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Number of moduli of irreducible families of plane curves with nodes and cusps

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Abstract

Let $\Sigma^n_{k,d} \subset \mathbb{P}^{n(n+3)/2}$ be the family of irreducible plane curves of degree n with d nodes and k cusps as singularities. Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component. We consider the natural rational map

$$\Pi_\Sigma : \Sigma \dashrightarrow \mathcal{M}_g,$$

from Σ to the moduli space of curves of genus $g = \binom{n-1}{2} - d - k$. We define the *number of moduli of* Σ as the dimension $\dim(\Pi_\Sigma(\Sigma))$. If Σ has the expected dimension equal to $3n + g - 1 - k$, then

$$\dim(\Pi_\Sigma(\Sigma)) \leq \min(\dim(\mathcal{M}_g), \dim(\mathcal{M}_g) + \rho - k), \qquad (1)$$

where $\rho := \rho(2, g, n) = 3n - 2g - 6$ is the Brill-Neother number of the linear series of degree n and dimension 2 on a smooth curve of genus g. We say that Σ has the expected number of moduli if the equality holds in (1). In this paper we construct examples of families of irreducible plane curves with nodes and cusps as singularities having expected number of moduli and with non-positive Brill-Noether number.

1. Introduction

In this paper we compute the number of moduli of certain families of irreducible plane curves with nodes and cusps as singularities. Let $\Sigma^n_{k,d} \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n))) := \mathbb{P}^N$,

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with $N = n(n+3)/2$, be the closure, in the Zariski's topology, of the locally closed set of reduced and irreducible plane curves of degree n with k cusps and d nodes. Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component of the variety $\Sigma^n_{k,d}$. We denote by Σ_0 the open set of Σ of points $[\Gamma] \in \Sigma$ such that Σ is smooth at $[\Gamma]$ and such that $[\Gamma]$ corresponds to a reduced and irreducible plane curve of degree n with d nodes, k cusps and no further singularities. Since the tautological family $\mathcal{S}_0 \to \Sigma_0$, parametrized by Σ_0, is an equigeneric family of curves, by normalizing the total space, we get a family

$$\begin{tikzcd} \mathcal{S}'_0 \arrow[r] \arrow[dr] & \mathcal{S}_0 \arrow[r, hook] \arrow[d] & \mathbb{P}^2 \times \Sigma_0 \arrow[dl] \\ & \Sigma_0 & \end{tikzcd}$$

of smooth curves of genus $g = \binom{n-1}{2} - k - d$. Because of the functorial properties of the moduli space $\mathcal{M}_g$ of smooth curves of genus g, we get a regular map $\Sigma_0 \to \mathcal{M}_g$, sending every point $[\Gamma] \in \Sigma_0$ to the isomorphism class of the normalization of the plane curve Γ corresponding to the point $[\Gamma]$. This map extends to a rational map

$$\Pi_\Sigma : \Sigma \dashrightarrow \mathcal{M}_g.$$

We say that Π_Σ is the *moduli map of* Σ and we set

$$\textit{number of moduli of } \Sigma := \dim(\Pi_\Sigma(\Sigma)).$$

Notice that, when $\Sigma^n_{k,d}$ is reducible, two different irreducible components of $\Sigma^n_{k,d}$ can have different number of moduli. We say that Σ has *general moduli* if Π_Σ is dominant. Otherwise, we say that Σ has *special moduli*.

DEFINITION 1.1 When Σ has the expected dimension equal to $3n + g - 1 - k$ and $g \geq 2$, we say that Σ has the expected number of moduli if

$$\dim(\Pi_\Sigma(\Sigma)) = \min(\dim(\mathcal{M}_g), \dim(\mathcal{M}_g) + \rho - k),$$

where $\rho := \rho(2, g, n) = 3n - 2g - 6$ is the number of Brill-Noether of the linear series of degree n and dimension 2 on a smooth curve of genus g.

As we shall see in the next section, when $g \geq 2$ and when Σ has the expected dimension equal to $3n + g - 1 - k$, the number of moduli of Σ is at most equal to the expected one. This happens in particular if $k < 3n$. If $k \geq 3n$, in general we have not an upper-bound for the dimension of Σ and we cannot provide an upper bound for the number of moduli of Σ, (see Lemma 2.2 and Remark 2.3). Moreover, by classical Brill-Neother theory when ρ is positive and by a well know result of Sernesi when $\rho \leq 0$ (see [18]), we have that $\Sigma^n_{0,d}$, (which is irreducible by [11]), has the expected number of moduli for every $d \leq \binom{n-1}{2}$. When $k > 0$ there are known results giving sufficient conditions for the existence of irreducible components Σ of $\Sigma^n_{k,d}$ with general moduli, (see Propositions 2.5 and 2.6 and Corollary 2.7). In this article we construct examples of families of irreducible plane curves with nodes and cusps with finite and expected number of moduli. A large part of this paper is obtained working out the main ideas and techniques that Sernesi uses in [18].

In Section 2.1 we introduce the varieties $\Sigma^n_{k,d}$ and we recall their main properties. In Section 2.2 we discuss on Definition 1.1 and we summarize known results on the number of moduli of families of irreducible plane curves with nodes and cusps. In Theorem 3.5 we prove the existence of plane curves with nodes and cusps as singularities whose singular points are in sufficiently general position to impose independent linear conditions to a linear system of plane curves of a certain degree. This result is related to the moduli problem by Lemma 3.2, Remark 3.4 and Proposition 4.1, where we find sufficient conditions in order that an irreducible component $\Sigma \subset \Sigma^n_{k,d}$ has the expected number of moduli. If Σ verifies the hypotheses of Proposition 4.1, then the Brill-Neother number ρ is not positive and Σ has finite number of moduli. Moreover, by Lemma 4.6 and Corollary 4.7, for every $k' \leq k$ and $d' \leq d+k-k'$, there is at least an irreducible component $\Sigma' \subset \Sigma^n_{k',d'}$, such that $\Sigma \subset \Sigma'$ and the general element $[D] \in \Sigma'$ corresponds to a plane curve D verifying hypotheses of Proposition 4.1 and so having the expected number of moduli. Finally, the main result of this paper is contained in Theorem 4.9, where, by using induction on the degree n and on the genus g of the general curve of the family, we construct examples of families of irreducible plane curves with nodes and cusps verifying the hypotheses of Proposition 4.1. In particular, we prove that, if $k \leq 6$ and $\rho \leq 0$, then $\Sigma^n_{k,d}$ has at least an irreducible component which is not empty and which has the expected number of moduli. This result may be improved and examples of families of curves showing that the condition $k \leq 6$ is not sharp are given in Remark 4.10. Notice that the previous theorem provides only examples of families of plane curves with nodes and cusps with expected number of moduli, when ρ is not positive. When the number of cusps k is very small, we expect it is possible to prove the existence of irreducible components of $\Sigma^n_{k,d}$ with expected number of moduli, for every value of ρ. For example, from a result of Eisenbud and Harris, it follows that $\Sigma^n_{1,d}$, (which is irreducible by [16]), has general moduli if $\rho \geq 2$, (see Corollary 2.7). In Theorem 4.11, by using induction on n we find that $\Sigma^n_{1,d}$ has general moduli also when $\rho = 1$. By recalling that, by Theorem 4.9, $\Sigma^n_{1,d}$ has expected number of moduli when $\rho \leq 0$, we conclude that $\Sigma^n_{1,d}$ has the expected number of moduli for every ρ or, equivalently, for every $d \leq \binom{n-1}{2} - 1$. We still don't know examples of irreducible components of $\Sigma^n_{k,d}$ having number of moduli smaller that the expected.

2. Preliminaries

2.1 On Severi-Enriques varieties

We shall denote by $\mathbb{P}^N = \mathbb{P}^{n(n+3)/2}$ the Hilbert scheme of plane curves of degree n, by $[\Gamma] \in \mathbb{P}^N$ the point parametrizing a plane curve $\Gamma \subset \mathbb{P}^2$ and by $\Sigma^n_{k,d} \subset \mathbb{P}^N$ the closure, in the Zariski topology, of the locally closed set parametrizing reduced and irreducible plane curves of degree n with d nodes and k cusps as singularities. These varieties have been introduced at the beginning of the last century by Severi and Enriques. In particular, the case $k = 0$ has been studied first by Severi and for this reason the varieties $\Sigma^n_{0,d}$ are usually called Severi varieties, while for $k > 0$ the varieties $\Sigma^n_{k,d}$ are called Severi-Enriques varieties. We recall that every irreducible component Σ of $\Sigma^n_{k,d}$ has dimension at least equal to

$$N - d - 2k = 3n + g - 1 - k,$$

where $g = \binom{n-1}{2} - k - d$. When the equality holds *we say that Σ has expected dimension.* Moreover, it is well known that *if $k < 3n$ then every irreducible component Σ of $\Sigma^n_{k,d}$ has expected dimension,* (see for Example [23] or [25]). On the contrary, when $k \geq 3n$, there exist examples of irreducible components of $\Sigma^n_{k,d}$ having dimension greater than the expected, (see [25]). Moreover, we recall that $\Sigma^n_{0,d}$ is not empty for every $d \leq \binom{n-1}{2}$ and it contains in its closure all points parameterizing irreducible plane curves of degree n and genus $g = \binom{n-1}{2} - d$, (see [24], [25] and [1]). Often, we shall denote $\Sigma^n_{0,d}$ by $V_{n,g}$. While the proof of the existence of $V_{n,g}$ is quite elementary and it is due to Severi, the irreducibility of $V_{n,g}$ remained an open problem for a long time and it has been proved by Harris only in 1986. Later, by using the same techniques of Harris, Kang has proved the irreducibility of $\Sigma^n_{k,d}$ with $k \leq 3$, see [11] and [16]. However, in general, $\Sigma^n_{k,d}$ is reducible and there exist values of n, d and k such that $\Sigma^n_{k,d}$ is empty, (see [25], [10], [20], [9] or Chapter 2 of [8] and related references). Finally, we recall that, if $\Sigma \subset \Sigma^n_{k,d}$ is a non-empty irreducible component of the expected dimension equal to $3n + g - 1 - k$, then, for every $k' \leq k$ and $d' \leq d + k - k'$, there exists a non-empty irreducible component $\Sigma' \subset \Sigma^n_{k',d'}$ such that $\Sigma \subset \Sigma'$. This happens in particular if $k < 3n$. More precisely, it is true that, if $\Gamma \subset \mathbb{P}^2$ is a reduced (possibly reducible) plane curve of degree n with $k < 3n$ cusps at points $q_1, \dots, q_k$, nodes at points $p_1, \dots, p_d$ and no further singularities, then, chosen arbitrarily k_1 cusps, say $q_1, \dots, q_{k_1}$ among the k cusps of Γ, k_2 cusps $q_{k_1+1}, \dots, q_{k_2}$ among $q_{k_1+1}, \dots, q_k$ and d_1 nodes $p_1, \dots, p_{d_1}$ among the nodes of Γ, there exists a family of reduced plane curves $\mathcal{D} \to B \subset \mathbb{P}^N$ of degree n, whose special fibre is $\mathcal{D}_0 = \Gamma$ and whose general fibre $\mathcal{D}_t = D$ has a node in a neighborhood of every marked node of Γ, a cusp in a neighborhood of each point $q_1, \dots, q_{k_1}$, a node in a neighborhood of each point $q_{k_1+1}, \dots, q_{k_2}$ and no further singularities, (see [25], Corollary 6.3 of [9] or Lemma 3.17 of Chapter 2 of [8]). To save space, we shall say that *the family $\mathcal{D} \to B$ is obtained from Γ by preserving the singularities $q_1, \dots, q_{k_1}$ and $p_1, \dots, p_{d_1}$, by deforming in a node each cusp $q_{k_1+1}, \dots, q_{k_2}$ and by smoothing the other singularities.*

2.2 Known results on the number of moduli of $\Sigma^n_{k,d}$

In order to explain the Definition 1.1, we need to recall some basics of Brill-Noether theory. Given a smooth curve C of genus g, the set $G^2_n(C)$ of linear series g^2_n on C of dimension 2 and degree n, is a projective variety which verifies the following properties:

(1) $G^2_n(C)$ is not empty of dimension at least ρ, if $\rho(2, n, g) = 3n - 2g - 6 \geq 0$, (see Theorem V.1.1 and Proposition IV.4.1 of [4]).

(2) Let g^2_n be a given linear series, let $H \in g^2_n$ be a divisor and let $W \subset H^0(C, H)$ be the three dimensional vector space corresponding to g^2_n. Denoting by $\omega_C = \mathcal{O}_C(K_C)$ the canonical sheaf of C and by

$$\mu_{o,C} : W \otimes H^0(C, \omega_C(-H)) \to H^0(C, \omega_C)$$

the natural multiplication map, also called the *Brill-Noether map* of the pair (C, W), we have that the dimension of the tangent space to $G^2_n(C)$ at the point $[g^2_n]$,corresponding to g^2_n, is equal to

$$\dim(T_{[g^2_n]}G^2_n(C)) = \rho + \dim(\ker(\mu_{0,C})),$$

(see [2] or Proposition IV.4.1 of [4] for a proof).

(3) Moreover, if C is a curve with general moduli (i.e. if $[C]$ varies in an open set of $\mathcal{M}_g$), the variety $G^2_n(C)$ is empty if $\rho < 0$, it consists of a finite number of points if $\rho = 0$ and it is reduced, irreducible, smooth and not empty variety of dimension exactly ρ, when $\rho \geq 1$, (see Theorem V.1.5 and Theorem V.1.6 of [4]). In the latter case, the general g^2_n on C defines a local Theorem 3.1 of [1] or Lemma 3.43 of [12]).

From (3), we deduce that, the Severi variety $\Sigma^n_{0,d} = V_{n,g}$ of irreducible plane curves of genus $g = \binom{n-1}{2} - d$, has general moduli when $\rho \geq 0$ and it has special moduli when $\rho < 0$. When $\rho < 0$, and then $g \geq 3$, by Definition 1.1, we expect that the image of $V_{n,g}$ into $\mathcal{M}_g$ has codimension exactly $-\rho$. Equivalently, recalling that, in this case,

$$\dim(V_{n,g}) = 3n + g - 1 = 3g - 3 + \rho + 8 = \dim(\mathcal{M}_g) + \rho + \dim(Aut(\mathbb{P}^2)),$$

we expect that on the smooth curve C, obtained by normalizing the plane curve corresponding to the general element of $V_{n,g}$, there is only a finite number of g^2_n mapping C to the plane as a nodal curve. This is a well known result proved by Sernesi in [18].

Theorem 2.1 (Sernesi, [18])

The Severi variety $V_{n,g} = \Sigma^n_{0,d}$ of irreducible plane curves of degree n and genus $g = \binom{n-1}{2} - d$ has number of moduli equal to

$$\min(\dim(\mathcal{M}_g), \dim(\mathcal{M}_g) + \rho).$$

What can we say about the number of moduli of an irreducible component Σ of $\Sigma^n_{k,d}$, when $k > 0$? In this case we need to distinguish the two cases $k < 3n$ and $k \geq 3n$. In the first case we have the following result.

Lemma 2.2

For every not empty irreducible component Σ of $\Sigma^n_{k,d}$, with $k < 3n$ and $g = \binom{n-1}{2} - k - d \geq 2$, the number of moduli of Σ is at most equal to

$$\min(\dim(\mathcal{M}_g),\ \dim(\mathcal{M}_g) + \rho - k),$$

where $\rho = 3n - 2g - 6$ is the Brill-Neother number of moduli of linear series of dimension 2 and degree n on a smooth curve of genus g.

Proof. We recall that an ordinary cusp P of a plane curve Γ corresponds to a simple ramification point p of the normalization map $\phi : C \to \Gamma$, i.e. to a simple zero of the differential map $d\phi$. If we denote by $G^2_{n,k}(C) \subset G^2_n(C)$ the set of g^2_n on C defining a birational morphism with k simple ramification points, then $G^2_{n,k}(C)$ is a locally closed subset of $G^2_n(C)$ and every irreducible component G of $G^2_{n,k}(C)$ has dimension at least equal to $\rho - k$, if it is not empty. In particular, if $F^2_{n,k}(C)$ is the variety whose points correspond to the pairs $([g^2_n], \{s_0, s_1, s_2\})$ where $[g^2_n] \in G^2_{n,k}(C)$ and $\{s_0, s_1, s_2\}$ is a frame of the three dimensional space associated to the linear series g^2_n, then every irreducible component of $F^2_{n,k}(C)$ has dimension at least equal to

$$\min(8, \rho - k + 8).$$

Now, let Σ be one of the irreducible components of $\Sigma^n_{k,d}$ and let $[\Gamma]$ be a general point of Σ. Then, if $\Gamma \subset \mathbb{P}^2$ is the corresponding plane curve and $\phi : C \to \Gamma$ is the normalization map, then the fibre over the point $[C] \in \mathcal{M}_g$ of the moduli map

$$\Pi_\Sigma : \Sigma \dashrightarrow \mathcal{M}_g$$

consists of an open set in one or more irreducible components of $F^2_{n,k}(C)$. In particular, *every irreducible component of the general fibre of* Π_Σ *has dimension at least equal to* $\min(8, \rho - k + 8)$. Moreover, if $k < 3n$ then Σ has the expected dimension equal to $N - d - 2k = 3n + g - 1 - k$, (see [25] or [23]). Finally, if $g = \binom{n-1}{2} - k - d \geq 2$, then

$$\dim(\Sigma) = 3n + g - 1 - k = 3g - 3 + \rho - k + 8.$$

This proves the statement. □

Remark 2.3 The proof of the previous lemma still holds if $k \geq 3n$ but Σ has the expected dimension. However in general, when $k \geq 3n$, we don't have a bound for $\dim(\Pi_\Sigma(\Sigma))$. Indeed, in this case the dimension of the general fibre of the moduli map of Σ is still at least equal to $\rho - k + 8$, but Σ may have dimension larger than $3n + g - 1 - k$. Anyhow, by the following proposition, every not empty irreducible component of $\Sigma^n_{k,d}$ has special moduli if $k \geq 3n$.

Proposition 2.4 (Arbarello-Cornalba, [1])

Let C be a general curve of genus $g \geq 2$ and let $\phi : C \to \mathbb{P}^2$ be a birational morphism, then the degree of the zero divisor of the differential map of ϕ is smaller than ρ. In particular, every irreducible component of $\Sigma^n_{k,d}$ has special moduli if $\rho = 3n - 2g - 6 < k$.

A sufficient condition for the existence of irreducible families of plane curves with nodes and cusps with general moduli is given by the following result.

Proposition 2.5 (Kang, [15])

$\Sigma^n_{k,d}$ is irreducible, not empty and with general moduli if $n > 2g - 1 + 2k$, where $g = \binom{n-1}{2} - d - k$.

Actually, in [15], Kang proves that if $n > 2g - 1 + 2k$, then $\Sigma^n_{k,d}$ is not empty and irreducible. But from his proof it follows that, under the hypothesis of Proposition 2.5, $\Sigma^n_{k,d}$ has general moduli because the general element of $\Sigma^n_{k,d}$ corresponds to a curve which is a projection of an arbitrary smooth curve C of genus g in $\mathbb{P}^{n-g}$, from a general $(n-3)$-plane intersecting the tangent variety of C in k different points. Another result which may be used to find examples of families of plane curves with nodes and cusps having general moduli is the following. Let g^r_n be a linear series on C

associated to a $(r+1)$-space $W \subset H^0(C, \mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf on C, and let $\{s_0, \dots, s_r\}$ be a basis of W, then the ramification sequence of the g^r_n at p is the sequence $b = (b_0, \dots, b_r)$ with $b_i = ord_p s_i - i$. Choosing another basis of W, the ramification sequence of g^r_n at p doesn't change. We say that the ramification sequence of the g^r_n at p is at least equal to $b = (b_0, \dots, b_r)$ if $b_i \le ord_p s_i - i$, for every i, and we write $(ord_p s_0, \dots, ord_p s_r - r) \ge (b_0, \dots, b_r)$.

Proposition 2.6 (Proposition 1.2 of [7])

Let C be a general curve of genus g, let p be a general point on C and let $b = (b_0, \dots, b_r)$ be any ramification sequence. There exists a g^r_n on C having ramification at least b at p if and only if

$$\sum_{i=0}^{r} (b_i + g - n + r)_+ \le g,$$

where $(-)_+ := \max(-, 0)$.

From Proposition 2.6, we easily deduce the following result.

Corollary 2.7

Suppose that $k \le 3$ and $\rho = 3n - 2g - 6 \ge 2k$. Then $\Sigma^n_{k,d}$ is not empty, irreducible and it has general moduli.

Proof. By [16], the variety $\Sigma^n_{k,d}$ is irreducible for every $k \le 3$ and $d \le \binom{n-1}{2} - k$. Moreover, by using classical arguments, one can prove that $\Sigma^n_{k,d}$ is not empty if $k \le 4$ and $d \le \binom{n-1}{2} - 4$, (see, for example, Corollary 3.18 of chapter two of [8]). Finally, by Theorem 1.1 of [21], by using the terminology of Proposition 2.6, under the hypothesis $k \le 3n - 4$, in particular if $k \le 3$, the variety $\Sigma^n_{k,d}$ contains every point of $\mathbb{P}^N$ corresponding to a plane curve Γ of genus $g = \binom{n-1}{2} - k - d$ such that the normalization morphism of Γ has at least a ramification point with ramification sequence $(b_0, b_1, b_2) \ge (0, k, k)$. Then, by Proposition 2.6, if $\rho \ge 2k$ and $k \le 3$, the moduli map of $\Sigma^n_{k,d}$ is surjective. $\square$

3. On the existence of certain families of plane curves with nodes and cusps in sufficiently general position

As we already observed, we don't have a complete answer for the existence problem of $\Sigma^n_{k,d}$. In this section we are interested in a little more specific existence problem. We shall prove the existence of plane curves with nodes and cusps as singularities whose singular points are in sufficiently general position to impose independent linear conditions to a linear system of plane curves of a certain degree.

DEFINITION 3.1 A projective curve $C \subset \mathbb{P}^r$ is said to be ***geometrically t-normal*** if the linear series cut out on the normalization curve $\tilde{C}$ of C by the pull-back to $\tilde{C}$ of the linear system of hypersurfaces of $\mathbb{P}^r$ of degree t is complete.

From a geometric point of view, a projective curve $C \subset \mathbb{P}^r$ is geometrically t-normal if and only if the image curve $\nu_{t,r}(C)$ of C by the Veronese embedding $\nu_{t,r} : \mathbb{P}^r \to \mathbb{P}^{\binom{r+t}{t}}$ of degree t, is not a projection of a non-degenerate curve living in a higher dimensional projective space. We shall say that a curve is ***geometrically linearly normal*** (***g.l.n.*** for short) if it is geometrically 1-normal. Every such a curve C is not a projection of a curve lying in a projective space of larger dimension.

The following result is proved under more general hypotheses in [5], Theorem 2.1.

Lemma 3.2

Let $\Gamma \subset \mathbb{P}^2$ be an irreducible and reduced plane curve of degree n and genus g with at most nodes and cusps as singularities. Let t be an integer such that $n-3-t<0$, then Γ is geometrically t-normal if and only if it is smooth. On the contrary, if $n-3-t \geq 0$, the plane curve Γ is geometrically t-normal if and only if its singular points impose independent linear conditions to plane curves of degree $n-3-t$.

We recall the following classical definition.

DEFINITION 3.3 Let $\Gamma \subset \mathbb{P}^2$ be a plane curve of degree n with d nodes at $p_1, \dots, p_d$ and k cusps at $q_1, \dots, q_k$ as singularities. Let $\phi : C \to \Gamma$ be the normalization of Γ. The adjoint divisor Δ of ϕ is the divisor on C defined by

$$\Delta = \sum_{i=1}^{d} \phi^{-1}(p_i) + \sum_{j=1}^{k} 2\phi^{-1}(q_j).$$

Proof of Lemma 3.2. Let Γ be a plane curve as in the statement of the lemma. Then, Γ is geometrically t-normal if and only if, by definition,

$$h^0(C, \mathcal{O}_C(t)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)) - h^0(\mathbb{P}^2, \mathcal{I}_\Gamma(t))$$

where $\mathcal{I}_\Gamma$ is the ideal sheaf of Γ in $\mathbb{P}^2$ and $\mathcal{O}_C(t) := \mathcal{O}_C(t\phi^*(H))$, where H is the general line of $\mathbb{P}^2$. By Riemann-Roch Theorem, Γ is geometrically t-normal if and only if

$$h^0(C, \omega_C(-t))) = -nt + g - 1 + \frac{(t+1)(t+2)}{2} - h^0(\mathbb{P}^2, \mathcal{I}_\Gamma(t)), \tag{2}$$

where g is the geometric genus of Γ and ω_C is the canonical sheaf of C. On the other hand, it is well known that

$$H^0(C, \omega_C(-t)) = H^0(C, \mathcal{O}_C(n-3-t)(-\Delta)),$$

where Δ is the adjoint divisor of ϕ, (see Definition 3.3 and [4], Appendix A). If $n-3-t < 0$ then $h^0(C, \mathcal{O}_C(n-3-t)) = 0$ and Γ is geometrically t-normal if and only if

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)) - h^0(\mathbb{P}^2, \mathcal{I}_\Gamma(t)) = nt - \frac{n^2-3n}{2} + \delta,$$

where $\delta = \binom{n-1}{2} - g = \deg(\Delta)/2$. This equality is verified if and only if $\delta = 0$, i.e. Γ is smooth. If $n-3 \geq t$, $h^0(\mathbb{P}^2, \mathcal{I}_\Gamma(t)) = 0$ and (2) is verified if and only if

$$h^0(C, \mathcal{O}_C(n-3-t)(-\Delta)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-3-t)) - \delta.$$

On the other hand, if $\psi : S \to \mathbb{P}^2$ is the blowing-up of the plane at the singular locus of Γ, denoting by $\sum_i E_i$ the pullback of the singular locus of Γ with respect to ψ and by $\mathcal{O}_S(r)$ the sheaf $\mathcal{O}_S(r\psi^*(H))$, we have that

$$\begin{aligned} h^0(C, \mathcal{O}_C(n-3-t)(-\Delta)) &= h^0\left(S, \mathcal{O}_S(n-3-t)\Big(-\sum_i E_i\Big)\right) \\ &= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-3-t) \otimes A) \end{aligned}$$

where A is the ideal sheaf of singular points of Γ. □

Remark 3.4 Notice that, if an irreducible and reduced plane curve Γ of degree n with only nodes and cusps as singularities is geometrically t-normal, with $t \leq n-3$, then it is geometrically r-normal for every $r \leq t$. Indeed, if a set of points imposes independent linear conditions to a linear system S, then it imposes independent linear conditions to every linear system S' containing S.

Theorem 3.5

Let $\Sigma^n_{k,d}$ be the variety of irreducible and reduced plane curves of degree n with d nodes and k cusps. Suppose that d, k, n and t are such that

$$d + k \leq \frac{n^2 - (3+2t)n + 2 + t^2 + 3t}{2} = h^0(\mathcal{O}_{\mathbb{P}^2}(n-t-3)) \tag{3}$$

$$t \leq n-3 \text{ if } k = 0, \tag{4}$$

$$k \leq 6 \text{ if } t = 1, 2 \text{ and} \tag{5}$$

$$k \leq 6 + \left[\frac{n-8}{3}\right], \text{ if } t = 3, \tag{6}$$

where $[-]$ is the integer part of $-$. Then the variety $\Sigma^n_{k,d}$ is not empty and there exists at least an irreducible component $W \subset \Sigma^n_{k,d}$ whose general element corresponds to a geometrically t-normal plane curve.

Remark 3.6 As we shall see in the next section, (see Proposition 4.1), the geometric linear normality of the plane curve corresponding to the general element of an irreducible component Σ of $\Sigma^n_{k,d}$, is related with the number of moduli of Σ. Another motivation for the previous theorem has been the family of irreducible plane sextics with six cusps. By [25], we know that $\Sigma^6_{6,0}$ contains at least two irreducible components Σ_1 and Σ_2. The general point of Σ_1 corresponds to a sextic with six cusps on a conic, whereas the general element of Σ_2 corresponds to a sextic with six cusps not on a conic. Note that, by the previous lemma the general element of Σ_2 parameterizes a geometric linearly normal sextic, unlike the general element of Σ_1, which corresponds to a projection of a canonical curve of genus four. Theorem 3.5, proves in particular that, under a suitable restriction, (see inequality (3)), on the genus of the curve corresponding to the general element of the family and, if the number of the cusps is small, the variety $\Sigma^n_{k,d}$ contains a not empty irreducible component whose general element corresponds to a curve which is not a projection of an other curve, lying in a projective space of larger dimension. We notice that the inequality (3) of the previous theorem can't be improved. Indeed, if $g = \binom{n-1}{2} - k - d$, then $k + d > h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-3-t))$ if and only if $g < \frac{2tn-t^2-3t}{2}$. On the other hand, by using the same notation as in Theorem (3.5), if $g < \frac{2tn-t^2-3t}{2}$, then, by Riemann-Roch Theorem, we have that

$$h^0(C, \mathcal{O}_C(t)) \geq tn - g + 1 > \frac{t^2+3t}{2} + 1 = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)).$$

On the contrary, inequalities (5) and (6) are not sharp, (see Example 3.7).

In the case of $k = 0$ and $t = 1$, Theorem 3.5 has been proved by Sernesi in [18], Section 4. The case $k = 0$ and $t \leq n - 3$ is already contained in [5]. To show Theorem 3.5, we proceed by induction on the degree n and on the number of nodes and cusps of the curve. The geometric idea at the base of the induction on the degree of the curve is, mutatis mutandis, the same as that of Sernesi.

Proof of Theorem 3.5. Let t be a positive integer such that $n-3-t \geq 0$ and let $W \subset \Sigma^n_{k,d}$ be an irreducible component of $\Sigma^n_{k,d}$. By standard semicontinuity arguments it follows that, *if there exists a point $[C] \in W$ corresponding to a geometrically t-normal curve with only k cusps and d nodes as singularities, then the general element of W corresponds to a geometrically t-normal plane curve*. Moreover, *if the theorem is true for fixed n, $t \leq n-3$, k as in (5) or in (6) and $k+d$ as in (3), then the theorem is true for n, t and any $k' \leq k$ and $d' \leq d + k - k'$*. Indeed, from the hypotheses (3), (5) and (6), it follows in particular that $k < 3n$. By Section 2.1, under this hypothesis, for every $k' \leq k$ and for every $d' \leq d+k-k'$, there exists a family of plane curves $\mathcal{C} \to \Delta$ of degree n, parametrized by a curve $\Delta \subset \Sigma^n_{k',d'}$, whose special fibre is $\mathcal{C}_0 = C$ and whose general fibre $\mathcal{C}_z$ has d' nodes and k' cusps as singularities. The statement follows by applying the semicontinuity theorem to the family $\tilde{\mathcal{C}} \to \tilde{\Delta}$, obtained by normalizing the total space of the pull-back family of $\mathcal{C} \to \Delta$ to the normalization curve $\tilde{\Delta}$ of Δ. *Finally, it's enough to show the theorem when the equality holds in (5), (6) and (3).*

First of all we consider the case $k = 0$. We will show the statement for any fixed t and by induction on n. Let, then $t \geq 1$ and $n = t + 3$. In this case the equality holds in (3) if $d = 1 = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})$. Since one point imposes independent linear conditions to regular functions, by using Lemma 3.2, we find that every irreducible plane curve of degree $n = t+3$ with one node and no further singularities is geometrically t-normal. So, the first step of the induction is proved. Suppose, now, that the theorem is true for $n = t + 3 + a$ and let $[\Gamma] \in V_{n,g}$ be a point corresponding to a geometrically t-normal curve with $\frac{a^2+3a+2}{2}$ nodes. Let D be a line which intersects transversally Γ and let $P_1, ..., P_{t+1}$ be $t+1$ marked points of $\Gamma \cap D$. If $\Gamma' = \Gamma \cup D \subset \mathbb{P}^2$, then $P_1, ..., P_{t+1}$ are nodes for Γ'. Let $C \to \Gamma$ be the normalization of Γ and $C' \to \Gamma'$ the partial normalization of Γ', obtained by smoothing all singular points of Γ', except $P_1, ..., P_{t+1}$. We have the following exact sequence of sheaves on C'

$$0 \to \mathcal{O}_D(t)(-P_1 - ... - P_{t+1}) \to \mathcal{O}_{C'}(t) \to \mathcal{O}_C(t) \to 0, \tag{7}$$

where $O_{C'}(t) := \mathcal{O}_{C'}(tH)$ and H is the pull-back with respect to $C' \to \Gamma'$ of general line of $\mathbb{P}^2$. Since $\deg(\mathcal{O}_D(t)(-P_1 - ... - P_{t+1})) < 0$, we get that

$$h^0(D, \mathcal{O}_D(t)(-P_1 - ... - P_{t+1})) = 0$$

and so

$$h^0(C', \mathcal{O}_{C'}(t)) = h^0(C, \mathcal{O}_C(t)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)). \tag{8}$$

Now, by Section 2.1, we can obtain Γ' as the limit of a 1-parameter family of irreducible plane curves

$$\psi : \mathcal{C} \to \Delta \subset \mathbb{P}^{(n+1)(n+4)/2}$$

of degree $n+1 = t+a+4$ with

$$\begin{aligned}\frac{a^2+3a+2}{2} + n - t - 1 &= \frac{(a+1)^2+3(a+1)+2}{2} \\ &= h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n+1-t-3))\end{aligned}$$

nodes specializing to nodes of Γ' different from the marked points $P_1, ..., P_{t+1}$. Moreover, one can prove that Δ is smooth, (see [24] or [25]). Normalizing $\mathcal{C}$, we obtain a family whose general fibre is smooth and whose special fibre is exactly C', and we conclude the inductive step by (8) and by semicontinuity theorem.

Now we consider the case $t = 1, 2$ or 3 and k as in (5) and in (6). Suppose the theorem is true for n and let $[\Gamma] \in \Sigma^n_{k,d}$ be a general point in one of the irreducible components of $\Sigma^n_{k,d}$. Then, let D be a smooth plane curve of degree t if $t = 1$, 2 or an irreducible cubic with a cusp if $t = 3$. By the generality of Γ, we may suppose that D intersects Γ transversally. Let $P_1, ..., P_{t^2+1}$ be t^2+1 fixed points of $\Gamma \cap D$. If $\Gamma' = \Gamma \cup D$, then $P_1, ..., P_{t^2+1}$ are nodes for Γ'. Let $C \to \Gamma$ be the normalization of Γ and $C' \to \Gamma'$ the partial normalization of Γ', obtained by smoothing all singular points except $P_1, ..., P_{t^2+1}$. By using the same notation and by arguing as before, from the following exact sequence of sheaves on C'

$$0 \to \mathcal{O}_D(t)(-P_1 - ... - P_{t^2+1}) \to \mathcal{O}_{C'}(t) \to \mathcal{O}_C(t) \to 0,$$

we deduce that

$$h^0(C', \mathcal{O}_{C'}(t)) = h^0(C, \mathcal{O}_C(t)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)). \tag{9}$$

Now, by Section 2.1, we can obtain Γ' as limit of a family of irreducible plane curves

$$\phi : \mathcal{C} \to \Delta$$

of degree $n+t$ with

$$d + nt - t^2 - 1 = \frac{(n+t)^2 + (3+2t)(n+t) + t^2 + 3t + 2}{2}$$

nodes specializing to nodes of Γ' different to $P_1, ..., P_{t^2+1}$, and $k + \frac{t^2-3t+2}{2}$ cusps specializing to cusps of Γ. We conclude by (9) and by semicontinuity, as before. Now we have to show the first step of the induction. For $t = 1$ the induction begins with the cases $(n,k) = (4,1)$, $(5,3)$, $(6,6)$. Trivially, if $n = 4$ and $k = 1$ one point imposes independent conditions to the linear system of regular functions. If $n = 5$ and $k = 3$ we have to show that there are irreducible quintics with three cusps not on a line. A quintic with three cusps is a projection of the rational normal quintic $C_5 \subset \mathbb{P}^5$ from a plane generated by three points lying on three different tangent lines to C_5. By Bezout theorem the three cusps of such a plane curve can't be aligned. If $n = k = 6$, one can repeat the classical argument used by Zariski, see [24] or Example 3.20 of Chapter 2 of [8]. For $t = 2$ we have to show the theorem for $(n, k) = (5, 1)$, $(6, 3)$, $(7, 6)$, $(8, 6)$, while for $t = 3$ we have to show the theorem for $(n, k) = (6, 1)$, $(7, 3)$, $(8, 6)$, $(9, 6)$, $(10, 6)$. The case $t = 2$ and $(n, k) = (5, 1)$ is trivial. When $t = 2$, $n = 6$ and $k = 3$ we have that $n - 3 - t = 1$. To show that there exists an irreducible sextic with three cusps not on a line, consider a rational quartic C_4 with three cusps, (see Corollary 3.18 of Chapter 2 of [8] for the existence). By Bezout Theorem, the three double points of C_4 can't be aligned. Then consider a sextic C_6 which is union of C_4 and a conic C_2 which intersects C_4 transversally. By Section 2.1, one can smooth the intersection points of C_4 and C_2 obtaining a family of sextics with three cusps not on a line. For $t = 2$, $n = 7$ and $k = 6$ we argue as in the previous case, by using a sextic C_6 with six cusps not on a conic and a line R with intersects C_6 transversally. Similarly for $t = 2$, $n = 8$ and $k = 6$ and $t = 3$ and $(n, k) = (6, 1)$, $(7, 3)$, $(8, 6)$, $(9, 6)$, $(10, 6)$. □

EXAMPLE 3.7 Inequalities (5) and (6) are not sharp. To see this, we can consider the example of curves of degree 10. We recall that we say that a plane curve is geometrically linearly normal (g.l.n. for short) if it is geometrically 1-normal. Theorem 3.5 ensures the existence of g.l.n. irreducible plane curves of degree 10 with $k \leq 6$ cusps and nodes as singularities. But, by using the same ideas as we used in Theorem 3.5, one can prove the existence of g.l.n. plane curves of degree 10 with nodes and $k \leq 9$ cusps. It is enough to consider a sextic Γ_6 with six cusps not on a conic and a rational quartic Γ_4 with three cusps intersecting Γ_6 transversally. We choose five points $P_1, \ldots, P_5$ of $\Gamma_4 \cap \Gamma_6$. If Γ_6' and Γ_4' are the normalization curves of Γ_6 and Γ_4 respectively and C' is the partial normalization of $\Gamma_6 \cup \Gamma_4$ obtained by normalizing all its singular points except $P_1, \ldots, P_5$, by considering the following exact sequence

$$0 \to \mathcal{O}_{\Gamma_4'}(1)(-P_1 - \cdots - P_5) \to \mathcal{O}_{C'}(1) \to \mathcal{O}_{\Gamma_6'}(1) \to 0$$

we find that $h^0(C', \mathcal{O}_{C'}(1)) = 3$. By using terminology of Section 2.1, the statement follows by smoothing the singular points $P_1, \ldots, P_5$ of $\Gamma_6 \cup \Gamma_4$, and by semicontinuity, as in the proof of Theorem 3.5. The bound on the number of cusps of Theorem 3.5 can be improved also for $t = 2$ or $t = 3$. For example, Theorem 3.5 ensures the existence of geometrically 3-normal curves of degree 12 with $k \leq 6$ and nodes as further singularities. But, by considering a geometrically 3-normal curve of degree 8 with six cusps and a quartic with 3 cusps and arguing as before, we can find geometrically 3-normal irreducible plane curves of degree 12 with nodes and $k \leq 9$ cusps.

4. Families of plane curves with nodes and cusps with finite and expected number of moduli

Let $\Sigma \subset \Sigma_{k,d}^n$ be an irreducible component of $\Sigma_{k,d}^n$. We want to give sufficient conditions for Σ to have the expected number of moduli. Let $[\Gamma] \in \Sigma$ be a general element, corresponding to a plane curve Γ with normalization map $\phi : C \to \Gamma$. We shall denote by ω_C the canonical sheaf of C and by $\mathcal{O}_C(1)$ the sheaf associated to the pullback to C of the divisor cut out on Γ from the general line of $\mathbb{P}^2$.

Proposition 4.1

Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component of $\Sigma^n_{k,d}$ and let $[\Gamma] \in \Sigma$ be a general element, corresponding to a plane curve Γ with normalization map $\phi : C \to \Gamma$. Suppose that Σ is smooth of the expected dimension equal to $3n + g - 1 - k$ at $[\Gamma]$. Moreover, suppose that:

(1) Γ is geometrically linearly normal, i.e. $h^0(C, \mathcal{O}_C(1)) = 3$,
(2) the Brill-Noether map

$$\mu_{o,C} : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C)$$

is surjective.

Then Σ has the expected number of moduli equal to $3g - 3 + \rho - k$.

Proof. The case $k = 0$ has been proved by Sernesi in [18], Section 4. We shall assume $k > 0$. Let Γ be a plane curve verifying the hypotheses of the proposition. By Lemma 1.5.(b) of [22], the hypothesis that Σ is smooth of the expected dimension at $[\Gamma]$ implies the vanishing $H^1(C, \mathcal{N}_\phi) = 0$, where $\mathcal{N}_\phi$ if the normal sheaf of ϕ. We recall that, denoting by Θ_C and $\Theta_{\mathbb{P}^2}$ the tangent sheaf of C and $\mathbb{P}^2$ respectively, then the normal sheaf of ϕ is defined as the cokernel of the differential map ϕ_* of ϕ

$$0 \to \Theta_C \stackrel{\phi_*}{\to} \phi^*\Theta_{\mathbb{P}^2} \to \mathcal{N}_\phi \to 0 \tag{10}$$

By Theorem 3.1 of [13], the vanishing $H^1(C, \mathcal{N}_\phi) = 0$ is a sufficient condition for the existence of a universal deformation family

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi"'] & \mathbb{P}^2 \\ B & \end{tikzcd}$$

of the normalization map ϕ, whose parameter space B is smooth at the point 0 corresponding to ϕ, with tangent space at 0 equal to $H^0(C, \mathcal{N}_\phi)$. On the contrary, by [3], p. 487, the Severi variety $V_{n,g} = \Sigma^n_{0,k+d}$ of irreducible plane curves of genus $g = \binom{n-1}{2} - d - k$ is singular at the point $[\Gamma]$ and the universal deformation space B of ϕ is a desingularization of $V_{n,g}$ at $[\Gamma]$. Moreover, by Corollary 6.11 of [2], if $B_k = F^{-1}(\Sigma)$ is the locus of points of B corresponding to a morphism with k ramification points, then the tangent space to B_k at 0 is a subspace W of $H^0(C, \mathcal{N}_\phi)$ of codimension k such that $W \cap H^0(C, \mathcal{K}_\phi) = 0$, where $\mathcal{K}_\phi$ is the torsion subsheaf of $\mathcal{N}_\phi$. By [3], p. 487, it follows that, if

$$F : B \to V_{n,g}$$

is the natural $(1:1)$-map from B to $V_{n,g}$, then the differential map

$$dF : H^0(C, \mathcal{N}_\phi) \to T_{[\Gamma]}V_{n,g}$$

restricts to an isomorphism between W and the tangent space $T_{[\Gamma]}\Sigma$ to Σ at $[\Gamma]$.

We can now go back to the number of moduli of Σ. From the exact sequence (10), by using that $H^1(C, \mathcal{N}_\phi) = 0$, we get the following long exact sequence

$$0 \to H^0(C, \Theta_C) \to H^0(C, \phi^*\Theta_{\mathbb{P}^2}) \to H^0(C, \mathcal{N}_\phi) \stackrel{\delta_C}{\to} H^1(C, \Theta_C)$$
$$\to H^1(C, \phi^*\Theta_{\mathbb{P}^2}) \to 0\,.$$

Recalling that the space $H^1(C, \Theta_C)$ is canonically identified with the tangent space $T_{[C]}\mathcal{M}_g$ to $\mathcal{M}_g$ at the point associated to the normalization C of Γ, the coboundary map $\delta_C : H^0(C, \mathcal{N}_\phi) \to H^1(C, \Theta_C)$ sends the Horikawa class of an infinitesimal deformation of ϕ to the Kodaira-Spencer class of the corresponding infinitesimal deformation of C. So, $\delta_C|W$ is the differential map at the point $0 \in B$ of the moduli map $\Pi_\Sigma \circ F : B_k = F^{-1}(\Sigma) \dashrightarrow \mathcal{M}_g$. Since the point $[\Gamma]$ is general in Σ, and recalling the isomorphism $dF : W \xrightarrow{\sim} T_{[\Gamma]}\Sigma$, we have that

$$\textit{the number of moduli of } \Sigma = \dim(\delta_C(W)).$$

Now, from the exact sequence (10), we have that

$$\dim(\delta_C(H^0(C, \mathcal{N}_\phi)) = 3g - 3 - h^1(C, \phi^*\Theta_{\mathbb{P}^2}).$$

Moreover, from the pull-back to C of the Euler exact sequence, we deduce the well known isomorphism

$$H^1(C, \phi^*\Theta_{\mathbb{P}^2}) \simeq \operatorname{coker}(\mu^*_{0,C}) \simeq (\ker(\mu_{0,C}))^*$$

and we conclude that

$$\dim(\delta_C(H^0(C, \mathcal{N}_\phi))) = 3g - 3 - \dim(\ker(\mu_{0,C})). \tag{11}$$

Notice that the previous equality is always true, even if Γ doesn't verify the hypothesis (1) or (2) of the statement. Moreover, if Γ is geometrically linearly normal, i.e. if $h^0(C, \mathcal{O}_C(1)) = 3$, we have that

$$\rho = 3n - 2g - 6 = \dim(\operatorname{coker}(\mu_{o,C})) - \dim(\ker(\mu_{o,C})).$$

When $\mu_{o,C}$ is surjective, $\rho = -\dim(\ker(\mu_{o,C}))$ and

$$\dim(\delta_C(H^0(C, \mathcal{N}_\phi)) = 3g - 3 + \rho = \dim(B) - 8 = \dim(V_{n,g}) - 8. \tag{12}$$

Since the dimension of the fibre of the moduli map

$$\Pi_{V_{n,g}} \circ F : B \dashrightarrow \mathcal{M}_g$$

has dimension at least equal to $8 = \dim(Aut(\mathbb{P}^2))$, from (12) we deduce that the differential map of $\Pi_{V_{n,g}} \circ F$ has maximal rank at 0 and, in particular, we have that $\dim((\Pi_{V_{n,g}} \circ F)^{-1}([C])) = 8$. Equivalently, there exist only finitely many g^2_n on C. It follows that there are only finitely many g^2_n on C mapping C to the plane as a curve with k cusps and d nodes. Then,

$$\dim(\delta_c(W)) = \dim(\Pi_\Sigma(\Sigma)) = 3g - 3 + \rho - k. \qquad \square$$

Remark 4.2 Arguing as in the proof of the previous proposition, it has been proved in [18] that, if Γ is a geometrically linearly normal plane curve with only d nodes as singularities and the Brill-Noether map $\mu_{o,C}$ of the normalization morphism of Γ is injective, then $\Sigma = \Sigma^n_{0,d}$ has general moduli. If $\Sigma \subset \Sigma^n_{k,d}$ and $[\Gamma] \in \Sigma$ verify the hypotheses of Proposition 4.1 but we assume that $\mu_{o,C}$ is injective, we may only conclude that $\Pi_{V_{n,g}} \circ F$ is dominant with surjective differential map at $[\Gamma]$. So $\dim(\Pi^{-1}_{V_{n,g}}([C])) = \rho + 8$. But this is not useful to compute the dimension of $\delta_C(W) = \delta_C(T_{[\Gamma]}\Sigma)$. However, in this case we get that

$$\delta_C(T_{[\Gamma]}\Sigma) + \delta_C(H^0(C, \mathcal{K}_\phi)) = \delta_C(H^0(C, \mathcal{N}_\phi)) = H^1(C, \Theta_C).$$

Then, by using that $\dim(\delta_C(H^0(C, \mathcal{K}_\phi))) \leq k$ and by recalling that if Σ has the expected dimension then the number of moduli of $\Sigma^n_{k,d}$ is at most the expected one (see Lemma 2.2 and Remark 2.3), we find that

$$3g - 3 - k \leq \textit{number of moduli of } \Sigma \leq 3g - 3 + \rho - k.$$

Remark 4.3 Notice that, if a plane curve Γ of genus g verifies the hypotheses (1) and (2) of the previous proposition, then the Brill-Noether number $\rho(2, g, n)$ is not positive and, in particular, $g \geq 3$. We don't know examples of complete irreducible families $\Sigma \subset \Sigma^n_{k,d}$ with the expected number of moduli whose general element $[\Gamma]$ corresponds to a curve Γ of genus g, with $\rho(2, g, n) \leq 0$, which doesn't verify properties (1) and (2).

Lemma 4.4 ([5], Corollary 3.4)

Let Γ be an irreducible plane curve of degree n with only nodes and cusps as singularities and let $\phi : C \to \Gamma$ be the normalization morphism of Γ. Suppose that Γ is geometrically 2-normal, i.e. $h^0(C, \mathcal{O}_C(2)) = 6$. Then the Brill-Noether map

$$\mu_{o,C} : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C)$$

is surjective.

Proof. By Lemma 3.2, the curve Γ is geometrically 2-normal if and only if the scheme N of the singular points of Γ imposes independent linear conditions to the linear system $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-5))$ of plane curves of degree $n-5$. Since

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-5)) \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-4)),$$

N imposes independent linear conditions plane curves of degree $n-4$, and, by using Lemma 3.2, we get that $h^0(C, \mathcal{O}_C(1)) = 3$, i.e. Γ is geometrically linearly normal. Now, denote by $\mathcal{I}_{N|\mathbb{P}^2}$ the ideal sheaf of N. Notice that the curve Γ is geometrically 2-normal if and only if the ideal sheaf $\mathcal{I}_{N|\mathbb{P}^2}(n-4)$ is 0-regular, (in the sense of Castelnuovo-Mumford). Indeed, since $h^2(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-6)) = 0$, the ideal sheaf $\mathcal{I}_{N|\mathbb{P}^2}(n-4)$ is 0-regular if and only if $h^1(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-5)) = 0$. Because of the 0-regularity of $\mathcal{I}_{N|\mathbb{P}^2}(n-4)$, we have the surjectivity of the natural map

$$H^0(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-4)) \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-3)),$$

(see [17]). Finally, by the geometric linear normality of Γ, the vertical maps of the following commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-4)) \arrow[r] \arrow[d] & H^0(\mathbb{P}^2, \mathcal{I}_{N|\mathbb{P}^2}(n-3)) \arrow[d] \\ H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \arrow[r, "\mu_{o,C}"] & H^0(C, \omega_C) \end{tikzcd}$$

are surjective and, hence, the Brill-Noether map $\mu_{o,C}$ is surjective too. $\square$

Corollary 4.5

Let $\Sigma \subset \Sigma^n_{k,d}$ be an irreducible component of $\Sigma^n_{k,d}$ of dimension equal to $3n + g - 1 - k$, such that the general point $[\Gamma] \in \Sigma$ corresponds to a geometrically 2-normal plane curve. Then Σ has the expected number of moduli equal to $3g - 3 + \rho - k$.

Proof. It follows from Proposition 4.1 and Lemma 4.4. $\square$

In order to produce examples of families of irreducible plane curves with nodes and cusps with the expected number of moduli, we study how increases the rank of the Brill-Noether map by smoothing a node or a cusp of the general curve of the family, (in the sense of Section 2.1).

Let $\Sigma \subset \Sigma^n_{k,d}$, with $n \geq 5$, be an irreducible component of $\Sigma^n_{k,d}$, let $[\Gamma] \in \Sigma$ be a general point of Σ and let $\phi : C \to \Gamma$ be the normalization of Γ. Choose a singular point $P \in \Gamma$ and denote by $\phi' : C' \to \Gamma$ the partial normalization of Γ obtained by smoothing all singular points of Γ, except the point P. If $\omega_{C'}$ is the dualizing sheaf of C' and

$$\mu_{o,C'} : H^0(C', \mathcal{O}_{C'}(1)) \otimes H^0(C', \omega_{C'}(-1)) \to H^0(C', \omega_{C'}),$$

is the natural multiplication map, we have the following result.

Lemma 4.6

If $h^0(C, \mathcal{O}_C(1)) = 3$ and the geometric genus g of C is such that $g > n - 2$, with $n \geq 5$, then $rk(\mu_{o,C'}) \geq rk(\mu_{o,C}) + 1$. In particular, if $h^0(C, \mathcal{O}_C(1)) = 3$, $n \geq 5$ and $\mu_{o,C}$ is surjective, then $\mu_{o,C'}$ is also surjective.

Proof. Let $\psi : C \to C'$ be the normalization map.

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[dr, "\phi"'] & C' \arrow[d, "\phi'"] \\ & \Gamma \end{tikzcd}$$

We recall that, if we set $\phi^*(P) := p_1 + p_2$ when P is a node and $\phi^*(P) = 2\phi^{-1}(P)$ when P is a cusp, then the dualizing sheaf of C' is a subsheaf of $\psi_*(\omega_C(\phi^*(P)))$, (see for Example [7], p. 80). In particular we have the following exact sequence

$$0 \to \omega_{C'} \to \psi_* \omega_C(\phi^*(P)) \to \mathbb{C}_P \to 0 \tag{13}$$

where $\mathbb{C}_P$ is the skyscraper sheaf on C with support at P. From this exact sequence, we deduce that

$$H^0(C', \omega_{C'}) \simeq H^0(C, \omega_C(\phi^*(P))).$$

Moreover, tensoring (13) by $\mathcal{O}_{C'}(-1)$, we find the exact sequence

$$0 \to \omega_{C'}(-1) \to \psi_* \omega_C(\phi^*(P))(-1) \to \mathbb{C}_P \to 0 \tag{14}$$

from which we get an injective map

$$H^0(C', \omega_{C'}(-1)) \to H^0(C, \omega_C(\phi^*(P))(-1)).$$

On the other hand

$$h^0(C', \omega_{C'}(-1)) = h^0(C, \omega_C(\phi^*(P))(-1)) = g - n + 3 \tag{15}$$

and so

$$H^0(C', \omega_{C'}(-1)) \simeq H^0(C, \omega_C(\phi^*(P))(-1)).$$

Moreover, from the hypothesis $h^0(C, \mathcal{O}_C(1)) = 3$, we have that

$$H^0(C, \mathcal{O}_C(1)) \simeq H^0(C', \mathcal{O}_{C'}(1)) \simeq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)).$$

Therefore, in the following commutative diagram

$$\begin{tikzcd}
H^0(C', \mathcal{O}_{C'}(1)) \otimes H^0(C', \omega_{C'}(-1)) \arrow[r, "\mu_{o,C'}"] \arrow[d] & H^0(C', \omega_{C'}) \arrow[d] \\
H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)(\phi^*(P))) \arrow[r, "\mu'_{o,C}"] & H^0(C, \omega_C(\phi^*(P)))
\end{tikzcd}$$

where we denoted by $\mu'_{o,C}$ the natural multiplication map, the vertical maps are isomorphisms. In particular,

$$rk(\mu_{o,C'}) = rk(\mu'_{o,C}).$$

In order to compute the rank of $\mu'_{o,C}$, we consider the following commutative diagram

$$\begin{tikzcd}
H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \arrow[r, "\mu_{o,C}"] \arrow[d] & H^0(C, \omega_C) \arrow[d, "G"] \\
H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)(\phi^*(P))) \arrow[r, "\mu'_{o,C}"] & H^0(C, \omega_C(\phi^*(P)))
\end{tikzcd}$$

where the vertical maps are injections. Notice that, since we supposed $n \geq 5$, $h^0(C, \mathcal{O}_C(1)) = 3$ and $g > n - 2 \geq 3$, the sheaf $\mathcal{O}_C(1)$ is special. We deduce that C is not hyperelliptic and, chosen a basis of $H^0(C, \omega_C)$, the associated map $C \to \mathbb{P}^{g-1}$ is an embedding. On the contrary, the sheaf $\omega_C(\phi^*(P))$ does not define an embedding on C. Choosing a basis of $H^0(C, \omega_C(\phi^*(P))$ and denoting by $\Phi : C \to \mathbb{P}^g$ the associated map, this will be an embedding outside $\phi^*(P)$. If P is a node of C and $\phi^*(P) = p_1 + p_2$, the image of C to $\mathbb{P}^g$, with respect to Φ, will have a node at the image point Q of p_1 and p_2. If $P \in \Gamma$ is a cusp, then $\Phi(C)$ will have a cusp at the image point

Q of $\phi^{-1}(P)$. The hyperplanes of $\mathbb{P}^g$ passing through Q cut out on C the canonical linear series $|\omega_C|$. Moreover, if we denote by $B \subset \mathbb{P}^g$ the subspace which is the base locus of the hyperplanes of $\mathbb{P}^g$ corresponding to $Im(\mu'_{o,C})$, then $Q \notin B$. Indeed, B intersects the curve C in the image of the base locus of

$$|\mathcal{O}_C(1)| + |\omega_C(\phi^*(P))(-1)| := \mathbb{P}(Im(\mu'_{0,C})),$$

which coincides with the base locus of $|\omega_C(\phi^*(P))(-1)|$, since $|\mathcal{O}_C(1)|$ is base point free. Now, by (15),

$$h^0(\omega_C(\phi^*(P))(-1)) = 3 + g - n = h^0(C, \omega_C(-1)) + 1.$$

Then $\phi^*(P)$ does not belong to the base locus of $|\omega_C(\phi^*(P))(-1)|$, and so

$$\dim(< Q, B >_{\mathbb{P}^g}) = \dim(B) + 1.$$

Finally, we find that

$$\begin{aligned} rk(\mu_{o,C}) = rk(G\mu_{o,C}) &\leq \dim(Im(G) \cap Im(\mu'_{o,C})) \\ &\leq g + 1 - \dim(< B, Q >_{\mathbb{P}^g}) - 1 \\ &= g - 1 - \dim(B) \\ &= rk(\mu'_{o,C}) - 1. \end{aligned}$$

□

Corollary 4.7

Let $\Sigma \subset \Sigma^n_{k,d}$ be a non-empty irreducible component of the expected dimension of $\Sigma^n_{k,d}$, with $n \geq 5$. Suppose that Σ has the expected number of moduli and that the general element $[\Gamma] \in \Sigma$ corresponds to a g.l.n. plane curve Γ of geometric genus g such that, if $C \to \Gamma$ is the normalization of Γ, then the map $\mu_{o,C}$ is surjective. Then, for every $k' \leq k$ and $d' \leq d + k - k'$, there is at least an irreducible component $\Sigma' \subset \Sigma^n_{k',d'}$, such that $\Sigma \subset \Sigma'$, the general element $[D] \in \Sigma'$ corresponds to a g.l.n. plane curve D of geometric genus g' with normalization $D^\nu \to D$ and the Brill-Noether map μ_{0,D^ν} surjective. In particular, Σ' has the expected number of moduli.

Proof. Let Γ be the curve corresponding to the general element $[\Gamma]$ of $\Sigma \subset \Sigma^n_{k,d}$. Since by hypothesis Σ is smooth of the expected dimension at $[\Gamma]$, by Section 2.1, for every $k' \leq k$ and for every $d' \leq d + k - k'$ there exists an irreducible component Σ' of $\Sigma^n_{k',d'}$ containing Σ. In order to prove the statement, it is enough to show it under the hypotheses $k' = k - 1$ and $d' = d + 1$, $k = k'$ and $d' = d - 1$ or $d = d'$ and $k' = k - 1$. If $k' = k - 1$ and $d' = d + 1$, then the statement follows by standard semicontinuity arguments. If $k = k'$ and $d' = d - 1$ or $d = d'$ and $k' = k - 1$, the statement follows by Lemma 4.6 and by standard semicontinuity arguments. □

The following lemma has been stated and proved by Sernesi in [18]. Actually, Sernesi supposes that Γ has only nodes as singularities. But, since his proof works for plane curves Γ with any type of singularities and, since we need it for curves with nodes and cusps, we state the lemma in a more general form.

Lemma 4.8 ([18], Lemma 2.3)

Let Γ be an irreducible and reduced plane curve of degree $n \geq 5$ with any type of singularities. Denote by C the normalization of Γ. Suppose that $h^0(C, \mathcal{O}_C(1)) = 3$ and the Brill-Noether map

$$\mu_{o,C} : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C),$$

has maximal rank. Let R be a general line and let P_1, P_2 and P_3 be three fixed points of $\Gamma \cap R$. We denote by C' the partial normalization of $\Gamma' = \Gamma \cup R$, obtained by smoothing all the singular points, except P_1, P_2 and P_3. Then $h^0(C', \mathcal{O}_{C'}(1)) = 3$ and, denoting by $\omega_{C'}$ the dualizing sheaf of C', the multiplication map

$$\mu_{o,C'} : H^0(C', \mathcal{O}_{C'}(1)) \otimes H^0(C', \omega_{C'}(-1)) \to H^0(C', \omega_{C'}),$$

has maximal rank.

Theorem 4.9

Let $\Sigma^n_{k,d}$ be the algebraic system of irreducible plane curves of degree $n \geq 4$ with k cusps, d nodes and geometric genus $g = \binom{n-1}{2} - k - d$. Suppose that:

$$n - 2 \leq g \ \text{ equivalently } \ k + d \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-4)) \tag{16}$$

and

$$k \leq 6 + \left[\frac{n-8}{3}\right] \ \text{ if } 3n - 9 \leq g \text{ and } n \geq 6, \tag{17}$$

$$k \leq 6 \quad \text{otherwise.} \tag{18}$$

Then $\Sigma^n_{k,d}$ has at least an irreducible component Σ which is not empty and such that, if $\Gamma \subset \mathbb{P}^2$ is the curve corresponding to the general element of Σ and C is the normalization curve of Γ, then $h^0(C, \mathcal{O}_C(1)) = 3$ and the map $\mu_{o,C}$ has maximal rank. In particular, when $\rho \leq 0$, the algebraic system Σ has the expected number of moduli equal to $3g - 3 + \rho - k$.

Proof. *Suppose that* (17) *holds.* Then, by observing that

$$g \geq 3n - 9 \ \text{ if and only if } \ k + d \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-6))$$

and by using Theorem 3.5 for $t = 3$, we have that there exists an irreducible component Σ of $\Sigma^n_{k,d}$ whose general element is a geometrically 3-normal plane curve Γ. By Remark 3.4, it follows that also the linear systems cut out on C by the conics and the lines are complete. The statement follows from Corollary 4.5.

In order to prove the theorem under the hypothesis (18), *we consider the following subcases:*

(1) $2n - 5 \leq g \leq 3n - 9$, i.e. $h^0(\mathcal{O}_{\mathbb{P}^2}(n-6)) \leq k + d \leq h^0(\mathcal{O}_{\mathbb{P}^2}(n-5))$ and $n \geq 5$,

(2) $n - 2 \leq g \leq 2n - 7$ and $n \geq 5$,

(3) $g = 2n - 6$ and $n \geq 4$.

Suppose that (1) holds. By Theorem 3.5 for $t = 2$, we know that, under this hypothesis, there exists a nonempty component $\Sigma \subset \Sigma^n_{k,d}$, whose general element is geometrically 2-normal. We conclude as in the previous case, by Corollary 4.5.

Now, suppose that g and n verify (2). We shall prove the theorem by induction on n and g. Set $g = 2n - 7 - a$, with $a \geq 0$ fixed. Suppose that the theorem is true for the pair (n, g), with $n \geq 7$. We shall prove the theorem for $(n + 1, g + 2)$, observing that $g + 2 = 2(n + 1) - 7 - a$. Let Γ be a g.l.n. irreducible plane curve of degree n and genus $g = 2n - 7 - a$ with $k \leq 6$ cusps, d nodes and no more singularities. Let C be the normalization of Γ. Suppose that the Brill-Noether map $\mu_{o,C}$ has maximal rank. Let $R \subset \mathbb{P}^2$ be a general line and let P_1, P_2 and P_3 be three fixed points of $\Gamma \cap R$. By Section 2.1, since $k \leq 6 < 3n$, one can smooth the singular points P_1, P_2, P_3 and preserve the other singularities of $\Gamma \cup R \subset \mathbb{P}^2$, obtaining a family of plane curves $\mathcal{C} \to \Delta$ whose general fibre is irreducible, has degree $n + 1$ and genus $g + 2$. We conclude by Lemma 4.8 and by standard semicontinuity arguments.

Now we prove the first step of the induction for $n \geq 7$. If $n = 7$, we get $0 \leq a \leq 2$. Let $a = 0$, i.e. $g = 2n - 7 - a = 7$. Let Γ be a g.l.n. irreducible plane curve of degree $n = 7$, of genus $g = n = 7$ with $k \leq 6$ cusps and nodes as singularities, such that no seven singular points of Γ lie on an irreducible conic. To prove that there exists such a plane curve, notice that, by applying Theorem 3.5 for $t = 1$, we get that, for any fixed $k \leq 6$, there exists a g.l.n. irreducible sextic D of genus four with k cusps and $d = 6 - k$ nodes. Let $R_1, \ldots, R_6$ be the singular points of D. Since the points $R_1, \ldots, R_6$ of D impose independent linear conditions to the conics, however we choose five singular points $R_{i_1}, \ldots, R_{i_5}$ of D, with $I = (i_1, \ldots, i_5) \subset (1, \ldots, 6)$, there exists only one conic C_I, passing through these points. Let us set $S = \bigcup_I C_I \cap D$ and let R be a line intersecting D transversally at six points out of S. By Bezout Theorem, no seven singular points of $\Gamma' = D \cup R$ belong to an irreducible conic. Moreover, if $\tilde{D}$ is the normalization of D, if $Q_1, \ldots, Q_4$ are four fixed points of $D \cap R$ and D' is the partial normalization of Γ' obtained by smoothing the singular points except $Q_1, \ldots, Q_4$, then, by the following exact sequence

$$0 \to \mathcal{O}_R(1)(-Q_1 - \cdots - Q_4) \to \mathcal{O}_{D'}(1) \to \mathcal{O}_{\tilde{D}}(1) \to 0 \tag{19}$$

we find that $h^0(D', O_{D'}(1)) = 3$. By Section 2.1, one can smooth the singularities $Q_1, \ldots, Q_4$ and preserve the other singularities of $D \cup R$, getting a family of irreducible septics $\mathcal{G} \to \Delta$ whose general fibre Γ is a geometrically linearly normal irreducible septic with k cusps and $8 - k$ nodes such that no seven singular point of Γ belong to an irreducible conic. Let, now, C be the normalization of Γ and let $\Delta \subset C$ be the adjoint divisor of the normalization map $\phi : C \to \Gamma$. We shall prove that $\ker(\mu_{o,C}) = 0$. Since Γ is geometrically linearly normal, we have that

$$h^0(C, \omega_C(-1)) = h^0(C, \mathcal{O}_C(3)(-\Delta))) = g - n + 2 = 2.$$

Then, by the base point free pencil trick, we find that

$$\ker(\mu_{o,C}) = H^0(C, \omega_C^*(B) \otimes \mathcal{O}_C(2)),$$

where B is the base locus of $|\omega_C(-1) = \mathcal{O}_C(3)(-\Delta)|$. Let $\mathcal{F}$ be the pencil of plane cubics passing through the eight double points $P_1, \ldots, P_8$ of Γ and let $B_{\mathcal{F}}$ be the base locus of the pencil $\mathcal{F}$. Let Γ_3 be the general element of $\mathcal{F}$. Suppose that $B_{\mathcal{F}}$ has dimension one. If $B_{\mathcal{F}}$ contains a line l, then, by Bezout theorem, at most three points among $P_1, \dots, P_8$, say $P_1, \dots, P_3$ can lie on l and the other points have to be contained in the base locus of a pencil of conics $\mathcal{F}'$. Using again Bezout theorem, we find that the curves of $\mathcal{F}'$ are reducible and the base locus of $\mathcal{F}'$ contains a line l'. But also l' contains at most three points of $P_4, \dots, P_6$. It follows that there is only one cubic through $P_1, \dots, P_8$. This is not possible by construction. Suppose that $B_{\mathcal{F}}$ contains an irreducible conic Γ_2. By Bezout theorem, at most seven points among $P_1, \dots, P_8$ may lie on Γ_2. On the other hand, since $\dim(\mathcal{F}) = 1$, there are exactly seven points of $P_1, \dots, P_8$, say $P_1, \dots, P_7$, on Γ_2 and the general cubic Γ_3 of $\mathcal{F}$ is union of Γ_2 and a line passing through P_8. Since, by construction, no seven singular points of Γ lie on an irreducible conic, also in this case we get a contradiction. So the general element Γ_3 of $\mathcal{F}$ is irreducible. Using again Bezout theorem, we find that Γ_3 is smooth and $\mathcal{F}$ has only one more base point Q. We consider the following cases:

a) Q doesn't lie on Γ;

b) Q lies on Γ, but $Q \neq P_1, \dots, P_8$;

c) Q is infinitely near to one of the points $P_1, \dots, P_8$, say $P_{\hat{i}}$, i.e. the cubics of $\mathcal{F}$ have at $P_{\hat{i}}$ the same tangent line l, but l is not contained in the tangent cone to Γ at $P_{\hat{i}}$;

d) Q is like in the case c), but l is contained in the tangent cone to Γ at $P_{\hat{i}}$.

Suppose that the case a) or c) holds. Thus $B = 0$ and

$$\ker(\mu_{o,C}) = H^0(C, \omega_C^* \otimes \mathcal{O}_C(2)) = H^0(C, \mathcal{O}_C(-2)(\Delta)).$$

By Riemann-Roch Theorem, $h^0(C, \mathcal{O}_C(-2)(\Delta)) = h^0(C, \mathcal{O}_C(6)(-2\Delta)) - 4$. One sees that $h^0(C, \mathcal{O}_C(6)(-2\Delta)) = 4$, by blowing-up the plane at $P_1, \dots, P_8$ and by using some standard exact sequences. Suppose now that the case b) holds. Thus $B = Q$ and

$$\dim(\ker(\mu_{o,C})) = h^0(C, \mathcal{O}_C(-2)(\Delta + Q)) = h^0(C, \mathcal{O}_C(6)(-2\Delta - Q)) - 3.$$

Also in this case one sees that $h^0(C, \mathcal{O}_C(6)(-2\Delta - Q)) = 3$ by blowing-up at $P_1, \dots, P_8$ and Q and by using standard exact sequences. Finally, we analyze the case d). Let $\Phi : S \to \mathbb{P}^2$ be the blow-up of the plane at $P_1, \dots, P_8$ with exceptional divisors $E_1, \dots, E_8$. Let $Q \in E_{\hat{i}}$ be the intersection point of $E_{\hat{i}}$ and the strict transform C_3 of the general cubic Γ_3 of the pencil $\mathcal{F}$. We denote by $\tilde{\Phi} : \tilde{S} \to S$ the blow-up of S at Q and by $\Psi : \tilde{S} \to \mathbb{P}^2$ the composition map of the maps Φ and $\tilde{\Phi}$. We still denote by $E_1, \dots, E_8$ their strict transforms on $\tilde{S}$, by C and C_3 the strict transforms of Γ and Γ_3 and by E_Q the new exceptional divisor of $\tilde{S}$. In this case we have that

$$\Psi^*(\Gamma) = C + 2\sum_i E_i + 3E_Q, \ \Psi^*(\Gamma_3) = C_3 + \sum_i E_i + 2E_Q.$$

Moreover, the divisor Δ is cut out on C by $\sum_i E_i + E_Q$ and the base locus B of the linear series $|\omega_C(-1)|$ coincides with the intersection point of E_Q and C. So, we have that

$$\begin{aligned}\dim(\ker(\mu_{o,C})) &= h^0\left(C, \mathcal{O}_C(-2)\Big(\sum_i E_i + 2E_Q\Big)\right)\\ &= h^0\left(C, \mathcal{O}_C(6)\Big(-2\sum_i E_i - 3E_Q\Big)\right) - 3.\end{aligned}$$

Moreover, from the following exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-1) \to \mathcal{O}_{\tilde{S}}(6)\Big(-2\sum_i E_i - 3E_Q\Big) \to \mathcal{O}_C(6)\Big(-2\sum_i E_i - 3E_Q\Big) \to 0$$

we find that

$$H^0\bigg(C, \mathcal{O}_C(6)\Big(-2\sum_i E_i - 3E_Q\Big)\bigg) = H^0\bigg(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)\Big(-2\sum_i E_i - 3E_Q\Big)\bigg).$$

In order to show that

$$h^0\bigg(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)\Big(-2\sum_i E_i - 3E_Q\Big)\bigg) = 3,$$

we consider the following exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_{\tilde{S}}(3)\Big(-\sum_i E_i - E_Q\Big) &\to \mathcal{O}_{\tilde{S}}(6)\Big(-2\sum_i E_i - 3E_Q\Big) \\ &\to \mathcal{O}_{C_3}(6)\Big(-2\sum_i E_i - 3E_Q\Big) \to 0. \end{aligned} \tag{20}$$

By Riemann-Roch Theorem, we have that

$$h^0\bigg(C_3, \mathcal{O}_{C_3}(6)\Big(-2\sum_i E_i - 3E_Q\Big)\bigg) = 1$$

and

$$h^1\bigg(C_3, \mathcal{O}_{C_3}(6)\Big(-2\sum_i E_i - 3E_Q\Big)\bigg) = 0.$$

Moreover, by Serre duality we have that

$$H^1\bigg(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)\Big(-\sum_i E_i - E_Q\Big)\bigg) = H^1\bigg(\tilde{S}, \mathcal{O}_{\tilde{S}}(-6)\Big(2\sum_i E_i + 3E_Q\Big)\bigg).$$

From the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-6)\Big(+2\sum_i E_i + 3E_Q\Big) \to \mathcal{O}_{\tilde{S}}(1) \to \mathcal{O}_C(1) \to 0 \tag{21}$$

by using that the map $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(1)) \to H^0(C, \mathcal{O}_C(1))$ is surjective and that $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(1)) = 0$, we find that

$$H^1\bigg(\tilde{S}, \mathcal{O}_{\tilde{S}}(-6)\Big(+2\sum_i E_i + 3E_Q\Big)\bigg) = H^1\bigg(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)\Big(-\sum_i E_i - E_Q\Big)\bigg) = 0.$$

Then, by (20),

$$h^0\Big(\tilde{S}, \mathcal{O}_{\tilde{S}}(6)\Big(-2\sum_i E_i - 3E_Q\Big)\Big) = h^0\Big(\tilde{S}, \mathcal{O}_{\tilde{S}}(3)\Big(-\sum_i E_i - E_Q\Big)\Big)$$
$$+ h^0\Big(C_3, \mathcal{O}_{C_3}(6)\Big(-2\sum_i E_i - 3E_Q\Big)\Big) = 3$$

and $\ker(\mu_{o,C}) = 0$. *The first step of induction for $g = n = 7$ and $k \leq 6$ is proved.*

We complete the proof of the first step of the induction, for n and g verifying (2). When $n = 7$ and $1 \leq a \leq 2$, the existence of a g.l.n. plane curve Γ follows from Theorem 3.5. Using the above notation, $h^0(C, \omega_C(-1)) = 1$ if $a = 1$ and $h^0(C, \omega_C(-1)) = 0$ if $a = 2$. In any case $\mu_{o,C}$ is injective. When $n \geq 8$ and $a \leq n - 6$ the theorem follows by induction from the case $n = 7$. For $n \geq 8$ and $a = n - 5$, we find that $g = n - 2$, or, equivalently, $k + d = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n - 4))$. In Theorem 3.5, we proved the existence of geometrically linearly normal plane curves of degree $n \geq 8$ and genus $g = n - 2$, with nodes and $k \leq 6$ cusps. For every such plane curve Γ, using the above notation, the Brill-Noether map $\mu_{o,C}$ is injective since $h^0(C, \omega_C(-1)) = 0$. The cases $n = 5$ and $n = 6$ are similar.

Suppose now that n and g verify (3). First of all we prove the theorem for $(n, g) = (4, 2)$, $(5, 4)$, $(6, 6)$. For $n = 4$ and $g = 2$, we find $n = g + 2$ and we argue as in the case $n \geq 8$ and $g = n - 2$. Similarly, for $(n, g) = (5, 4)$. For $n = 6$ and $g = 6$ in Theorem 3.5 we proved the existence of geometrically linearly normal plane curves Γ with $k \leq 4$ cusps and nodes as singularities. For every such a plane curve Γ, denoting by C its normalization, we get that $h^0(C, \omega_C(-1)) = 2$, i.e. the linear system $\mathcal{F}$ of conics passing through the four singular points $P_1, \dots, P_4$ of Γ is a pencil which cuts out on C the complete linear series $|\omega_C(-1)|$. We have two possibilities: either the general element of this pencil is irreducible or it consists of a line containing exactly three singular points P_1, P_2, P_3 of Γ and a line passing through P_4. In any case the base locus of $\mathcal{F}$ intersects Γ only at $P_1, \dots, P_4$ and the linear series $|\omega_C(-1)|$ has no base points. Then, by the base point free pencil trick, we find that $\ker(\mu_{o,C}) = H^0(C, \omega_C^* \otimes \mathcal{O}(2)) = H^0(C, \mathcal{O}_C(-1)(\Delta))$, where $\Delta \subset C$ is the adjoint divisor of the normalization map $C \to \Gamma$. By Riemann-Roch Theorem, we have that $h^0(C, \mathcal{O}_C(-1)(\Delta)) = h^0(C, \mathcal{O}_C(4)(-2\Delta)) - 3$. By blowing-up at $P_1, \dots, P_4$, one can see that $h^0(C, \mathcal{O}_C(4)(-2\Delta)) = 3$, as we wanted.

Finally, we show the theorem under the hypothesis (3) for $n \geq 7$, by using induction on n. In order to prove the inductive step we may use Lemma 4.8, exactly as we did in the case (2). We prove the first step of induction. If $n = 7$ we have that $g = 8$. On pages 337 and 338 we proved the existence of geometrically linearly normal plane curves Γ of degree 7 and genus 7 with $k \leq 6$, such that, if $P_1, \dots, P_8$ are the singular points of Γ, then no seven points among $P_1, \dots, P_8$ lie on a conic. In particular, we proved that, for every such a plane curve Γ, the general element of the pencil of cubics passing through $P_1, \dots, P_8$ is irreducible and, if $\phi : C \to \Gamma$ is the normalization of Γ, then the Brill-Noether map $\mu_{o,C}$ is injective. Let C' be the partial normalization of Γ which we get by smoothing all the singular points of Γ except a node, say P_8. By using the same notation and by arguing exactly as in the proof of Lemma 4.6, we get the following commutative diagram

$$\begin{tikzcd} H^0(C', \mathcal{O}_{C'}(1)) \otimes H^0(C', \omega_{C'}(-1)) \arrow[r, "\mu_{o,C'}"] \arrow[d] & H^0(C', \omega_{C'}) \arrow[d] \\ H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)(\phi^*(P_8))) \arrow[r, "\mu'_{o,C}"] & H^0(C, \omega_C(\phi^*(P_8))) \end{tikzcd}$$

where $\mu'_{o,C}$ is the multiplication map and the vertical maps are isomorphisms. We want to prove that the map $\mu_{o,C'}$ is surjective. By the previous diagram it is enough to prove that $\mu'_{o,C}$ is surjective. Since $h^0(C, \omega_C(\phi^*(P_8))) = 8$ and

$$h^0(C, \mathcal{O}_C(1))h^0(C, \omega_C(-1)(\phi^*(P_8))) = 3(7 - 7 + 3) = 9,$$

we have that $\dim(\ker(\mu_{o,C'})) \geq 1$ and $\mu_{o,C'}$ is surjective if $\dim(\ker(\mu_{o,C'})) = 1$. By recalling that Γ is geometrically linearly normal, we have that, if Z is the scheme of the points $P_1, \dots, P_7$ and $\mathcal{I}_{Z|\mathbb{P}^2}$ is the ideal sheaf of Z in $\mathbb{P}^2$, then in the following commutative diagram

$$\begin{tikzcd} H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)(\phi^*(P_8))) \arrow[r, "\mu'_{o,C}"] \arrow[d] & H^0(C, \omega_C(\phi^*(P_8))) \arrow[d] \\ H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{I}_{Z|\mathbb{P}^2}(3)) \arrow[r, "\mu"] & H^0(\mathbb{P}^2, \mathcal{I}_{Z|\mathbb{P}^2}(4)) \end{tikzcd}$$

the vertical maps are isomorphisms. Hence, it is enough to prove that the kernel of the multiplication map μ has dimension one. Let $\{f_0, f_1, f_2\}$ be a basis of the vector space $H^0(\mathbb{P}^2, \mathcal{I}_{Z|\mathbb{P}^2}(3))$. Since the general cubic passing through $P_1, \dots, P_8$ is irreducible, we may assume that f_0, f_1 and f_2 are irreducible. Suppose, by contradiction, that there exist at least two linearly independent vectors in the kernel of μ. Then, there exist sections u_0, u_1, u_2 and v_0, v_1, v_2 of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ such that the sections $\sum_i u_i \otimes f_i$ and $\sum_i v_i \otimes f_i$ are linearly independent in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathbb{P}^2, \mathcal{I}_{Z|\mathbb{P}^2}(3))$ and

$$\left\{ \begin{array}{rcl} \sum_{i=0}^3 u_i f_i & = & 0 \\ \sum_{i=0}^3 v_i f_i & = & 0. \end{array} \right. \tag{22}$$

We can look at (22) as a linear system in the variables f_0, f_1, f_2. The space of solutions of (22) is generated by the vector

$$(u_1 v_2 - u_2 v_1, u_3 v_0 - u_0 v_3, u_0 v_1 - u_1 v_0).$$

In particular, if we set $q_i = (-1)^{1+i} u_i v_j - v_i u_j$, we find that $f_j q_i = f_i q_j$, for every $i \neq j$. But this is not possible since f_1, f_2 and f_3 are irreducible. We deduce that

$$\dim(\ker(\mu)) = \dim(\ker(\mu_{o,C'})) = 1$$

and $\mu_{o,C'}$ is surjective. The existence of a plane septic of genus 8 with $k \leq 6$ cusps and nodes as singularities, with injective Brill-Neother map, follows now by smoothing the node P_8 (in the sense of Section 2.1) and by standard semicontinuity arguments. $\square$

Remark 4.10 Notice that the conditions which we found in Theorem 4.9 in order that $\Sigma^n_{k,d}$ has at least an irreducible component with the expected number of moduli, are not sharp, even if we suppose $\rho \leq 0$. To see this, notice that in Remark 3.6 we proved the existence of an irreducible component Σ of $\Sigma^{12}_{9,0}$ whose general element corresponds to a 3-normal plane curve. By Remark 3.4 and Corollary 4.5, we have that Σ has the expected number of moduli.

Theorem 4.11

$\Sigma^n_{1,d}$ has the expected number of moduli, for every $d \leq \binom{n-1}{2} - 1$.

Proof. First of all, we recall that, by [16], $\Sigma^n_{1,d}$ is irreducible for every $d \leq \binom{n-1}{2} - 1$. Moreover, from Theorem 4.9 and from Corollary 2.7, we know that $\Sigma^n_{1,d}$ is not empty and it has the expected number of moduli if either $\rho \leq 0$ or $\rho \geq 2$. Next we shall prove that, if $\rho = 1$, then the algebraic system

$$\Sigma^n_{1,d} = \Sigma^n_{1,(n-3)^2/2-1}$$

has general moduli. Equivalently, we will show that, if $[\Gamma] \in \Sigma^n_{1,d}$ is a general point and $g = \binom{n-1}{2} - 1 - d = \frac{3n-7}{2}$, then, on the normalization curve C of Γ there are only finitely many linear series g^2_n with at least a ramification point. Notice that, if $g = \binom{n-1}{2} - 1 - d = \frac{3n-7}{2}$, then n is odd and $n \geq 5$. We prove the statement by induction on n.

If $n = 5$ then $g = 4$. Let $C \subset \mathbb{P}^3$ be the canonical model of a general curve of genus four and let $2P + Q$, with $P \neq Q$ be a divisor in a g^1_3 on C. This divisor is cut out on C by the tangent line to C at P. The projection of C from Q is a plane quintic of genus four with a cusp. This proves that $\Sigma^5_{1,1}$ has general moduli.

Now we suppose that the theorem is true for n and we prove the theorem for $n+2$. Let $\Gamma \subset \mathbb{P}^2$ be the plane curve with a cusp and $\frac{(n-3)^2}{2} - 1$ nodes corresponding to a general point $[\Gamma] \in \Sigma^n_{1,\frac{(n-3)^2}{2}-1}$ and let C_2 be an irreducible conic intersecting Γ transversally. By Section 2.1, the point $[C_2 \cup \Gamma]$ belongs to $\Sigma^{n+2}_{1,\frac{(n+2-3)^2}{2}-1}$. In particular, however we choose four points $P_1, \dots, P_4$ of intersection between Γ and C_2, there exists an analytic branch $\mathcal{S}_{P_1,\dots,P_4}$ of $\Sigma^{n+2}_{1,\frac{(n-1)^2}{2}-1}$, passing through $[C_2 \cup \Gamma]$ and whose general point corresponds to an irreducible plane curve of degree $n+2$ with a cusp in a neighborhood of the cusp of Γ and a node at a neighborhood of every node of $C_2 \cup \Gamma$ different from $P_1, \dots, P_4$. Moreover, $\mathcal{S} := \mathcal{S}_{P_1,\dots,P_4}$ is smooth at the point $[C_2 \cup \Gamma]$, (see [8], Chapter 2). Let

$$\Pi : \Sigma^{n+2}_{1,(n-1)^2/2-1} \dashrightarrow \mathcal{M}_{3(n+2)-7/2}$$

be the moduli map of $\Sigma^{n+2}_{1,\frac{(n-1)^2}{2}-1}$. In order to prove that Π is dominant it is sufficient to show that $\overline{\Pi(\mathcal{S})} = \mathcal{M}_{\frac{3n-1}{2}}$. By Section 2.1, there exist an analytic open sets

$$\mathcal{S}^i \subset \Sigma^{n+2}_{1,(n-3)^2/2-1+2n-i},$$

with $i = 1, 2, 3$, such that

$$\mathcal{S}^0 := \mathcal{S} \cap \left(\mathbb{P}^5 \times \Sigma^n_{1,(n-3)^2/2-1}\right) \subset \mathcal{S}^1 \subset \mathcal{S}^2 \subset \mathcal{S}^3 \subset \mathcal{S}.$$

Every $\mathcal{S}^i$, with $i = 1, 2, 3$, has $\binom{4}{4-i}$ irreducible components, passing through $[C_2 \cup \Gamma]$ and intersecting transversally at $[C_2 \cup \Gamma]$, (see [8], Chapter 2 or [25]). Moreover, the general point of every irreducible component of $\mathcal{S}^i$, with $i = 1, 2, 3$, corresponds to an irreducible plane curve Γ_i of degree $n+2$ with a cusp in a neighborhood of the cusp of Γ, a node in a neighborhood of every node of $C_2 \cup \Gamma$ different from $P_1, \dots, P_4$ and $4-i$ nodes specializing to $4-i$ fixed points among $P_1, \dots, P_4$, as Γ_i specializes to $C_2 \cup \Gamma$. Now, notice that the moduli map Π is not defined at the point $[C_2 \cup \Gamma]$, but, if $\mathcal{S}$ is sufficiently small, then the restriction of Π to $\mathcal{S}$ extends to a regular function on $\mathcal{S}$. More precisely, let $\mathcal{C} \to \Delta$ be any family of curves, parametrized by a projective curve $\Delta \subset \mathcal{S}$, passing through the point $[C_2 \cup \Gamma]$ and whose general point corresponds to an irreducible plane curve of degree $n+2$ of genus $\frac{3n-1}{2} = \frac{3(n+2)-7}{2}$ with a cusp and nodes as singularities. If we denote by $\mathcal{C}' \to \Delta$ the family of curves obtained from $\mathcal{C} \to \Delta$ by normalizing the total space, we have that the general fibre of $\mathcal{C}' \to \Delta$ is a smooth curve of genus $\frac{3n-1}{2}$, corresponding to the normalization of the general fibre of $\mathcal{C} \to \Delta$, whereas the special fibre $\mathcal{C}'_0$ is the partial normalization of $C_2 \cup \Gamma$, obtained by normalizing all the singular points, except $P_1, \dots, P_4$. Then, the map $\Pi_{|\mathcal{S}}$ is defined at $[C_2 \cup \Gamma]$ and it associates to the point $[C_2 \cup \Gamma]$ the isomorphism class of $\mathcal{C}'_0$. Similarly, if $[\Gamma_i]$ is a general point in one of the irreducible components of $\mathcal{S}^i$, with $i = 1, 2, 3$, then $\Pi_{|\mathcal{S}}([\Gamma_i])$ is the partial normalization of Γ_i obtained by smoothing all the singular points except for the $4-i$ nodes of Γ_i tending to $4-i$ fixed points among $P_1, \dots, P_4$ as Γ_i specializes to $C_2 \cup \Gamma$. It follows that, if we denote by $\mathcal{M}^j_{\frac{3n-1}{2}}$ the locus of $\mathcal{M}_{\frac{3n-1}{2}}$ parametrizing j-nodal curves, then $\Pi_{\mathcal{S}}(\mathcal{S}^i) \subseteq \mathcal{M}^{4-i}_{\frac{3n-1}{2}}$, for every $i = 0, \dots, 4$, and $\Pi_{\mathcal{S}}(\mathcal{S}^i) \subsetneq \Pi_{\mathcal{S}}(\mathcal{S}^{i+1})$. In particular, we find that

$$\dim(\Pi_{|\mathcal{S}}(\mathcal{S})) \geq \dim(\Pi_{|\mathcal{S}}(\mathcal{S}^0)) + 4.$$

In order to compute the dimension of $\Pi_{|\mathcal{S}}(\mathcal{S}^0)$ we consider the rational map

$$F : \Pi_{|\mathcal{S}}(\mathcal{S}^0) \dashrightarrow \mathcal{M}_{(3n-7)/2}$$

forgetting the rational tail. By the hypothesis that $\Sigma^n_{1,\frac{(n-3)^2}{2}-1}$ has general moduli and hence F is dominant. Moreover, if C is the normalization curve of Γ, by the generality of $[\Gamma]$ in $\Sigma^n_{1,\frac{(n-3)^2}{2}-1}$, we may assume that C is general in $\mathcal{M}_{\frac{3n-7}{2}}$. We want to show that $\dim(F^{-1}([C])) = 5$. In order to see this, we recall that, by the hypothesis that $\Sigma^n_{1,\frac{(n-3)^2}{2}-1}$ has general moduli, on C there exist only finitely many linear series of degree n and dimension two, mapping C to the plane as curve with a cusp and nodes as singularities. Let g^2_n be one of these linear series, let $\{s_0, s_1, s_2\}$ be a basis of g^2_n and $\phi' : C \to \Gamma' \subset \mathbb{P}^2$ the associated morphism. If $Q_1, \dots, Q_4$ are four general points of Γ', then the linear system of conics through $Q_1, \dots, Q_4$ is a pencil $\mathcal{F}(Q_1, \dots, Q_4)$. Let C_2 and D_2 be two general conics of $\mathcal{F}(Q_1, \dots, Q_4)$. We claim that, if $\eta : \mathbb{P}^1 \to C_2$ and $\beta : \mathbb{P}^1 \to D_2$ are isomorphisms between $\mathbb{P}^1$ and C_2 and D_2 respectively, then the points $\eta^{-1}(Q_1), \dots, \eta^{-1}(Q_4)$ are not projectively equivalent to the points $\beta^{-1}(Q_1), \dots, \beta^{-1}(Q_4)$. In order to prove this, it is enough to prove that there are at least two conics in the pencil $\mathcal{F}(Q_1, \dots, Q_4)$ which verify the claim. Let $D \subset \mathbb{P}^2$ be a conic. If we choose two sets of points $p_1, \dots, p_4$ and $q_1, \dots, q_4$ of

D not projectively equivalent on D, we may always find projective automorphisms $A : \mathbb{P}^2 \to \mathbb{P}^2$ and $A' : \mathbb{P}^2 \to \mathbb{P}^2$ such that $A(p_i) = Q_i$ and $A'(q_i) = (Q_i)$, for every i. By construction, the conics $C_2 = A(D)$ and $D_2 = A'(D)$ belong to the pencil $F(Q_1, \ldots, Q_4)$ and verify the claim. This implies that the partial normalizations C' and D' of $\Gamma' \cup C_2$ and $\Gamma' \cup D_2$, obtained by smoothing all the singular points except $Q_1, \ldots, Q_4$, are not isomorphic. Now, let C_2' be a general conic of $\mathcal{F}(Q_1, \ldots, Q_4)$ and let $R_1, \ldots, R_4$ be four general points of Γ', different from $Q_1, \ldots, Q_4$. If D_2' is a general conic of the pencil $\mathcal{F}(R_1, \ldots, R_4)$, then the partial normalization C' and D' of $\Gamma' \cup C_2'$ and $\Gamma' \cup D_2'$ obtained, respectively, by smoothing all the singular points except $Q_1, \ldots, Q_4$ and $R_1, \ldots, R_4$, are not isomorphic. Indeed, since C is a general curve of genus $\frac{3n-7}{2} \geq 7$, the only automorphism of C is the identity. This proves that $\dim(F^{-1}([C])) = 5$. In particular, we deduce that

$$\dim\left(\Pi_{|\mathcal{S}}(\mathcal{S}^0)\right) = 3\frac{3n-7}{2} - 3 + 5$$

and

$$\dim(\Pi_{|\mathcal{S}}(\mathcal{S}) \geq 3\frac{3n-7}{2} - 3 + 9 = 3\frac{3(n+2)-7}{2} - 3. \qquad \square$$

Remark 4.12 We expect that it is possible to prove that $\Sigma^n_{k,d}$ has expected number of moduli for every ρ also when $k = 2$ or $k = 3$. By Corollary 2.7 and Theorem 4.9, $\Sigma^n_{k,d}$ is not empty, irreducible and it has expected number of moduli for $\rho \leq 0$ and $\rho \geq 2k$. In order to extend Theorem 4.11 to the case $k = 2$ and $k = 3$ one needs to consider a finite number of cases.

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