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Some remarks on the unified characterization of reproducing systems

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Abstract

The affine systems generated by $\Psi \subset L^2(\mathbb{R}^n)$ are the systems

$$\mathcal{A}_A(\Psi) = \{D_A^j\, T_k\, \Psi : j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

where T_k are the translations, and D_A the dilations with respect to an invertible matrix A. As shown in [5], there is a simple characterization for those affine systems that are a Parseval frame for $L^2(\mathbb{R}^n)$. In this paper, we correct an error in the proof of the characterization result from [5], by redefining the class of not-necessarily expanding dilation matrices for which this characterization result holds. In addition, we examine the connection between the eigenvalues of the dilation matrix A and the characterization equations of the affine system $\mathcal{A}_A(\Psi)$ that are Parseval frames. Our observations go in the same directions as other recent results in the literature that show that, when A is not expanding, the information about the eigenvalues alone is not sufficient to characterize or to determine existence of those affine systems that are Parseval frames.

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1. Introduction

Let $A \in GL_n(\mathbb{R})$ and $\Psi = \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^n)$. The **affine systems** generated by Ψ are the systems of the form

$$\mathcal{A}_A(\Psi) = \{D_A^j \, T_k \, \Psi : \, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}, \tag{1.1}$$

where the **translation operators** T_y, $y \in \mathbb{R}^n$, is defined by $T_y \, f(x) = f(x-y)$, and the **dilation operators** D_A, $A \in GL_n(\mathbb{R})$, is defined by $(D_A \, f)(x) = |\det A|^{1/2} \psi(Ax)$, for $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Of particular interest are those functions Ψ for which the system $\mathcal{A}_A(\Psi)$ is an orthonormal (ON) basis or, more generally, a Parseval frame for $L^2(\mathbb{R}^n)$. In particular, Ψ is an **ON wavelet** if the set $\mathcal{A}_A(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ and is a **Parseval frame wavelet** if $\mathcal{A}_A(\Psi)$ is a Parseval frame for $L^2(\mathbb{R}^n)$.

There are relatively simple equations that completely characterize those functions Ψ for which these systems are Parseval frames or ON bases for $L^2(\mathbb{R}^n)$. This problem has been investigated in several papers, including [3, 12, 8, 9, 1, 2, 7] (see [6] for more insight about this problem). The most general result in this direction is obtained in [5]:

Theorem 1.1 ([5])

Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. Then the system $\mathcal{A}_A(\Psi)$, given by (1.1), is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{\ell=1}^{L} \sum_{j \in \mathcal{P}_m} \hat{\psi}^\ell(B^{-j}\xi) \, \overline{\hat{\psi}^\ell(B^{-j}(\xi+m))} = \delta_{m,0} \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{1.2}$$

and all $m \in \mathbb{Z}^n$, where $\mathcal{P}_m = \{j \in \mathbb{Z} : \, B^{-j} m \in \mathbb{Z}^n\}$.

We are now going to recall the definition of *expanding on a subspace*. Before doing this, let us observe that, until very recently, all dilation matrices A considered in the literature were assumed to be **expanding**, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$. As shown in [5, Section 5], this is equivalent to the existence of constants k and γ, satisfying $0 < k \leq 1 < \gamma < \infty$, such that

$$|A^j \, x| \; \geq \; k \, \gamma^j \, |x| \tag{1.3}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \geq 0$, and

$$|A^j \, x| \; \leq \; \frac{1}{k} \, \gamma^j \, |x| \tag{1.4}$$

when $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, $j \leq 0$. The dilation matrices which are considered in [5] are more general than the expanding ones and contain, for example, matrices with eigenvalues of modulus one. Also observe that it was shown in [5] and more recently in [4] that there exist some classes of non-expanding matrices for which very useful wavelets exist.

We recall the following definition from [5]:

Definition 1.2 Given $M \in GL_n(\mathbb{R})$ and a non-zero linear subspace F of $\mathbb{R}^n$, we say that M **is expanding on** F if there exists a complementary (not necessarily orthogonal) linear subspace E of $\mathbb{R}^n$ with the following properties:

(i) $\mathbb{R}^n = F + E$ and $F \cap E = \{0\}$;
(ii) $M(F) = F$ and $M(E) = E$, that is, F and E are invariant under M;
(iii) conditions (1.3) and (1.4) hold for all $x \in F$;
(iv) given $r \in \mathbb{N}$, there exists $C = C(M, r)$ such that, for all $j \in \mathbb{Z}$, the set

$$\mathcal{Z}_r^j(E) = \{m \in E \cap \mathbb{Z}^n : |M^j\, m| < r\}$$

has less than C elements.

Unfortunately, there is a mistake in an argument of Lemma 5.11 in [5], so that Theorem 1.1 cannot be deduced under the assumption that $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. This mistake was pointed out by X. Yu, from McMaster University. As we will show in this paper, there is a slight change in Definition 1.2, that will allow us to correct the argument in Lemma 5.11 from [5] and so prove Theorem 1.1.

The paper is organized as follows. In Section 2 we construct an example to show that Lemma 5.11 from [5] is false. In Section 3 we modify the definition of *expanding on a subspace*, so that Theorem 1.1 can be obtained. In Section 4 we show several examples of non-trivial matrices that are expanding on a subspace. Finally, in Section 5, we examine in more detail the relationship between eigenvalues of a dilation matrix and characterization equations.

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2. Counterexample

We begin by showing that Lemma 5.11 from [5] is false by producing a counterexample. We need to introduce some notation: Given two complementary linear subspaces of $\mathbb{R}^n$, according to Definition 1.2, for any $x \in \mathbb{R}^n$, there exist unique $x_F \in F$ and $x_E \in E$ such that $x = x_F + x_E$. For $r \in \mathbb{R}$, define

$$\widetilde{Q}(r) = \{x = x_F + x_E : x_F \in F,\, x_E \in E,\, |x_F| < r,\, |x_E| < r\}.$$

We recall Lemma 5.11 from [5]:

Lemma 2.1 ([5])

Let $M \in GL_n(\mathbb{R})$ be expanding on a subspace F of $\mathbb{R}^n$, $r \in \mathbb{R}$, and E be a complementary subspace of F as in Definition 1.2. There exists $C = C(M, r) \in \mathbb{R}$ such that

$$\#\{m \in \mathbb{Z}^n \setminus E : M^j m \in \widetilde{Q}(r)\} \le C|\det M|^{-j}$$

for all $j \in \mathbb{Z}$.

Consider the matrix

$$M = \begin{pmatrix} 2/3 & 0 \\ \sqrt{2} & 2 \end{pmatrix}.$$

Let F be the eigenspace corresponding to the eigenvalue 2, that is

$$F = \left\{ \begin{pmatrix} 0 \\ t \end{pmatrix} : t \in \mathbb{R} \right\},$$

and E be the eigenspace corresponding to the eigenvalue $2/3$, that is

$$E = \left\{ \begin{pmatrix} \frac{-4}{(3\sqrt{2})}\, t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}.$$

It is clear that that the complementary subspaces F and E are invariant subspaces, and that conditions (1.3) and (1.4) hold for all $x \in F$. In addition, since $\mathbb{Z}^2 \bigcap E = \{0\}$, condition (iv) in Definition 1.2 is also satisfied. Therefore, M is expanding on F, in the sense of Definition 1.2.

Now let $m = \begin{pmatrix} 0 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2$. We have that $m \in F$, and, thus, $m \in \mathbb{Z}^2 \setminus E$. Observe that $M^j\, m = \begin{pmatrix} 0 \\ 2^j\, m_2 \end{pmatrix}$, and, thus, $|M^j\, m| = 2^j\, |m_2|$. Therefore, if $j < 0$, there exists $C = C(M, r) \in \mathbb{R}$

$$\#\left\{ m = \begin{pmatrix} 0 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2 : M^j m \in \widetilde{Q}(r) \right\} = \#\{m_2 \in \mathbb{Z} : 2^j\, |m_2| < r\} \geq C\, 2^{-j}. \tag{2.5}$$

On the other hand, since $\det M = 4/3$, by Lemma 2.1 we should have that

$$\#\{m \in \mathbb{Z}^2 \setminus E : M^j m \in \widetilde{Q}(r)\} \leq C\, (4/3)^{-j} \tag{2.6}$$

for all $j \in \mathbb{Z}$. This shows that the statement of Lemma 2.1 is false.

3. New definition

As we have shown, Lemma 5.11 from [5] does not hold. We will now propose a new definition of the notion of *expanding on a subspace* and will prove Theorem 1.1 using this new definition.

Observe that, by condition (i) in Definition 1.2, given any $x \in \mathbb{R}^n$, there exist unique $x_E \in E$ and $x_F \in F$ such that $x = x_E + x_F$. Let us replace condition (iv) in Definition 1.2 with the new condition:

(iv') *For any* $j \geq 0$, *there exists* $k_1 = k_1(M) > 0$ *such that,* $|x_E| \leq k_1\, |M^j\, x_E|$.

Remark. Using the change of variables $y_E = M^j\, x_E$, the last inequality gives $|M^{-j}\, y_E| \leq k_1\, |y_E|$. Thus, condition (iv') is equivalent to:

(iv') *For any* $j \leq 0$, *there exists* $k_1 = k_1(M) > 0$ *such that,* $|M^j\, x_E| \leq k_1\, |x_E|$.

Remark. If M is expanding on a subspace $F \subset \mathbb{R}^n$, according to Definition (1.2)', then all eigenvalues λ of M satisfy $|\lambda| \geq 1$. To show that this is the case, suppose that

$M\,u = \alpha\,u$, for $u \neq 0$, with $|\alpha| < 1$. Observe that $M^j u = M^j u_E + M^j u_F$, where $M^j u_F \in F$ and $M^j u_E \in E$. We also have that $M^j u = \alpha^j u = \alpha^j u_F + \alpha^j u_E$, and thus, by the uniqueness on the decomposition into the E and F subspaces, $M^j u_F = \alpha^j u_F$ and $M^j u_E = \alpha^j u_E$. Since we assume that $|\alpha| < 1$, then $|M^j u_F| = |\alpha^j u_F| \to 0$, for $j \to \infty$ and $|M^j u_E| = |\alpha^j u_E| \to 0$, for $j \to \infty$. On the other hand, since M is expanding on F, it follows that $|M^j u_F| \geq k\gamma^j |u_F|$, with $k > 0, \gamma > 1$, and, by (iv') in Definition (1.2)', $|M^j\, u_E| \geq k_1^{-1}|u_E|$. This is a contradiction, since $u \neq 0$. Therefore one cannot have any eigenvalue $\lambda = \alpha$ with $|\alpha| < 1$.

Observe, in particular, that the matrix $M = \begin{pmatrix} 2/3 & 0 \\ \sqrt{2} & 2 \end{pmatrix}$, used in Section 2 as a counterexample for Lemma 5.11 from [5], is not expanding on a subspace according with Definition (1.2)' (while it is expanding on a subspace according with Definition (1.2)).

We also observe that, if all eigenvalues λ of M satisfy $|\lambda| \geq 1$, this *does not* imply that M is expanding on a subspace (cf. Example 4 in Section 4).

A matrix M satisfying conditions (i)–(iii) from Definition (1.2) and the new condition (iv)' will still be called *expanding on* F. We will refer to such new definition as Definition (1.2)', to distinguish it from the previous one.

The goal is to prove the following proposition:

Proposition 3.1

Let $\Psi = \{\psi^1, \cdots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and $A \in GL_n(\mathbb{R})$ be such that the matrix $B = A^t$ is expanding on a subspace F of $\mathbb{R}^n$. If

$$\sum_{\ell=1}^{L} \sum_{j\geq 0} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \leq C \qquad \text{for a.e.} \quad \xi \in \mathbb{R}^n \tag{3.7}$$

for some $C > 0$, then there is a dense subspace $\mathcal{D} \subset L^2(\mathbb{R}^n)$ such that

$$L(f) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{\mathrm{supp}\hat{f}} |\hat{f}(\xi + B^j m)|^2\, |\hat{\psi}^\ell(B^{-j}\xi)|^2\, d\xi < \infty \tag{3.8}$$

for all $f \in \mathcal{D}$.

If Proposition 3.1 holds, then we say that A satisfies the **Local Integrability Condition (LIC)**. This condition plays a critical role in the proof of Theorem 1.1 (see [5] for details).

In order to prove this proposition, we need to introduce some new notation and two lemmas. For $r, s \in \mathbb{R}$, $r > 1, s > 0$, define

$$Q(r,s) = \Big\{x = x_F + x_E : x_F \in F,\ x_E \in E,\ \frac{1}{r} < |x_F| < r,\ |x_E| < s\Big\}, \tag{3.9}$$

and

$$\widetilde{Q}(r,s) = \{x = x_F + x_E : x_F \in F,\, x_E \in E,\, |x_F| < r,\, |x_E| < s\}.$$

In addition, we write $Q(r) = Q(r,r)$, and $\widetilde{Q}(r) = \widetilde{Q}(r,r)$.

The following lemma is proved in [5, Lemma 5.10] (observe that its proof only uses (i), (ii) and (iii) from Definition 1.2).

Lemma 3.2

Let $M \in GL_n(\mathbb{R})$ be expanding on a subspace F of $\mathbb{R}^n$, and $r \in \mathbb{R}$. There exists $N = N(M, r) \in \mathbb{N}$ such that

$$\#\{j \in \mathbb{Z} : M^j \eta \in Q(r)\} \leq N \quad \text{for all } \eta \in \mathbb{R}^n. \tag{3.10}$$

The following lemma is adapted from Lemma 5.11 in [5]:

Lemma 3.3

Let $M \in GL_n(\mathbb{R})$ be expanding on a subspace F of $\mathbb{R}^n$, $r \in \mathbb{R}$, and E be a complementary subspace of F as in Definition (1.2)'. There exist $C = C(M, r) > 0$ and $\widetilde{C} = \widetilde{C}(M, r) > 0$ such that

$$\#\{m \in \mathbb{Z}^n : M^j m \in \widetilde{Q}(r)\} \leq C \quad \text{for all } j \geq 0, \tag{3.11}$$

and

$$\#\{m \in \mathbb{Z}^n : M^j m \in \widetilde{Q}(r)\} \leq \widetilde{C}\,|\det M|^{-j} \quad \text{for all } j < 0. \tag{3.12}$$

Proof. Assume $j \geq 0$. For $m \in \mathbb{Z}^n$, write $m = m_F + m_E$ with $m_F \in F$, $m_E \in E$. If $M^j m \in \widetilde{Q}(r)$, then $|M^j m_F| < r$, $|M^j m_E| < r$. By condition (iv)' in Definition (1.2)', $|m_E| < k_1\,|M^j m_E| \leq k_1 r$, for some $k_1 = k_1(M)$. Also, since M is expanding on F, it follows that $|m_F| < r_2$, for some $r_2 = r_2(M, r)$. Thus, for all $j \geq 0$,

$$\{m \in \mathbb{Z}^n : M^j m \in \widetilde{Q}(r)\} \subset \{m \in \mathbb{Z}^n : m \in \widetilde{Q}(r, k_1 r_2)\}.$$

Since the last quantity is a finite set, then one can find $C = C(M, r) > 0$ such that (3.11) holds.

Consider now the case $j < 0$. For $\xi \in [0, 1)^n$, write $\xi = \xi_F + \xi_E$ with $\xi_F \in F$ and $\xi_E \in E$. For any $m \in \mathbb{Z}^n$ with $M^j m \in \widetilde{Q}(r)$, since M is expanding on F, we have:

$$|M^j(m_F + \xi_F)| \leq |M^j(m_F)| + |M^j(\xi_F)| \leq r + \frac{1}{k}|\xi_F| \leq r + \frac{1}{k}S_1 \equiv R_1$$

where $S_1 = \sup\{|\xi_F| : \xi \in [0, 1)^n\}$. Also, by condition (iv)' in Definition (1.2)', there is a $k_1 = k_1(M)$ such that

$$|M^j(m_E + \xi_E)| \leq |M^j(m_E)| + |M^j(\xi_E)| \leq r + k_1|\xi_E| \leq r + k_1 S_2 \equiv R_2$$

where $S_2 = \sup\{|\xi_E| : \xi \in [0, 1)^n\}$.

We have just shown that

$$\{m \in \mathbb{Z}^n : M^j m \in \widetilde{Q}(r)\} \subset \{m \in \mathbb{Z}^n : m + [0, 1)^n \subset M^{-j}(\widetilde{Q}(R_1, R_2))\} \equiv \mathcal{M}^j_{R_1, R_2}.$$

Since the sets $m + [0, 1)^n$, $m \in \mathbb{Z}^n$, are disjoint,

$$\begin{aligned}\#\{m \in \mathbb{Z}^n : M^j m \in \tilde{Q}(r)\} \leq \#\mathcal{M}^j_{R_1, R_2} &= \Big|\bigcup_{m \in \mathcal{M}^j_{R_1, R_2}} (m + [0, 1)^n)\Big| \\ &\leq \big|M^{-j}(\widetilde{Q}(R_1, R_2))\big| = |\widetilde{Q}(R_1, R_2)|\,|\det M|^{-j}\end{aligned} \tag{3.13}$$

Equation (3.12) then follows from (3.13) by taking $\widetilde{C} = |\widetilde{Q}(R_1, R_2)|$. □

We can now prove Proposition 3.1

Proof of Proposition 3.1 Let $f \in \mathcal{D}_E$, where $\mathcal{D}_E$ is a dense subspace of $L^2(\mathbb{R}^n)$, defined by

$$\mathcal{D}_E = \{f \in L^2(\mathbb{R}^n) \colon \hat{f} \in L^\infty(\mathbb{R}^n) \text{ and } \operatorname{supp} \hat{f} \text{ is compact in } \mathbb{R}^n \setminus E\}.$$

Choose $r \in \mathbb{N}$ such that $\operatorname{supp} \hat{f} \subset Q(r)$. Then

$$L(f) \leq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}^n} \int_{Q(r)} |\hat{f}(\xi + B^j m)|^2 \, |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi. \tag{3.14}$$

For $m \in \mathbb{Z}^n$, if $\xi \in Q(r)$ and $\xi + B^j m \in Q(r)$, then, for $j \in \mathbb{Z}$, we have

$$|B^j m| \leq |\xi_E + B^j m| + |\xi_E| < r + r = 2r\,,$$

where $\xi = \xi_F + \xi_E$, with $\xi_F \in F$ and $\xi_E \in E$. Thus, we have:

$$\{m \in \mathbb{Z}^n : \xi \in Q(r) \text{ and } \xi + B^j m \in Q(r)\} \subset \{m \in \mathbb{Z}^n : B^j m \in Q(2r)\} = \mathcal{Z}^j_{2r} \tag{3.15}$$

for every $j \in \mathbb{Z}$, and

$$L(f) \leq \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \#(\mathcal{Z}^j_{2r}) \int_{Q(r)} \|\hat{f}\|^2_\infty \, |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi. \tag{3.16}$$

We write $L^+(f)$ for the sum of the terms in (3.14) for which $j \geq 0$, and $L^-(f)$ for the sum of the terms in the same expression for which $j < 0$. Then, $L(f) = L^+(f) + L^-(f)$.

We first estimate $L^+(f)$. By (3.11) in Lemma 3.3, $\#(\mathcal{Z}^j_{2r}) \leq C(A, 2r)$, and, thus, from (3.16) we obtain

$$L^+(f) \leq C(A, 2r) \, \|\hat{f}\|^2_\infty \sum_{\ell=1}^{L} \sum_{j\geq 0} \int_{Q(r)} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, d\xi.$$

Using (3.7), it follows that

$$L^+(f) \leq C(A, 2r) \|\hat{f}\|^2_\infty |Q(r)| < \infty. \tag{3.17}$$

We now estimate $L^-(f)$. Using (3.12) from Lemma 3.3 in (3.16) and then the change of variables $\eta = B^{-j}\xi$ we obtain:

$$\begin{aligned} L^-(f) &\leq \widetilde{C}(A, 2r) \, \|\hat{f}\|^2_\infty \sum_{\ell=1}^{L} \sum_{j<0} \int_{Q(r)} |\hat{\psi}^\ell(B^{-j}\xi)|^2 \, |\det B|^{-j} \, d\xi \\ &= \widetilde{C}(A, 2r) \|\hat{f}\|^2_\infty \sum_{\ell=1}^{L} \sum_{j<0} \int_{B^j\eta\in Q(r)} |\hat{\psi}^\ell(\eta)|^2 \, d\eta. \end{aligned}$$

By Lemma 3.2, the number of $j \in \mathbb{Z}$ such that $B^j\eta \in Q(r)$ does not exceed a fixed number, $N(B,r)$, independently of $\eta \in \mathbb{R}^n$. Hence,

$$L_1(f) \leq \widetilde{C}(B,2r)\|\hat{f}\|_\infty^2 \, N(B,r) \sum_{\ell=1}^{L} \|\hat{\psi}^\ell\|_2^2 < \infty. \tag{3.18}$$

Therefore, from (3.17), and (3.18) we deduce that, if $f \in \mathcal{D}_E$, then $L(f) < \infty$. □

4. Examples

We describe in the following a number of examples of matrices $M \in GL_n(\mathbb{R})$ that satisfy Definition (1.2)'.

The first observation is that if M is expanding, then it is also expanding on a subspace.

EXAMPLE 1 When M is an expanding matrix, Definition (1.2)', is satisfied with $F = \mathbb{R}^n$ and $E = \{0\}$.

The following examples show that there are matrices $M \in GL_n(\mathbb{R})$ that are expanding on a subspace, but are not expanding.

EXAMPLE 2 For $a \in \mathbb{R}$, $|a| > 1$, the matrix

$$M = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

has eigenvalues a and 1. Letting F be the eigenspace corresponding to the eigenvalue a, and E the eigenspace corresponding to the eigenvalue 1, it is clear that M is expanding on F, in the sense of Definition (1.2)'. It is easy to obtain analogous, higher dimensional, diagonal matrices, even allowing some of the elements of the diagonal to be -1, that satisfy "expanding on F".

EXAMPLE 3 For $a \in \mathbb{R}$, $|a| > 1$, and $\theta \in \mathbb{R}$, consider the matrix

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

which corresponds to a dilation on the X–axis and a rotation around the origin in the YZ–plane. The matrix M is expanding on $F = \mathbb{R} \times \{0\} \times \{0\}$, with $E = \{0\} \times \mathbb{R} \times \mathbb{R}$.

The following example is not expanding on a subspace, but we will show that Theorem 1.1 is still valid in this case.

EXAMPLE 4 For $a \in \mathbb{R}$, $|a| > 1$, consider

$$M = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

With $F = \mathbb{R} \times \{0\} \times \{0\}$, and $E = \{0\} \times \mathbb{R} \times \mathbb{R}$, properties (i), (ii), and (iii) of Definition (1.2)' are obvious. However, condition (iv)' fails. In fact, for $x_E = (0, x, y) \in E$, one finds that $M^j x_E = (0, x + jy, y)$, and $|M^j x_E|$ diverges for $j \to \infty$ (this violates (iv)').

On the other hand, one can show that Theorem 1.1 is still valid in this case. In fact, we will show that one can prove Proposition 3.1 in this case and, as a consequence, prove Theorem 1.1. Observe that, in order to prove Proposition 3.1, it is sufficient to prove condition (3.10) from Lemma 3.2 and conditions (3.11) and (3.12) from Lemma 3.3, since these conditions imply the LIC (eq. (3.8)). Our first observation is that condition (3.10) is still valid in this case, since Lemma 3.2 does not depend on condition (iv)' of Definition (1.2)'. To prove (3.11) and (3.12) we need the following argument. Let

$$m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

and assume that $M^j m \in \widetilde{Q}(r)$. Since

$$M^j m = \begin{pmatrix} a^j & 0 & 0 \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} m,$$

it follows that

$$|a^j m_1| < r \quad \text{and} \quad \left| \begin{pmatrix} m_2 + j\, m_3 \\ m_3 \end{pmatrix} \right| < r.$$

This shows that, if $j \geq 0$, then $\#\{m_1 \in \mathbb{Z} : |a^j m_1| < r\} \leq N_1$, for some $N_1 = N_1(r)$; if $j < 0$, then $\#\{m_1 \in \mathbb{Z} : |a^j m_1| < r\} \leq a^{-j}\, N_1$. Also, since

$$\left| \begin{pmatrix} m_2 + j\, m_3 \\ m_3 \end{pmatrix} \right| \simeq |m_2 + j\, m_3| + |m_3|,$$

then $|m_2 + j\, m_3| < Cr$ and $|m_3| < Cr$, for some $C > 0$. This shows that $\#\{m_3 \in \mathbb{Z} : |m_3| < r\} \leq N_2$, for some $N_2 = N_2(Cr)$, and that $\#\{m_2 \in \mathbb{Z} : |m_2 + j\, m_3| < r\} \leq N_3$, for some $N_3 = N_3(Cr)$ (in fact, for each j and m_3 fixed, the number of such m_2 is independent of j, m_3). Combining these observations, we obtain that there are constants $N = N(M, r), \tilde{N} = \tilde{n}(M, r)$ such that

$$\#\{m \in \mathbb{Z}^3 : M^j m \in \widetilde{Q}(r)\} \leq N, \qquad \text{for } j \geq 0$$

$$\#\{m \in \mathbb{Z}^3 : M^j m \in \widetilde{Q}(r)\} \leq \tilde{N}\, |\det M|^{-j}, \qquad \text{for } j \geq 0.$$

This proves the conditions (3.11) and (3.12), and, as a consequence, shows that Theorem 1.1 holds in this case.

Example 4 might suggest to the reader that Theorem 1.1 holds whenever the dilation matrix M has all eigenvalues $|\lambda_k| \geq 1$ and at least one eigenvalue $|\lambda_1| > 1$. However, this is not the case. In the next section we show that there are examples of dilation matrices having the same eigenvalues as in Example 4, namely $\lambda_1 = a > 1$

and $\lambda_2 = \lambda_3 = 1$, for which the LIC fails. This shows that the information about the eigenvalues of M alone is not sufficient to determine the LIC. This suggests that the information about the eigenvalues alone is not sufficient to determine the characterization equation (1.2) for affine systems with non-expanding dilation matrices. A similar observation concerning the role of the eigenvalues in the existence of wavelets with non-expanding dilations can be found in [11], where there are examples of affine systems having dilation matrices with the same eigenvalues for which wavelets do or do not exist.

5. Some observations about the role of eigenvalues

Let $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. As we observed in the previous section, Theorem 1.1 is satisfied for this choice of dilation matrix. In this section, we show that if B is replaced by $M = N^{-1}BN$, with $N \in GL_3(\mathbb{R})$, then there are choices of N for which the LIC (defined after formula (3.8)) fails, and, as a consequence, Theorem 1.1 cannot be proved.

Theorem 5.1

Let $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix}$, *with* $0 < a < 1$ *and* $\frac{a^2}{a-1} \in \mathbb{R} \setminus \mathbb{Q}$. *Then the LIC fails for*

$$M = N^{-1}BN = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+a & 1 \\ 0 & -a^2 & 1-a \end{pmatrix}. \tag{5.19}$$

In order to prove this theorem, we need some lemmas.

Lemma 5.2

Let $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Given* $\delta > 0$ *and a lattice* $\Lambda = P\,\mathbb{Z}^2$, *where* $P = \begin{pmatrix} 1+a & 1 \\ -a^2 & 1-a \end{pmatrix}$, *with* $0 < a < 1$ *and* $\frac{a^2}{a-1} \in \mathbb{R} \setminus \mathbb{Q}$, *there exist* $j = j(\Lambda) \in \mathbb{Z}^+$ *and* $\lambda \in \Lambda \setminus \{0\}$ *such that* $|b^{-j}\lambda| \leq \delta$.

Proof. For $m, n \in \mathbb{Z}$, a direct calculation gives that

$$b^{-j}\,P \begin{pmatrix} m \\ -n \end{pmatrix} = \begin{pmatrix} \alpha - j\beta \\ \beta \end{pmatrix},$$

where $\alpha = (1-a)m - n$ and $\beta = -a^2 m - (1-a)n$. By the properties of infinite continued fractions [10, Section 12.3], we can find infinitely many $m, n \in \mathbb{Z}$ with $\gcd(\mathrm{m}, \mathrm{n}) = 1$ such that $m < 0$, $n > 0$, $\frac{n}{m} < \frac{a^2}{a-1}$ and

$$0 \neq \left| \frac{n}{m} - \frac{a^2}{a-1} \right| < \frac{1}{m^2}. \tag{5.20}$$

Observe that, since $m < 0$, $n > 0$, then $\alpha < 0$, and since $\frac{n}{m} < \frac{a^2}{a-1}$, then also $\beta < 0$. The inequalities (5.20) imply that

$$0 \neq |(a-1)n - a^2 m| < \frac{1-a}{m}.$$

By choosing m large enough, so that $\frac{1}{m} < \frac{\delta}{\sqrt{2}(1-a)}$, we have that

$$0 \neq |\beta| = |-a^2 n - (1-a)m| < \frac{\delta}{\sqrt{2}}.$$

Since $\beta \neq 0$, with m, n chosen as above, we can find $j \in \mathbb{Z}^+$ such that

$$j\,\beta \leq \alpha < (j-1)\,\beta.$$

From this expression we obtain that

$$0 \leq \alpha - j\,\beta < -\beta < \delta/\sqrt{2}.$$

Therefore, we have that

$$\left|\begin{pmatrix} \alpha - j\beta \\ \beta \end{pmatrix}\right| < \sqrt{\delta^2/2 + \delta^2/2} = \delta,$$

and, thus:

$$\left| b^{-j}\, P \begin{pmatrix} m \\ -n \end{pmatrix} \right| < \delta,$$

for some $j \in \mathbb{Z}^+$. This completes the proof with $\lambda = P \begin{pmatrix} m \\ -n \end{pmatrix}$. □

Observe that the argument of the above lemma goes through if P is replaced by $k\,P$, for $k \in \mathbb{N}$.

Lemma 5.3

Let $A = P^{-1} b P$, where P and b are as in Lemma 5.2, and let

$$C_j(A) = \#\Big\{ m \in \mathbb{Z}^2 : |A^j m| \leq \frac{1}{4} \Big\}.$$

Then $\sup_{j \in \mathbb{Z}^+} C_{-j}(A) = \infty$.

Proof. Since P is invertible, there is a $\delta > 0$ such that $B(0,\delta) \subset P\,B(0,1/4)$. Thus:

$$C_{-j}(A) \geq \#\{\lambda \in P\mathbb{Z}^2 : |b^{-j}\lambda| \leq \delta\}.$$

Now choose $k \in \mathbb{N}$, and apply Lemma 5.2, with the lattice $\Lambda = k\,P\mathbb{Z}^2$. Then there is a $j = j(k) \in \mathbb{Z}^+$ and a $\lambda \in \Lambda \setminus \{0\}$ such that $|b^{-j}\lambda| < \delta$. Let $\lambda_l = \frac{l}{k}\,\lambda$, for $l = 1, 2, \ldots, k$. It follows that $\lambda_l \in \Lambda$ for each l, and $\lambda_{l_1} \neq \lambda_{l_2}$ if $l_1 \neq l_2$. Moreover:

$$|b^{-j}\lambda_l| = \frac{l}{k}\,|b^{-j}\lambda| \leq \delta, \quad l = 1, 2, \ldots, k.$$

This shows that, for any given $k \in \mathbb{N}$, there is a $j(k)$ such that $C_{-j(k)}(A) \geq k$, which implies that $\sup_{j \in \mathbb{Z}^+} C_{-j}(A) = \infty$. □

From Lemma 5.3 the following observation follows easily.

Lemma 5.4

Let $M = \begin{pmatrix} 2 & 0 \\ 0 & A \end{pmatrix}$, *where* A *is given in Lemma 5.3. Then, for* $j \in \mathbb{Z}^+$, *we have*

$$K_{-j}(M) \geq 2^{j-1}\, C_{-j}(A),$$

where

$$K_j(M) = \#\Big\{m \in \mathbb{Z}^3 : |M^j m| \leq \frac{1}{2}\Big\}.$$

We can now prove Theorem 5.1

Proof of Theorem 5.1 Write the matrix M, given by (5.19) as in Lemma 5.4. Let $Q = [1,2) \times [1,2)^2$, $T = [\frac{1}{2}, \frac{5}{2}) \times [\frac{1}{2}, \frac{5}{2})^2$, and $E = \cup_{j=1}^{\infty} E_j$, where $E_j = [2^{j+1} - 1, 2^{j+1} - \frac{1}{2}) \times A^j[1,2)^2$. Observe that, for all $j \in \mathbb{Z}^+$, $E_j \subset M^j Q$, $E_j \cap E_{j'} = \varnothing$, if $j \neq j'$, and that, for any $i \in \mathbb{Z}$, $M^i E \cap E = \varnothing$. By Lemma 5.3, $\sup_{j \in \mathbb{Z}^+} C_{-j}(A) = \infty$, and, thus, there is a subsequence $\{C_{-j_k}(A)\} \subset \{C_{-j}(A)\}$, with $j_1 \leq j_2 \leq \dots$ and $C_{-j_k}(A) \geq 1$, such that

$$\sum_{k=1}^{\infty} \frac{1}{C_{-j_k}(A)} < \infty. \tag{5.21}$$

Define $\hat{\psi}(\xi)$ by $\hat{\psi}(\xi) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{C_{-j_k}(A)}} \chi_{E_{j_k}}(\xi)$. Then, by (5.21), we have that

$$\int_{\mathbb{R}^3} |\hat{\psi}(\xi)|^2\, d\xi = \sum_{k=1}^{\infty} \frac{1}{2} \frac{1}{C_{-j_k}(A)} < \infty,$$

and, since $M^i E \cap E = \varnothing$,

$$\sum_{i \in \mathbb{Z}} |\hat{\psi}(M^i \xi)|^2 \leq 1.$$

This shows that the assumptions of Proposition 3.1 are satisfied and, thus, in order to show that the LIC fails, we only need to show that $L(f)$, given by (3.8) is unbounded, for our special choice of M. Let $\hat{f} = \chi_T$. Then, using Lemma 5.4, we have

$$\begin{aligned} L(f) &\geq \sum_{j \in \mathbb{Z}^-} \sum_{m \in \mathbb{Z}^3} \int_{\operatorname{supp} \hat{f}} |\hat{f}(\xi + M^j m)|^2\, |\hat{\psi}(M^{-j}\xi)|^2\, d\xi \\ &\geq \sum_{j \geq 0} c\, 2^j C_{-j}(A) \int_Q |\hat{\psi}(M^j \xi)|^2\, d\xi, \end{aligned}$$

for some $c > 0$. Using the change of variables $\eta = M^j \xi$ (notice that $\det M^j = 2^j$), we have

$$\begin{aligned} L(f) &\geq c \sum_{j \geq 0} C_{-j}(A) \int_{M^j Q} |\hat{\psi}(\eta)|^2\, d\eta \\ &\geq c \sum_{k=1}^{\infty} C_{-j_k}(A) \int_{M^{j_k} Q} |\hat{\psi}(\eta)|^2\, d\eta \\ &\geq c \sum_{k=1}^{\infty} C_{-j_k}(A) \int_{E_{j_k}} |\hat{\psi}(\eta)|^2\, d\eta \\ &= c \sum_{k=1}^{\infty} C_{-j_k}(A) \frac{1}{C_{-j_k}(A)} = \infty. \end{aligned}$$

□

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