda > 0\}$ is the Orlicz space. It is well known that L_M is a Banach space with the Luxemburg norm defined as follows:

$$\|x\|_{L_M} = \inf\left\{\lambda > 0 : \int_\Omega M(|x|/\lambda)\, d\mu \le 1\right\}.$$

Take $\overline{X} = \overline{Y} = \big(L_1(\mathbb{R}_+, m_1), L_\infty(\mathbb{R}_+, m_1)\big)$, $\overline{Z} = \big(L_1(\mathbb{R}^2_+, m_2), L_\infty(\mathbb{R}^2_+, m_2)\big)$ where m_1 and m_2 are Lebesgue measures on $\mathbb{R}_+$ and $\mathbb{R}^2_+$, respectively. Define the map $T: \Sigma(\overline{X}) \times \Sigma(\overline{Y}) \to \Sigma(\overline{Z})$ as follows

$$T(x,y)(s,t) = (x \otimes y)(s,t) = x(s)\, y(t), \qquad (s,t) \in \mathbb{R}^2_+.$$

We have

$$\big\|T(x,y)\big\|_{Z_0} = \int_{\mathbb{R}^2_+} \big|x(s)y(t)\big|\, dm_2 = \|x\|_{X_0}\|y\|_{Y_0}$$

for $(x,y) \in X_0 \times Y_0$. Furthermore

$$\big\|T(x,y)\big\|_{Z_1} = \operatorname{supess}\{|x(s)y(t)|; (s,t) \in \mathbb{R}^2_+\} \le \|x\|_{X_1}\|y\|_{Y_1}$$

for $(x,y) \in X_i \times Y_i$. Thus T is a bilinear bounded operator from $\Sigma(\overline{X}) \times \Sigma(\overline{Y})$ into $\Sigma(\overline{Z})$. Now if we suppose that the bilinear theorem holds for φ_ℓ, then by (10) and (11) it follows that for some $C > 0$

$$\big\|T(f,g)\big\|_{L_M(m_2)} \le C\|f\|_{L_M(m_1)}\|g\|_{L_M(m_1)} \tag{12}$$

for any $(f,g) \in L_M(m_1) \times L_M(m_1)$.

Taking $f = \chi_{(0,u)}$ and $g = \chi_{(0,v)}$, where χ_A is the characteristic function of the set A, we obtain by simple calculation that

$$\big\|f \otimes g\big\|_{L_M(m_2)} = 1/M^{-1}(1/uv)$$

and

$$\|f\|_{L_M(m_1)} = 1/M^{-1}(1/u), \quad \|g\|_{L_M(m_1)} = 1/M^{-1}(1/v).$$

From (12) it follows that

$$1/M^{-1}(1/uv) \le C\, 1/M^{-1}(1/u)\, 1/M^{-1}(1/v).$$

Thus the proof is finished by (+). □

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