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Ultrasymmetric sequence spaces in approximation theory

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Abstract

Let $\varphi(t)$ be a positive increasing function and let $\widetilde{E}$ be an arbitrary sequence space, rearrangement-invariant with respect to the atomic measure $\mu(n) = 1/n$. Let $\{a_n^*\}$ mean the decreasing rearrangement of a sequence $\{|a_n|\}$. A sequence space $\ell_{\varphi,E}$ with symmetric (quasi)norm $\|\{\varphi(n)a_n^*\}\|_{\widetilde{E}}$ is called *ultrasymmetric*, because it is not only intermediate but also interpolation between the corresponding Lorentz and Marcinkiewicz spaces Λ_φ and M_φ. We study properties of the spaces $\ell_{\varphi,E}$ for all admissible parameters φ, E and use them for the definition of *ultrasymmetric approximation spaces* $X_{\varphi,E}$, which essentially generalize most of classical approximation spaces. At the same time we show that the spaces $X_{\varphi,E}$ possess almost all properties of classical prototypes, such as equivalent norms, representation, reiteration, embeddings, transformation etc. Special attention is paid to interpolation properties of these spaces. At last, we apply our results to ultrasymmetric operator ideals.

1. Introduction

The ultrasymmetric function spaces $L_{\varphi,E}$ with the (quasi)norm $\|f\| = \|\varphi(t)\, f^*(t)\|_{\widetilde{E}}$ were introduced in [14] and shown to form a large class of spaces, including L_p, Lorentz spaces L_{pq}, Lorentz-Zygmund spaces $L^{pr}(\log L)^\alpha$ and many other classical examples. In spite of their generality, these new spaces can be investigated deeply and in detail with rather sharp results that open a simple way to various applications. All these possibilities appear owing to the following main property: *any space $L_{\varphi,E}$ is interpolation*

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between the Lorentz space Λ_φ and the Marcinkiewicz space M_φ with the same fundamental function $\varphi(t)$. While the classical spaces depend only on two-three numerical parameters, the ultrasymmetric spaces may be defined for an arbitrary positive increasing function $\varphi(t)$ and an arbitrary space $\widetilde{E}$ which is rearrangement-invariant (symmetric) with respect to the measure dt/t.

The simple form and the rich properties of ultrasymmetric spaces make them tempting for use in more general situations, via the replacement of $f^*(t)$ by other decreasing functions connected with f. For example, we may involve the K-functional of J. Peetre (cf. Section 5 below), substituting $K(t, f, A_0, A_1)/t$ in place of $f^*(t)$ and obtaining various new interpolation spaces for the (quasi)Banach couple (A_0, A_1). We can take the E-functional $E(t, f, A_0, A_1) = \inf \|f - f_0\|_{A_1},\ \|f_0\|_{A_0} \le t$, and thus obtain various new approximation spaces.

Unfortunately, the theory of E-spaces, elaborated in [10] within the framework of abstract Abelian group theory (see also [1, Chapter 7]), gives explicit practical results only for power-type parameters $\varphi(t) = t^\theta$ and $E = L_q$, while the further generalization becomes rather complicated and implicit (see [4, Section 4.2.C]). At the same time, some other investigations show that it could be more fruitful to use, instead of the E-functional, the so-called *approximation numbers* (besides the classical work [12], see, e.g., [6, 7, 8], where the parameter functions are $\varphi(t) = t^\theta(1 + |\ln t|)^\alpha$). However an immediate extension of this approach to arbitrary ultrasymmetric spaces from [14] is impossible, since the sequence of approximation numbers should be considered in ultrasymmetric *sequence* spaces which were not defined and considered in [14].

At first sight, the transition from function to sequence spaces is not problematic, requiring only a standard change of measure. In fact, this is not so, because the main tool of [14] —the close connection between spaces $\widetilde{E}$ (with measure dt/t) and E (with usual Lebesgue measure)— does not work in the discrete case. Consequently, all properties of ultrasymmetric spaces, based on this connection, require a special care in the case of sequence spaces. For example, this concerns such important properties of approximation spaces as equivalent exponential form of the norm, the representation theorem and some others.

The next section of the present paper is just devoted to the exposition and proof of all needed properties of ultrasymmetric sequence spaces $\ell_{\varphi,E}$. These properties are used then in Sections 3 and 4 for stating the main facts about the corresponding approximation spaces, following the line of the paper [12] (theorems of equivalence, representation, reiteration, embedding, transformation etc.). In Section 5 we establish the interpolation properties of ultrasymmetric approximation spaces, reflecting the corresponding properties of ultrasymmetric sequence spaces. And in the last section we consider some applications to operator ideals.

Throughout the paper we do not differentiate spaces with equivalent (quasi)norms and use the same letter C for various constants, even in the same chain of inequalities (as a rule, these constants depend only on the parameter functions φ, ψ etc.). We write $a \sim b$ if $(1/C)a \le b \le Ca$ for some positive constant C.

2. Sequence spaces

A rearrangement-invariant (another name — symmetric) Banach sequence space E is an interpolation space[1] in the Banach couple (ℓ_1, ℓ_∞), i.e., any sequence space which can be represented as $E = \mathcal{F}(\ell_1, \ell_\infty)$ for some (real) interpolation functor $\mathcal{F}$ (about this and other basic concepts of interpolation theory see, e.g., [1, 4]). It can be described also as a Banach space of sequences $a = \{a_n\}$ with the properties

i) $|a_n| \le |b_n|$ for all $n \implies \|\{a_n\}\|_E \le \|\{b_n\}\|_E$,

ii) $\|\{a_n\}\|_E = \|\{a_n^*\}\|_E$, where $\{a_n^*\}$ means the decreasing rearrangement of $\{|a_n|\}$.

(Formally speaking, these properties give something more, since they may be used also for the definition of quasi-Banach symmetric spaces which are outside of the couple (ℓ_1, ℓ_∞).)

Given an arbitrary $E = \mathcal{F}(\ell_1, \ell_\infty)$, we define another sequence space $\widetilde{E} = \mathcal{F}(\widetilde{\ell}_1, \ell_\infty)$, where

$$\|\{a_n\}\|_{\widetilde{\ell}_1} = \sum_{n=1}^{\infty} \frac{1}{n}|a_n|.$$

As a simplest example, we obtain the spaces $\widetilde{\ell}_p$, $1 \le p \le \infty$, with the norm

$$\|\{a_n\}\|_{\widetilde{\ell}_p} = \Big(\sum_{n=1}^{\infty} \frac{1}{n}|a_n|^p\Big)^{1/p},$$

(which, of course, makes sense as a quasinorm for $p < 1$ too).

The spaces $\widetilde{E}$ also are symmetric but with respect to another (atomic) measure $\mu(n) = 1/n$. Such spaces will be used as one of the parameters in the definition of ultrasymmetric sequence spaces. As the second parameter we will use an arbitrary positive increasing function $\varphi(t)$ with positive and finite extension indices

$$0 < \pi_\varphi := \lim_{s\to 0} \frac{\ln m_\varphi(s)}{\ln s} \le \rho_\varphi := \lim_{s\to\infty} \frac{\ln m_\varphi(s)}{\ln s} < \infty,$$

where $m_\varphi(s) = \sup_t \varphi_E(ts)/\varphi_E(t)$. In what follows, any given parameter function $\varphi(t)$ could be replaced by an equivalent one, thus we always may assume that it is strictly monotone, smooth and such that $\varphi'(t) \sim \varphi(t)/t$.

Definition 2.1 A symmetric sequence space G is called ***ultrasymmetric*** if its (quasi) norm $\|\{a_n\}\|_G$ is equivalent to $\|\{\varphi(n)a_n^*\}\|_{\widetilde{E}}$ for some function φ and space $\widetilde{E}$ described above. Further on we will use the notation $G = \ell_{\varphi,E}$.

Notice that $\ell_{\varphi,E}$ is a Banach space if $\rho_\varphi < 1$, independently of the parameter space E. In the case of $\rho_\varphi > 1$ it is only quasi-Banach and in the remaining case $\rho_\varphi = 1$ the character of $\ell_{\varphi,E}$ depends on additional conditions. As a well-studied example of such spaces we can mention the Lorentz-Zygmund spaces $\ell^{pr}(\log \ell)^\alpha$, appearing when $\varphi(t) = t^{1/p}(1 + \ln t)^\alpha$ and $\widetilde{E} = \widetilde{\ell}_r$ (see, e.g., [7]).

[1] Recall that a space E is called interpolation between spaces E_0, E_1 if any linear operator, bounded on E_0, E_1, is also bounded on E.

Most properties of the ultrasymmetric sequence spaces can be obtained analogously to the properties of ultrasymmetric function spaces considered in [14]. The main assertion is that *a space G is ultrasymmetric if and only if it is a (real) interpolation space between the extreme spaces Λ_φ and M_φ with the (quasi)norms*

$$\|\{a_n\}\|_{\Lambda_\varphi} = \sum_{n=1}^{\infty} \frac{\varphi(n)a_n^*}{n}, \qquad \|\{a_n\}\|_{M_\varphi} = \sup_n \varphi(n)a_n^*,$$

so that $G = \mathcal{F}(\Lambda_\varphi, M_\varphi) = \ell_{\varphi,E}$, where $E = \mathcal{F}(\ell_1, \ell_\infty)$ with the same functor $\mathcal{F}$. Moreover, a space $\ell_{\varphi,E}$ is interpolation between two other such spaces ℓ_{φ,E_0} and ℓ_{φ,E_1} if and only if the space E is interpolation between E_0 and E_1.

Interpolation of spaces $\ell_{\varphi,E}$ with different parameter functions φ is more complicated. The following result can be readily obtained from some general theorems of real interpolation (see, e.g., [4, Theorem 4.3.1]).

Proposition 2.2

If $\rho_{\varphi_0} < \pi_\varphi \leq \rho_\varphi < \pi_{\varphi_1}$ then the space $\ell_{\varphi,E}$ is (real) interpolation between ℓ_{φ_0,E_0}, ℓ_{φ_1,E_1}, independently of the spaces E, E_0, E_1. In particular, the inequalities $1/p_0 < \pi_\varphi \leq \rho_\varphi < 1/p_1$ are sufficient for the space $\ell_{\varphi,E}$ to be interpolation between ℓ_{p_0} and ℓ_{p_1}.

Another problem, important for the applications, is comparing ultrasymmetric sequence spaces one with another; the corresponding results also are analogous to those from [14].

Proposition 2.3

The embedding $\ell_{\varphi,E} \hookrightarrow \ell_{\psi,F}$ is independent of the spaces E, F, if and only if

$$\int_1^\infty \frac{\psi(t)}{t\varphi(t)}\, dt < \infty. \tag{2.1}$$

In order for this embedding to be always valid when $E \hookrightarrow F$, it is necessary and sufficient that $\varphi(t) \geq C\psi(t)$, for some positive constant C and all $t \geq 1$.

Proposition 2.4

Let $r(n) = \psi(2^n)/\varphi(2^n)$ and let E^d be the cone of nonnegative and nonincreasing sequences from the space E. Let, as usual, r^* mean the nonincreasing rearrangement of the sequence r. If the linear operator $Qa = \{r^*(n)a_n\}$ is bounded from E^d to F, then $\ell_{\varphi,E} \hookrightarrow \ell_{\psi,F}$. Moreover, in the case of the function $r(n)$ being (almost) decreasing (that is, when $r(n) \sim r^*(n)$), the condition $Q : E^d \to F$ is necessary for such an embedding.

Let us illustrate the last proposition by the following example.

EXAMPLE 2.5 Let $E = \ell_p,\ F = \ell_q,\ p > q$ and let $\psi(t) = \varphi(t)(1 + |\ln t|)^\varepsilon$, with some $\varepsilon < 0$. Then $Qa \sim \{n^\varepsilon a_n\}$ and by the Hölder inequality we obtain that

$$\|Qa\|_{\ell_q} \sim \Big(\sum_{n=1}^{\infty}(n^\varepsilon a_n)^q\Big)^{1/q} \le \Big(\sum_{n=1}^{\infty} a_n^p\Big)^{1/pq}\Big(\sum_{n=1}^{\infty} n^{\varepsilon qp/(p-q)}\Big)^{(p-1)/pq} < \infty,$$

for any $a \in E^d$, if $\varepsilon pq/(p-q) < -1$, that is, $\varepsilon < 1/p - 1/q$. For any such ε, we get an embedding $\ell_{\varphi,E} \hookrightarrow \ell_{\psi,F}$. Considering the sequence $a = \{n^{-1/p}\ln^{-\sigma} n\}$ with $1/p < \sigma < 1/q$, it is easy to ascertain that the operator Q does not act from ℓ_p to ℓ_q if $\varepsilon = 1/p - 1/q$. Hence the condition $\varepsilon < 1/p - 1/q$ is also necessary for the above mentioned embedding.

At last we present the following corollary from Proposition 2.4.

Proposition 2.6

$\ell_{\varphi,E} = \ell_{\psi,F}$ if and only if $\varphi(t) \sim \psi(t)$, for $t \ge 1$ and $E = F$.

Many properties of ultrasymmetric spaces $\ell_{\varphi,E}$ (e.g., Proposition 2.4 above) require simultaneous consideration of the parameter spaces $\widetilde{E}$ and E. In the case of function spaces this is easy because of the simple connection between these spaces: $f(t) \in \widetilde{E}(1,\infty)$ if and only if $g(u) = f(e^u) \in E(0,\infty)$; moreover, $\|f\|_{\widetilde{E}} = \|g\|_E$. The corresponding relation does not exist in the case of sequence spaces, and the norms $\|\{a_n\}\|_{\widetilde{E}}$ and $\|\{a_{2^n}\}\|_E$ may be even not equivalent. However the needed connection exists in some important situations, and the following lemmas will be used for showing this.

For the first assertion, let us define a linear operator T such that

$$(Ta)_n = a_{2^k}, \quad \text{for } n = 2^k,\ 2^k+1, \dots, 2^{k+1}-1, \quad k = 0, 1, \dots$$

and the weight function $\delta(n)$ equal to 1 when $n = 2^k$, for some $k = 0, 1, \dots$ and equal to 0 otherwise. For any E, the weight space $E(\delta)$ will be defined, as usual, as having the norm $\|a\|_{E(\delta)} = \|\{\delta(n)a_n\}\|_E$.

Lemma 2.7

$T:\ E(\delta) \to \widetilde{E}$ for any symmetric sequence space E.

Proof. If $E = \ell_\infty$ then

$$\|Ta\|_{\ell_\infty} = \sup_k |a_{2^k}| = \sup_n \delta(n)|a_n| = \|a\|_{\ell_\infty(\delta)},$$

which implies that $T:\ \ell_\infty(\delta) \to \ell_\infty$. If $E = \ell_1$ then

$$\|Ta\|_{\widetilde{\ell_1}} = |a_1| + \Big(\frac{1}{2} + \frac{1}{3}\Big)|a_2| + \cdots + \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n}\,|a_{2^k}| + \cdots$$

But, for any k,

$$\sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} \le 2\int_{2^k}^{2^{k+1}} \frac{dx}{x} = 2\ln 2,$$

so that
$$\|Ta\|_{\widetilde{\ell}_1} \le 2\ln 2 \sum_{k=0}^{\infty} |a_{2^k}| = 2\ln 2\, \|\{\delta(n)a_n\}\|_{\ell_1} = 2\ln 2\, \|a\|_{\ell_1(\delta)},$$
which means that $T:\ \ell_1(\delta) \to \widetilde{\ell}_1$.

Let now $E = \mathcal{F}(\ell_1, \ell_\infty)$ for some interpolation functor $\mathcal{F}$. It is easy to show that the same relation is true for the corresponding weight spaces with arbitrary weight; in particular, $E(\delta) = \mathcal{F}(\ell_1(\delta), \ell_\infty(\delta))$. Since, by definition, also $\widetilde{E} = \mathcal{F}(\widetilde{\ell}_1, \ell_\infty)$, the required boundedness of the operator T follows immediately; moreover, the norm of operator T can be estimated independently of E. □

For the second assertion, we define a linear operator S such that $(Sa)_1 = a_1$, $(Sa)_2 = a_2/2$ and
$$(Sa)_{n+1} = \sum_{k=2^{n-1}+1}^{2^n} \frac{a_k}{k}, \qquad \text{for } \ n = 2, 3, \ldots$$

Lemma 2.8

$S:\ \widetilde{E} \to E$ for any symmetric sequence space E.

Proof. Again we start by showing that $S:\ \widetilde{\ell}_1 \to \ell_1$ and $S:\ \ell_\infty \to \ell_\infty$. For ℓ_1, we have immediately that
$$\|Sa\|_{\ell_1} \le \sum_{n=1}^{\infty} \frac{1}{n}\, |a_n| = \|a\|_{\widetilde{\ell}_1}.$$
For the second space, we have
$$\|Sa\|_{\ell_\infty} = \sup\Big\{|a_1|, \frac{1}{2}\,|a_2|, \ldots, \Big|\sum_{k=2^{n-1}+1}^{2^n} \frac{a_k}{k}\Big|, \ldots\Big\}$$
$$\le \sup\Big\{|a_1|, \frac{1}{2}\,|a_2|, \ldots, \max_{2^{n-1}+1 \le k \le 2^n} |a_k| \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k}, \ldots\Big\}.$$
As before,
$$\sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k} \le \int_{2^{n-1}}^{2^n} \frac{dx}{x} = \ln 2, \qquad n = 2, 3, \ldots,$$
which gives
$$\|Sa\|_{\ell_\infty} \le 2\ln 2\ \sup\{|a_1|, |a_2|, \ldots\} = 2\ln 2\, \|a\|_{\ell_\infty},$$
so that the required assertion follows from interpolation properties of the spaces E and $\widetilde{E}$. □

Theorem 2.9

Let $\{a_n\}$ be a nonincreasing sequence of nonnegative numbers. Then $\|a\|_{\widetilde{E}} \sim \|\{a_{2^n}\}\|_E$ for any symmetric sequence space E, with equivalence constant independent of E.

Proof. We already know from Lemma 2.7 that $\|Ta\|_{\widetilde{E}} \le C\,\|a\|_{E(\delta)}$. The given properties of the sequence $\{a_n\}$ imply that $(Ta)_n \ge a_n$ for each n, thus the lattice properties of any space $\widetilde{E}$ give that $\|a\|_{\widetilde{E}} \le C\,\|a\|_{E(\delta)} = C\,\|\{a_{2^n}\}\|_E$. On the other hand, Lemma 2.8 asserts that $\|Sa\|_E \le C\,\|a\|_{\widetilde{E}}$ and, for our sequences,

$$(Sa)_{n+1} = \sum_{k=2^{n-1}+1}^{2^n} \frac{a_k}{k} \ge a_{2^n} \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k} \ge \frac{1}{2}(\ln 2)\, a_{2^{n+1}}\,, \quad n \ge 2,$$

so that $\|Sa\|_E \ge \frac{1}{2}\ln 2\,\|\{a_{2^n}\}\|_E$ and thus $\|\{a_{2^n}\}\|_E \le C\,\|a\|_{\widetilde{E}}$. □

3. Approximation spaces

Let X be a quasi-Banach space and let $\{G_n\},\ n = 1, 2, \ldots$, be an ***approximation family*** of subsets from X, that is, arbitrary sets having the properties

a) $G_1 \subset G_2 \subset \ldots \subset X, \quad G_0 = \{0\}$,

b) $\lambda G_n \subseteq G_n$ for all $\lambda \in \mathbb{R}$ and all n,

c) $G_m + G_n \subseteq G_{m+n}$ for all $m,\ n$.

The ***approximation numbers*** of arbitrary $f \in X$ are defined as

$$a_n(f, X) = \inf_{g \in G_{n-1}} \|f - g\|_X\,,$$

so that $a_1(f, X) = \|f\|_X$. Evidently, $\{a_n(f, X)\}$ is a nonincreasing sequence of nonnegative numbers.

The main idea of approximation theory is to classify the elements $f \in X$ in accordance with the properties of their approximation numbers. In particular, we can construct various ***approximation spaces*** of f, requiring that $\{a_n(f, X)\} \in A$ for special types of sequence spaces A. A general description of such an approach was given in [2]; the papers [3, 12] contain effective realization and applications for the case of weight ℓ_p-spaces taken as A. In the present paper we consider as A all ultrasymmetric spaces.

Definition 3.1 The ultrasymmetric approximation space $X_{\varphi,E}$ is defined as the set of all $f \in X$ such that $\{a_n(f, X)\} \in \ell_{\varphi,E}$. We put

$$\|f\|_{X_{\varphi,E}} := \|\{a_n(f, X)\}\|_{\ell_{\varphi,E}} = \|\{\varphi(n) a_n(f, X)\}\|_{\widetilde{E}}\,, \tag{3.1}$$

so that the space $X_{\varphi,E}$ with this quasinorm becomes a quasi-Banach space.

Evidently, any embedding $\ell_{\varphi,E} \hookrightarrow \ell_{\psi,F}$ entails an analogous embedding $X_{\varphi,E} \hookrightarrow X_{\psi,F}$, thus we may use the conditions of Propositions 2.3 and 2.4 (and also Example 2.5) as sufficient for the last embedding. The necessity of these conditions depends on completeness of the considered interpolation scheme, that is, the existence of functions $f \in X$ with prescribed sequences of approximation numbers (up to equivalence).

As a first step for studying ultrasymmetric approximation spaces, let us pass to another form of the quasinorm (3.1), containing the parameter space E instead of $\widetilde{E}$.

Theorem 3.2

The quasinorm (3.1) is equivalent to

$$\|f\|^{\exp}_{X_{\varphi,E}} := \|\{\varphi(2^n)a_{2^n}(f,X)\}\|_E, \tag{3.2}$$

with equivalence constant dependent only on φ.

Proof. We show that $\|\{\varphi(n)a_n\}\|_{\widetilde{E}} \sim \|\{\varphi(2^n)a_{2^n}\}\|_E$ for all nonincreasing sequences of nonnegative numbers $\{a_n\}$ and for any parameter function $\varphi(t)$, with $0 < \pi_\varphi \le \rho_\varphi < \infty$. As is known, the last inequalities provide that $\varphi(t) \sim \varphi(2t)$, for all $t \ge 1$. Let T be as in Lemma 2.7 and let $b = T\{\varphi(n)a_n\}$. Then $b_n = \varphi(2^k)a_{2^k}$, for $n = 2^k, \dots, 2^{k+1}-1$, $k = 0, 1, \dots$ Therefore $b_n \ge \varphi(2^k)a_n \sim \varphi(2^{k+1})a_n \ge \varphi(n)a_n$. But Lemma 2.7 implies that

$$\|b\|_{\widetilde{E}} \le C\,\|\{\varphi(2^n)a_{2^n}\}\|_E \quad \Longrightarrow \quad \|\{\varphi(n)a_n\}\|_{\widetilde{E}} \le C(\varphi)\,\|\{\varphi(2^n)a_{2^n}\}\|_E.$$

For the opposite inequality, set $c = S\{\varphi(n)a_n\}$ with the same operator S as in Lemma 2.8. Then for any $n \ge 2$,

$$c_{n+1} = \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k}\,\varphi(k)a_k \ge \varphi(2^{n-1})\,a_{2^n} \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k} \ge C(\varphi)\,\varphi(2^n)a_{2^n},$$

(of course, $c_{n+1} \ge C(\varphi)\,\varphi(2^n)a_{2^n}$ for $n = 0, 1$ as well). As a consequence,

$$\|\{\varphi(2^n)a_{2^n}\}\|_E \le C(\varphi)\,\|\{\varphi(n)a_n\}\|_{\widetilde{E}},$$

and we are done. ☐

For various applications of approximation spaces, it is important to have a special representation of any element $f \in X_{\varphi,E}$ via elements from the subsets G_n (the *representation theorem*, see [12]).

Theorem 3.3

The space $X_{\varphi,E}$ consists of those (and only those) elements $f \in X$ which can be represented as $f = \sum g_n$, for some elements $g_n \in G_{2^n}$, $n = 0, 1, \dots$, such that $\{\varphi(2^n)\|g_n\|_X\} \in E$. Moreover, the norm of f in $X_{\varphi,E}$ is equivalent to

$$\|f\|^{\mathrm{rep}}_{X_{\varphi,E}} := \inf\|\{\varphi(2^n)\|g_n\|_X\}\|_E\,,$$

where the infimum is taken over all possible representations $f = \sum g_n$, $g_n \in G_{2^n}$, converging in X.

Proof. a) Let $f \in X_{\varphi,E}$. For each $n = 0, 1, \dots$, define $f_n \in G_{2^n-1}$ such that $\|f - f_n\|_X \le 2\,a_{2^n}(f,X)$ (in particular, $f_0 = 0$). For any $n \ge 2$, set $g_n = f_{n-1} - f_{n-2}$, so that $g_n \in G_{2^{n-1}-1} + G_{2^{n-2}-1} \subseteq G_{2^n}$; for the remaining n, set $g_0 = g_1 = 0$. Then

$$\Big\|f - \sum_{n=0}^{k} g_n\|_X = \|f - f_{k-1}\Big\|_X \le 2\,a_{2^{k-1}} \to 0,$$

as $k \to \infty$, due to the definition of the space $X_{\varphi,E}$. Hence, the series $\sum g_n$ converges to f in X as needed.

Furthermore, for any $n \geq 2$,

$$\|g_n\|_X \leq C\,(\|f - f_{n-1}\|_X + \|f - f_{n-2}\|_X) \leq 4C\, a_{2^{n-2}}(f, X),$$

hence

$$\varphi(2^n)\|g_n\|_X \leq 4C\,\varphi(2^n)\, a_{2^{n-2}}(f, X) \leq C(X, \varphi)\,\varphi(2^{n-2})\, a_{2^{n-2}}(f, X).$$

Due to symmetricity of the space E, this implies that

$$\|\{\varphi(2^n)\|g_n\|_X\}\|_E \leq C(X, \varphi)\, \|\{\varphi(2^n)\, a_{2^n}(f, X)\}\|_E \sim \|f\|_{X_{\varphi,E}},$$

which proves the ***necessity*** part of the theorem.

b) Now let a representation $f = \sum_{n=0}^{\infty} g_n,\ g_n \in G_{2^n}$ be given so that

$$\{\varphi(2^n)\|g_n\|_X\} \in E.$$

For any n, we have that $\sum_{k=0}^{n-1} g_k \in G_1 + \cdots + G_{2^{n-1}} \subseteq G_{2^n-1}$. Thus

$$a_{2^n}(f, X) \leq \|f - \sum_{k=0}^{n-1} g_k\|_X \leq \sum_{k=n}^{\infty} \|g_k\|_X. \tag{3.3}$$

Therefore it is enough to prove the following assertion.

Lemma 3.4

Let $b_n \geq 0,\ n \in \mathbb{N}_0$ and let $\{\varphi(2^n)b_n\} \in E$. If $c_n = \sum_{k=n}^{\infty} b_k$ then $\{\varphi(2^n)c_n\} \in E$ for any positive function $\varphi(t)$ with low extension index $\pi_\varphi > 0$.

Proof. We show first that the series $\sum_{k=0}^{\infty} b_k$ converges and all c_n are defined. Indeed, any symmetric sequence space $E \hookrightarrow \ell_\infty$, so that $\varphi(2^n)b_n \leq M$, for some M and all n, that is, $b_n \leq M/\varphi(2^n)$. Since $\pi_\varphi > 0$, there exists a constant $C = C(\varphi)$ such that $m_\varphi(t) \leq C\, t^{\pi_\varphi/2}$ for all $t > 0$. This implies that $\varphi(1)/\varphi(2^n) \leq m_\varphi(2^{-n}) \leq C\, 2^{-\pi_\varphi n/2}$ and thus

$$\sum_{k=0}^{\infty} b_k \leq \frac{CM}{\varphi(1)} \sum_{k=0}^{\infty} 2^{-\pi_\varphi n/2} < \infty.$$

Let us define a linear operator $Q\{b_n\} = \{c_n\}$ and show that it is bounded on the weight space $E(\varphi(2^n))$, for any symmetric sequence space E. If $\{b_n\} \in \ell_\infty(\varphi(2^n))$ with unit norm, then $b_n \leq 1/\varphi(2^n)$. At the same time,

$$\frac{\varphi(2^n)}{\varphi(2^k)} \leq m_\varphi(2^{n-k}) \leq C\, 2^{-\pi_\varphi(k-n)/2},$$

so that

$$c_n \leq \sum_{k=n}^{\infty} \frac{1}{\varphi(2^k)} \leq \frac{C}{\varphi(2^n)} \sum_{k=0}^{\infty} 2^{-\pi_\varphi k/2} \leq \frac{C}{\varphi(2^n)}.$$

This implies that $\{\varphi(2^n)c_n\} \in \ell_\infty$ with norm not exceeding C.

If now $\{b_n\} \in \ell_1(\varphi(2^n))$, then

$$\|\{\varphi(2^n)c_n\}\|_{\ell_1} = \sum_{n=0}^{\infty} \varphi(2^n) \sum_{k=n}^{\infty} b_k = \sum_{k=0}^{\infty} b_k \sum_{n=0}^{k} \varphi(2^n).$$

But

$$\sum_{n=0}^{k} \varphi(2^n) \le C\,\varphi(2^k) \sum_{n=0}^{k} 2^{-\pi_\varphi(k-n)/2} \le C\varphi(2^k),$$

so that

$$\|\{\varphi(2^n)c_n\}\|_{\ell_1} \le C \sum_{k=0}^{\infty} \varphi(2^k)\, b_k.$$

Therefore the operator Q is bounded on both spaces $\ell_1(\varphi(2^n))$ and $\ell_\infty(\varphi(2^n))$, and the assertion of Lemma 3.4 follows from the interpolation properties of the spaces $E(\varphi(2^n))$. □

In order to finish the proof of Theorem 3.3 we should now put $b_n = \|g_n\|_X$, recall that $a_{2^n}(f, X) \le c_n$ and apply Lemma 3.4. This gives us that $\{\varphi(2^n)a_{2^n}(f, X)\} \in E$, that is, $f \in X_{\varphi,E}$, and the *sufficiency* part of the theorem is also proved. □

4. Iteration and comparison of approximation schemes

Due to Theorem 3.3, the representation $f = \sum_{n=0}^{\infty} g_n,\; g_n \in G_{2^n}$ can be chosen so that

$$\|\{\varphi(2^n)\|g_n\|_X\}\|_E \le C\,\|f\|_{X_{\varphi,E}} \quad\Longrightarrow\quad \|g_n\|_X \le \frac{C}{\varphi(2^n)}\,\|f\|_{X_{\varphi,E}}, \text{ for any } n.$$

Using inequality (3.3), we obtain that

$$a_{2^n}(f, X) \le C \sum_{k=n}^{\infty} \frac{1}{\varphi(2^k)}\,\|f\|_{X_{\varphi,E}} \le \frac{C}{\varphi(2^n)}\,\|f\|_{X_{\varphi,E}}, \tag{4.1}$$

which may be regarded as an analog of the famous Jackson inequality.

Another inequality, which will be used below, is analogous to the Bernstein inequality from the theory of trigonometric approximation. Given an arbitrary $g \in G_{2^n}$, we may apply Theorem 3.3 to the element $f = g$, considering this equality a possible representations for f. This implies that

$$\|g\|_{X_{\varphi,E}} \le C\,\varphi(2^n)\,\|g\|_X, \qquad \text{for any } g \in G_{2^n}. \tag{4.2}$$

Due to the definition of approximation spaces, $G_n \subset X_{\varphi,E}$ for any n and any parameters φ, E. Thus we can define the approximation numbers with respect to the space $X_{\varphi,E}$ instead of X:

$$a_n(f, X_{\varphi,E}) = \inf_{g\in G_{n-1}} \|f - g\|_{X_{\varphi,E}},$$

and then construct the *iterated* approximation spaces $(X_{\varphi,E})_{\psi,F}$. Our aim is to compare these new spaces with the initial ones.

First of all, let us compare the approximation numbers.

Lemma 4.1

There exists a constant $C = C(\varphi)$ *such that*

$$a_{2^{n+1}}(f, X) \leq \frac{C}{\varphi(2^n)}\, a_{2^n}(f, X_{\varphi,E}). \tag{4.3}$$

Proof. Let two elements $g_1, g_2 \in G_{2^n-1}$ be chosen such that, for some $\varepsilon > 0$,

$$\|f - g_1\|_{X_{\varphi,E}} \leq a_{2^n}(f, X_{\varphi,E}) + \varepsilon$$
$$\|(f - g_1) - g_2\|_X \leq a_{2^n}(f - g_1, X) + \varepsilon\,.$$

Then $g_1 + g_2 \in G_{2(2^n-1)} \subset G_{2^{n+1}-1}$, and hence by (4.1)

$$\begin{aligned} a_{2^{n+1}}(f, X) &\leq \|f - (g_1 + g_2)\|_X \leq a_{2^n}(f - g_1, X) + \varepsilon \\ &\leq \frac{C}{\varphi(2^n)}\, \|f - g_1\|_{X_{\varphi,E}} + \varepsilon \leq \frac{C}{\varphi(2^n)}\, (a_{2^n}(f, X_{\varphi,E}) + \varepsilon) + \varepsilon\,. \end{aligned}$$

Since ε is arbitrary, this leads to (4.3). □

Theorem 4.2

The ultrasymmetric approximation scheme is stable with respect to iteration, namely, $(X_{\varphi,E})_{\psi,F} = X_{\varphi\psi,F}$ *for any parameter functions* φ, ψ *and any parameter spaces* E, F.

Proof. If $f \in (X_{\varphi,E})_{\psi,F}$, then by Lemma 4.1

$$\begin{aligned} \|f\|_{(X_{\varphi,E})_{\psi,F}} &\sim \|\{\psi(2^n)\, a_{2^n}(f, X_{\varphi,E})\}\|_F \geq C\, \|\{\psi(2^n)\varphi(2^n)\, a_{2^{n+1}}(f, X)\}\|_F \\ &\geq C\, \|\{\psi(2^{n+1})\varphi(2^{n+1})\, a_{2^{n+1}}(f, X)\}\|_F \geq C\, \Big(\|f\|_{X_{\varphi\psi,F}} - \|f\|_X\Big), \end{aligned}$$

so that

$$\|f\|_{X_{\varphi\psi,F}} \leq C\, \Big(\|f\|_X + \|f\|_{(X_{\varphi,E})_{\psi,F}}\Big) \leq C\, \|f\|_{(X_{\varphi,E})_{\psi,F}}.$$

Conversely, let $f \in X_{\varphi\psi,F}$. By Theorem 3.3, there exists a representation $f = \sum g_n$, $g_n \in G_{2^n}$, converging in X and such that

$$\|\{\psi(2^n)\varphi(2^n)\|g_n\|_X\}\|_F \leq C\, \|f\|_{X_{\varphi\psi,F}}.$$

In particular, due to the inequality (4.2), this implies that for any $n = 0, 1, \ldots$

$$\|g_n\|_{X_{\varphi,E}} \leq C\, \varphi(2^n)\|g_n\|_X \leq \frac{C}{\psi(2^n)}\, \|f\|_{X_{\varphi\psi,F}}.$$

But

$$\sum_{n=0}^{\infty} \frac{1}{\psi(2^n)} \leq \frac{1}{\psi(1)} \sum_{n=0}^{\infty} m_\psi(2^{-n}) \leq C \sum_{n=0}^{\infty} 2^{-\pi_\psi n/2} < \infty,$$

and hence the series $\sum g_n$ converges to f in the space $X_{\varphi,E}$ as well. Moreover,

$$\{\psi(2^n)\|g_n\|_{X_{\varphi,E}}\} \leq C\, \{\psi(2^n)\varphi(2^n)\|g_n\|_X\} \in F, \tag{4.4}$$

which means that $f \in (X_{\varphi,E})_{\psi,F}$ with inequality $\|f\|_{(X_{\varphi,E})_{\psi,F}} \leq C\, \|f\|_{X_{\varphi\psi,F}}$. □

Both one-sided embeddings in this theorem depend only on the relations (4.2) or (4.3), which may connect in the same manner arbitrary spaces and even different approximation schemes. This allows us to compare approximation spaces of various nature one with another (see, e.g., [12] for the case of ℓ_p-spaces as parameters).

Theorem 4.3

Let (X, Y) be a quasi-Banach couple[2] *with common approximation family $\{G_n\} \subset X \cap Y$. Suppose that there exist a constant C and a nonnegative nonincreasing function $\psi(t)$, $t > 0$, such that*

$$\|g\|_Y \leq C\,\psi(n)\,\|g\|_X, \qquad \text{for all } g \in G_n \text{ and } n = 1, 2, \ldots$$

Then $X_{\varphi\psi,E} \subseteq Y_{\varphi,E}$, for every parameters $\varphi(t)$ and E.

Proof. The proof of this theorem is essentially the same as for the second part of the preceding theorem, only the inequality (4.4) should now be changed to

$$\{\varphi(2^n)\|g_n\|_Y\} \leq C\,\{\varphi(2^n)\psi(2^n)\|g_n\|_X\} \in E,$$

which gives the required embedding. □

Another kind of comparing different approximation schemes was also proposed in [12] as *transformation* theorem.

Theorem 4.4

Consider two approximation families $\{G_n\} \subset X$ and $\{H_n\} \subset Y$ and let a linear operator $T : X \to Y$ be such that $T(G_m) \subseteq H_n$, whenever $n \geq \psi(m)$, for some nonnegative nonincreasing function $\psi(t)$ with positive and finite extension indices. Then $T(X_{\varphi(\psi),E}) \subseteq Y_{\varphi,E}$, for every parameters $\varphi(t)$ and E.

Proof. For a given n, let m be such an integer that $\psi(m) \leq n < \psi(m+1)$ (without loss of generality, we may assume that the function $\psi(m)$ is strictly monotone and thus this number always exists and is unique). Then $T(G_m) \subseteq H_n$ and thus

$$a_{n+1}(Tf, Y) = \inf_{h \in H_n} \|Tf - h\|_Y \leq \inf_{g \in G_m} \|Tf - Tg\|_Y \leq \|T\|\,a_{m+1}(f, X). \tag{4.5}$$

At the same time, from the condition ρ_ψ, $\rho_\varphi < \infty$, it follows that $\psi(m+1) \sim \psi(m)$ and $\varphi(Cn) \sim \varphi(n)$ for any constant C; thus $\varphi(n) \leq \varphi(\psi(m+1)) \sim \varphi(\psi(m))$. Consequently, $\varphi(n)\,a_n(Tf, Y) \leq C\,\varphi(\psi(m))\,a_m(f, X)$ for any $n \in \mathbb{N}$.

Before comparing the norms of the corresponding sequences, it should be mentioned that any number m may correspond to several different numbers n entering into the same interval $(\psi(m), \psi(m+1))$. Let us show that this multiplicity can be ignored in our estimates, that is, any number m may be taken only once.

[2] Two spaces X, Y form a couple if they are continuously embedded into some common linear topological Hausdorff space.

Lemma 4.5

For arbitrary positive sequence $a = \{a_n\}$, let $b_n = a_m$ whenever $\psi(m) \le n < \psi(m+1)$ and let $\psi(t)$, $\varphi(t)$, E be as described above. Then

$$\|\{\varphi(n)b_n\}\|_{\widetilde{E}} \le C\,\|\{\varphi(\psi(m))a_m\}\|_{\widetilde{E}}.$$

with some constant C dependent only on functions φ, ψ.

Proof. As mentioned before, $n \sim \psi(m)$ and $\varphi(n) \sim \varphi(\psi(m))$, hence it remains to compare $\{b_n\}$ with $\{a_m\}$. The cardinality of the set $\{n : n \in (\psi(m), \psi(m+1))\}$ is equivalent to $\psi(m+1) - \psi(m) \sim \psi'(m) \sim \psi(m)/m$, provided the last ratio does not vanish at infinity (otherwise this cardinality becomes eventually no greater than 1, in which case the assertion of the lemma is obvious). Therefore

$$\|\{\varphi(n)b_n\}\|_{\widetilde{\ell}_1} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n}\, b_n \sim \sum_{m=1}^{\infty} \frac{\varphi(\psi(m))}{m}\, a_m,$$

which implies that the linear operator $Ra = b$ acts boundedly from $\widetilde{\ell}_1(\varphi(\psi))$ to $\widetilde{\ell}_1(\varphi)$. The second action $R : \ell_\infty(\varphi(\psi)) \to \ell_\infty(\varphi)$ is obviously independent of the above mentioned cardinality and the assertion of the lemma follows by interpolation. □

Returning to the proof of Theorem 4.4, denote by m_n the value of m corresponding to n as described above. Then, from the inequality (4.5) and the last lemma, we obtain that

$$\|\{\varphi(n)a_n(Tf,Y)\}\|_{\widetilde{E}} \le \|T\|\,\|\{\varphi(n)a_{m_n}(f,X)\}\|_{\widetilde{E}} \le C\,\|\{\varphi(\psi(m))a_m(f,X)\}\|_{\widetilde{E}},$$

which was required. □

Remark. In the same manner as above, it is possible to generalize other assertions from [12, Section 3], such as theorems of ***composition, commutation*** and so on.

5. Interpolation of approximation spaces

In this section we show that the ultrasymmetric approximation spaces are rather suitable for applications of real interpolation methods. Given a quasi-Banach couple (X, Y), we denote by $(X, Y)^K_\Phi$ the interpolation space with the (quasi)norm $\|f\| = \|(1/t)K(t, f, X, Y)\|_\Phi$, where Φ may be an arbitrary Banach function space and $K(t, f, X, Y)$ is the K-functional of J. Peetre:

$$K(t, f, X, Y) = \inf_{f=f_1+f_2} (\|f_1\|_X + t\,\|f_2\|_Y).$$

In the case of $Y \hookrightarrow X$ this (quasi)norm is equivalent to $\|(1/t)K(t, f, X, Y)\chi_{(0,1)}(t)\|_\Phi$. Another equivalent expression can be obtained by discretization of the parameter space Φ (see, e.g., [4, p. 358]). For instance, if $\rho_\varphi < 1$, we may take $\Phi = \ell_{\overline{\varphi},E}$, $\overline{\varphi}(t) = t\varphi(1/t)$, getting (again for the case of $Y \hookrightarrow X$) the (quasi)norm of $(X, Y)^K_\Phi$ in the form

$$\|f\|_{K,\varphi,E} := \|\{\varphi(n)\,K(1/n, f, X, Y)\}\|_{\widetilde{E}} \sim \|\{\varphi(2^n)\,K(2^{-n}, f, X, Y)\}\|_E, \qquad (5.1)$$

(the last equivalence is due to Theorem 3.2). Remark that, instead of 2^n, we may take here q^n with any $q > 1$.

In order to involve maximum of approximation spaces into one interpolation scheme, we should take an appropriate couple (X, Y) with maximally large space X and minimally small space Y. The most natural large space X is the basic space of the approximation scheme which contains all approximation spaces by definition[3]. The choice of a small space Y is not so simple, since the most natural candidate $Y = \bigcup_{n=1}^{\infty} G_n$ is neither ultrasymmetric approximation nor quasi-Banach space at all. The famous paper [10] uses this kind of a space Y within the framework of theory of quasinormed Abelian groups. Unfortunately, interpolation theory of these groups, elaborated in [10], gives explicit results only for the spaces $X_{\varphi,E}$ with $\varphi(t) = t^\theta$ and $E = \ell_p$.

For our purposes, we will take a space Y not so small, but sufficient for consideration of all needed spaces $X_{\varphi,E}$. Moreover, this space will depend on a parameter p which can be taken as big as needed. Namely, we define $Y = Y_p = X_{\varphi,E}$ with $\varphi = t^p$ and $E = \ell_1$; as follows from Proposition 2.3, this space is contained in any other $X_{\varphi,E}$ with $\rho_\varphi < p$ and arbitrary E.

Lemma 5.1

Let $p \geq 1$ be fixed, then for any $n = 0, 1, \dots$ and any $f \in X$,

$$(1/C)\, a_{2^n}(f, X) \leq K(2^{-np}, f, X, Y_p) \leq C \sum_{k=0}^{n} 2^{p(k-n)} a_{2^k}(f, X). \tag{5.2}$$

Proof. For arbitrary $h \in Y_p$, we get

$$\begin{aligned} a_{2^n}(f, X) &= \inf_{g \in G_{2^n-1}} \|f - g\|_X \leq C\Big(\|f - h\|_X + \inf_{g \in G_{2^n-1}} \|h - g\|_X\Big) \\ &\leq C\big(\|f - h\|_X + a_{2^n}(h, X)\big). \end{aligned}$$

By the inequality (4.1), we obtain that $a_{2^n}(h, X) \leq (C/2^{np})\, \|h\|_{Y_p}$, whence

$$a_{2^n}(f, X) \leq C\, \big(\|f - h\|_X + 2^{-np} \|h\|_{Y_p}\big).$$

Passing to infimum over all $h \in Y_p$, we obtain the first inequality in (5.2).

To prove the second inequality, for any given $n = 0, 1, \dots$, let us define a function $g_n \in G_{2^n-1}$ such that $\|f - g_n\|_X \leq 2a_{2^n}(f, X)$, and then put $f_1 = g_0$, $f_n = g_{n-1} - g_{n-2}$ $(n \geq 2)$. Then,

$$\begin{aligned} &\|f_1\|_X \leq C\big(\|f - g_0\|_X + \|f\|_X\big) \leq C\big(2a_1(f, X) + \|f\|_X\big) = 3C\, a_1(f, X), \\ &\|f_n\|_X \leq C\big(\|f - g_{n-1}\|_X + \|f - g_{n-2}\|_X\big) \leq 4C\, a_{2^{n-2}}(f, X), \quad \text{for all } n \geq 2. \end{aligned}$$

Since $g_n \in Y_p$ for each n, we may write that

$$K(2^{-np}, f, X, Y_p) \leq \|f - g_n\|_X + 2^{-np} \|g_n\|_{Y_p} \leq 2\, a_{2^n}(f, X) + 2^{-np} \|g_n\|_{Y_p},$$

[3] Remark that the space X itself is an ultrasymmetric approximation space $X_{\varphi,E}$ with $\varphi(t) \equiv 1$ and $E = \ell_\infty$.

and then, by the inequality (4.2),

$$\|g_n\|_{Y_p} = \Big\| \sum_{k=1}^{n+1} f_k \Big\|_{Y_p} \le C \sum_{k=1}^{n+1} 2^{kp} \|f_k\|_X \le C_1 \sum_{k=0}^{n} 2^{kp} a_{2^k}(f, X).$$

Therefore,

$$K(2^{-np}, f, X, Y_p) \le C_1 \Big(a_{2^n}(f, X) + 2^{-np} \sum_{k=0}^{n} 2^{kp} a_{2^k}(f, X) \Big) \sim \sum_{k=0}^{n} 2^{p(k-n)} a_{2^k}(f, X),$$

and we are done. □

Theorem 5.2

Let $p > \max(1, \rho_\varphi)$ *and* $\psi(t) = \varphi(t^{1/p})$. *Then*

$$X_{\varphi,E} = (X, Y_p)^K_\Phi, \qquad \text{where } \ \Phi = \ell_{\overline{\psi},E}.$$

Proof. By the definition of the function $\psi(t)$, we have that $\rho_\psi < 1$, which allows us to write the norm of $(X, Y)^K_\Phi$ in a discrete form like (5.1), namely,

$$\|f\|_{K,\psi,E} \sim \|\{\psi(2^{np}) K(2^{-np}, f, X, Y_p)\}\|_E. \tag{5.3}$$

The theorem will be proved if we show that $\|f\|_{K,\psi,E} \sim \|f\|_{X_{\varphi,E}}$.

Using the first inequality in (5.2), we obtain immediately that

$$\begin{aligned} \|f\|_{K,\psi,E} &\ge (1/C)\, \|\{\psi(2^{np})\, a_{2^n}(f, X)\}\|_E \\ &= (1/C)\, \|\{\varphi(2^n)\, a_{2^n}(f, X)\}\|_E \\ &= (1/C)\, \|f\|_{X_{\varphi,E}}. \end{aligned}$$

The second inequality from (5.2) gives that

$$\|f\|_{K,\psi,E} \le C \Big\| \Big\{ \psi(2^{np}) \sum_{k=0}^{n} 2^{p(k-n)} a_{2^k}(f, X) \Big\} \Big\|_E.$$

At the same time, for any $k \le n$,

$$\psi(2^{np}) \le \psi(2^{kp})\, m_\psi(2^{p(n-k)}) \le C(\varepsilon)\, 2^{p(n-k)(\rho_\psi+\varepsilon)} \psi(2^{kp}).$$

Taking $\varepsilon = (1 - \rho_\psi)/2$, we obtain that

$$\psi(2^{np}) \le C\, 2^{p(n-k)(1-\varepsilon)} \psi(2^{kp}),$$

whence

$$\|f\|_{K,\psi,E} \le C \Big\| \Big\{ \sum_{k=0}^{n} 2^{\varepsilon p(k-n)} \varphi(2^k) a_{2^k}(f, X) \Big\} \Big\|_E.$$

For finishing the proof, we need the following:

Lemma 5.3

The linear operator $Pa = b$, defined by the equalities

$$b_n = \sum_{k=0}^{n} 2^{\varepsilon p(k-n)} a_k, \qquad n = 0, 1, \ldots,$$

is bounded in any symmetric sequence space E.

Proof. As usual, it is enough to show that P is bounded in ℓ_1 and ℓ_∞. We have that

$$\|Pa\|_{\ell_1} \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} 2^{\varepsilon p(k-n)} |a_k| = \sum_{k=0}^{\infty} |a_k| \sum_{n=k}^{\infty} 2^{-\varepsilon p(n-k)} \leq C \sum_{k=0}^{\infty} |a_k|,$$

and also

$$\|Pa\|_{\ell_\infty} \leq \sup_n \sum_{k=0}^{n} 2^{\varepsilon p(k-n)} |a_k| \leq \sup_n \Big(\max_{k \leq n} |a_k| \cdot \sum_{k=0}^{n} 2^{-\varepsilon p(n-k)} \Big) \leq C \sup_n |a_n|,$$

as required. $\square$

Returning to the proof of Theorem 5.2, we obtain that

$$\|f\|_{K,\psi,E} \leq C \, \|\{\varphi(2^n) a_{2^n}(f, X)\}\|_E = C \, \|f\|_{X_{\varphi,E}},$$

and the theorem is proved. $\square$

Corollary 5.4

A space $X_{\varphi,E}$ is interpolation between the spaces X_{φ_0,E_0} and X_{φ_1,E_1} if and only if the space $\ell_{\varphi,E}$ is interpolation between ℓ_{φ_0,E_0} and ℓ_{φ_1,E_1}. Moreover, for any real interpolation functor $\mathcal{F}$ and any number $p > \max(\rho_{\varphi_0}, \rho_{\varphi_1})$, one has

$$\mathcal{F}(X_{\varphi_0,E_0}, X_{\varphi_1,E_1}) = (X, Y_p)^K_\Psi,$$

where $\Psi = \mathcal{F}(\ell_{\overline{\psi}_0,E_0}, \ell_{\overline{\psi}_1,E_1}),\ \psi_i(t) = \varphi_i(t^{1/p}),\ i = 0, 1.$

This assertion follows immediately from Theorem 5.2, using general *reiteration* theorems of interpolation (see, e.g., [4, p. 357]).

For applying this corollary, we should return to the problem, already posed in Section 2: when is the ultrasymmetric space $\ell_{\varphi,E}$ interpolation between two other spaces ℓ_{φ_0,E_0} and ℓ_{φ_1,E_1}? The solution is simple for the case of interpolation with numerical parameters (for the definition and properties of this method, see, e.g., [1, Section 3.1]). For example, under conditions of Proposition 2.2, we get that

$$(X_{\varphi_0,E_0}, X_{\varphi_1,E_1})_{\theta,q} = X_{\varphi,\ell_q}, \quad \varphi(t) = \big(\varphi_0(t)\big)^{1-\theta} \big(\varphi_1(t)\big)^{\theta},$$

whenever $0 < \theta < 1,\ 1 \leq q \leq \infty$.

Interpolation with functional parameters is much more difficult; apparently, the most general interpolation theorem of such a type for ultrasymmetric function spaces is Theorem 5.2 from [14]. An analogous result can be proved for sequence spaces as well, leading to the following assertion for approximation spaces.

Proposition 5.5

Let the parameter functions $\varphi_0(t)$, $\varphi_1(t)$ and $\sigma(t)$ be such that $\pi_{\varphi_1/\varphi_0} > 0,\ 0 < \pi_\sigma \le \rho_\sigma < 1$ and let $\varphi = \varphi_0\sigma(\varphi_1/\varphi_0)$. Then the space $X_{\varphi,E}$ is interpolation between the spaces X_{φ_0,E_0} and X_{φ_1,E_1} independently of parameter spaces E, E_0, E_1; moreover,

$$X_{\varphi,E} = (X_{\varphi_0,E_0}, X_{\varphi_1,E_1})^K_\Psi, \quad \Psi = \ell_{\overline{\sigma},E}.$$

Of course, interpolation between ultrasymmetric sequence spaces becomes much simpler if the parameter function $\varphi(t)$ does not change —a space $\ell_{\varphi,E}$ is interpolation between ℓ_{φ,E_0} and ℓ_{φ,E_1} if and only if the space E is interpolation between E_0 and E_1. It might be useful to combine this result with embedding assertions like Proposition 2.4. For instance, we obtain the following assertion.

Proposition 5.6

Let the functions $\varphi(t), \psi(t)$ and the symmetric sequence spaces E, F satisfy the conditions of Proposition 2.4 and let the space E be interpolation between E_0, E_1. Then $T : X_{\varphi,E} \to X_{\psi,F}$ for any (quasi)linear operator T which is bounded on the spaces X_{φ,E_0} and X_{φ,E_1}.

6. Ultrasymmetric operator ideals

The ultrasymmetric approximation spaces can be used for various applications, formerly known only for spaces with numerical parameters (such as X^ρ_u from [12] and some others). Due to their generality, they give results with more delicate distinction of approximation characteristics for considered functions. The spaces $X_{\varphi,E}$ could be particularly effective for solution of problems with non-power conditions on the growth of n. A more or less complete exposition of such applications requires special study and separate papers. In the present paper we consider only some consequences for operator ideals.

Let X be the Banach space $\mathcal{L}(A,B)$ of all bounded linear operators T between the Banach spaces A and B and let G_n denote the set of all operators $S \in \mathcal{L}(A,B)$ such that $\operatorname{rank} S \le n$. The corresponding approximation numbers of linear operators

$$a_n(T) = \inf_{\operatorname{rank} S < n} \|T - S\|_{\mathcal{L}(A,B)},$$

were introduced by A. Pietsch in 1963 and studied then by many authors. Let us denote the approximation space $X_{\varphi,E}$, with these numbers, by $\mathcal{L}_{\varphi,E}(A,B)$. As follows from general theory of s-numbers elaborated in [11, Section 14], the spaces $\mathcal{L}_{\varphi,E}(A,B)$ are operator ideals; by analogy we may call them *ultrasymmetric ideals*. In fact, all operators from these ideals are compact, since, due to Theorem 3.3, any $T \in \mathcal{L}_{\varphi,E}(A,B)$ can be approximated by finite dimensional operators.

Now any result, obtained in the previous sections, can be reformulated so as to get some new property of operator ideals. For example, if $p > q$ and $\psi(t)/\varphi(t) = (1 + |\ln t|)^\varepsilon$, then $\mathcal{L}_{\varphi,\ell_p}(A,B) \hookrightarrow \mathcal{L}_{\psi,\ell_q}(A,B)$ for any $\varepsilon < 1/p - 1/q$. Another fact: the ideal $\mathcal{L}_{\varphi,E}(A,B)$ is interpolation between two other such ideals $\mathcal{L}_{\varphi_i,E_i}(A,B),\ i = 0,1$, if and only if the sequence space $\ell_{\varphi,E}$ is interpolation between ℓ_{φ_0,E_0} and ℓ_{φ_1,E_1}.

As a more interesting result, let us show that the ultrasymmetric operator ideals have a *multiplication* property, which again generalizes an analogous property of ideals X_u^ϱ, stated in [12]. We say that the symmetric spaces E_0, E_1 are *multipliers* of a given symmetric space E if, for any sequences $\{a_n\} \in E_0$ and $\{b_n\} \in E_1$, their product sequence $\{a_n b_n\}$ belongs to E (as shown in [9, Lemma 13.5], this always implies the existence of a constant C such that $\|\{a_n b_n\}\|_E \le C\,\|\{a_n\}\|_{E_0}\|\{b_n\}\|_{E_1}$). Such a relation between symmetric spaces is well-studied not only for ℓ_p, but also for Lorentz, Marcinkiewicz, Lorentz-Zygmund, Orlicz and many other spaces.

Theorem 6.1

Let E_0, E_1 be multipliers of E and let $\varphi(t) = \varphi_0(t)\varphi_1(t)$ for $t \ge 1$. Then

$$\mathcal{L}_{\varphi_0,E_0}(D,B) \circ \mathcal{L}_{\varphi_1,E_1}(A,D) \subseteq \mathcal{L}_{\varphi,E}(A,B), \tag{6.1}$$

i.e., for any $S \in \mathcal{L}_{\varphi_0,E_0}(D,B)$ and $T \in \mathcal{L}_{\varphi_1,E_1}(A,D)$, the composition $ST \in \mathcal{L}_{\varphi,E}(A,B)$.

Proof. Using the proof of Theorem 3.2, it is easy to show that the spaces ℓ_{φ_0,E_0} and ℓ_{φ_1,E_1} are also multipliers of the space $\ell_{\varphi,E}$. Due to symmetricity and the lattice property of the norm in $\ell_{\varphi,E}$, for any nonnegative nonincreasing sequence $\{a_n\}$, one has that

$$\begin{aligned}\|\{a_n\}\|_{\ell_{\varphi,E}} &\le C\left(\|\{a_1,0,a_3,0,\dots\}\|_{\ell_{\varphi,E}} + \|\{0,a_2,0,a_4,\dots\}\|_{\ell_{\varphi,E}}\right)\\ &= C\left(\|\{a_1,a_3,\dots\}\|_{\ell_{\varphi,E}} + \|\{a_2,a_4,\dots\}\|_{\ell_{\varphi,E}}\right) \le 2C\,\|\{a_{2n-1}\}\|_{\ell_{\varphi,E}},\end{aligned}$$

and thus

$$\|\{\varphi(n)\,a_n(ST)\}\|_{\widetilde{E}} \le C\,\|\{\varphi(n)\,a_{2n-1}(ST)\}\|_{\widetilde{E}}. \tag{6.2}$$

On the next step we will use the following multiplication property of approximation numbers (which is also due to A. Pietsch, see [13, Proposition 2.3.12]: $a_{m+n-1}(ST) \le a_m(S)\,a_n(T)$. By this property we obtain that

$$\|\{\varphi(n)\,a_n(ST)\}\|_{\widetilde{E}} \le C\,\|\{a_n(S)a_n(T)\}\|_{\ell_{\varphi,E}} \le C\,\|\{a_n(S)\}\|_{\ell_{\varphi_0,E_0}}\|\{a_n(T)\}\|_{\ell_{\varphi_1,E_1}},$$

which implies (6.1). □

Remark. A stronger, so-called *factorization* property appears if the embedding (6.1) turns into equality (with equivalent norms), which means that any operator from $\mathcal{L}_{\varphi,E}(A,B)$ can be factorized by some operators from $\mathcal{L}_{\varphi_0,E_0}(D,B)$ and $\mathcal{L}_{\varphi_1,E_1}(A,D)$. The factorization property of ultrasymmetric operator ideals depends on the similar property of symmetric spaces E, E_0, E_1, which in the abstract case occurs rather seldom.

The remarkable inequalities, proved by B. Carl in [5], allow us to estimate the behavior of entropy numbers and eigenvalues of operators from ultrasymmetric ideals. Containing arbitrary parameter functions $\varphi(t)$ rather than only powers t^θ and arbitrary symmetric sequence spaces E rather than only ℓ_p, these estimates would be more subtle and more accurate than the previously existent.

Let us recall that the nth entropy number $e_n(T)$ of an operator $T \in \mathcal{L}(A,B)$ is defined as the infimum of all $\varepsilon > 0$ such that there exist elements $g_1, \dots, g_r$, with $r \le 2^{n-1}$, giving that

$$T(U_A) \subseteq \bigcup_{i=1}^{r} \{g_i + \varepsilon U_B\},$$

where U_A and U_B are closed unit balls of A and B respectively. As shown in [5, Theorem 1], for any $p > 0$, there exists a constant $C = C(p)$ such that

$$n\,(e_n(T))^p \le C \sum_{k=1}^{n} (a_k(T))^p, \qquad n = 1, 2, \dots \tag{6.3}$$

Theorem 6.2

If $T \in \mathcal{L}_{\varphi,E}(A,B)$, then $\{\varphi(n)\, e_n(T)\} \in \widetilde{E}$.

Proof. First of all, let us show that the operator $Pa = b$, such that $b_n^p = (1/n) \sum_{k=1}^n a_k^p$, with fixed number p, is bounded in any space $\ell_{\varphi,E}$ with $\rho_\varphi < 1/p$. In the case of $p = 1$ this operator is linear, namely, the classical Hardy operator, which is known to be bounded in any symmetric space with upper Boyd index less than 1. Otherwise, this operator is only quasilinear, but this property is also sufficient for application of real interpolation. Due to Proposition 2.2, it is enough to show that P is bounded on some ℓ_{p_1} with $\rho_\varphi < 1/p_1 < 1/p$ (as the second space ℓ_{p_0} we can always take ℓ_∞, where the boundedness of P is evident). Setting $\overline{a}_n = a_n^p$ and $\overline{b}_n = b_n^p$, we obtain that the operator $P_1 \overline{a} = \overline{b}$ is again the classical Hardy operator, and thus it is bounded in ℓ_r with $r = p_1/p$. This means that

$$\|\{b_n\}\|_{\ell_{p_1}} = \|\{\overline{b}_n\}\|_{\ell_r}^{1/p} \le C\, \|\{\overline{a}_n\}\|_{\ell_r}^{1/p} = C\, \|\{a_n\}\|_{\ell_{p_1}},$$

and we are done.

If now the ideal $\mathcal{L}_{\varphi,E}(A,B)$ is given, we take arbitrary $p < 1/\rho_\varphi$, obtaining that

$$\Big\| \Big\{ \Big(\frac{1}{n} \sum_{k=1}^{n} a_n(T)^p \Big)^{1/p} \Big\} \Big\|_{\ell_{\varphi,E}} \le C\, \|\{a_n(T)\}\|_{\ell_{\varphi,E}},$$

and it remains only to use inequality (6.3). □

Proceeding to the characterization of eigenvalues of operators from ultrasymmetric ideals, we should observe that the set of these numbers is at most countable, since all such operators are compact. Let a compact operator T act in a complex Banach space A. Denote by $\{\lambda_n(T)\}$ the sequence of all eigenvalues of this operator, counted according to their multiplicities and such that $|\lambda_1(T)| \ge |\lambda_2(T)| \ge \ldots \ge 0$. If T has only a finite number n_0 of eigenvalues, we put $\lambda_n(T) = 0$ for all $n > n_0$.

As shown in [5, Theorem 4], the eigenvalues of a compact operator T, acting in a complex Banach space, are connected with its entropy numbers by the inequality $|\lambda_n(T)| \le \sqrt{2}\, e_n(T)$, $n = 1, 2, \dots$ Therefore Theorem 6.2 entails immediately an analogous assertion for the eigenvalues.

Theorem 6.3

Let $T \in \mathcal{L}_{\varphi,E}(A,A)$. Then $\{\varphi(n)|\lambda_n(T)|\} \in \widetilde{E}$.

To finish, let us consider a simple application of this result.

EXAMPLE 6.4 Let $h(t)$ be a locally integrable 2π-periodical function. Consider the corresponding convolution operator

$$T\,f(t) = \int_0^{2\pi} h(t-s)\,f(s)\,ds. \tag{6.4}$$

It is easy to see that $T \in \mathcal{L}(L_1, L_1)$ with norm equal to $\|h\|_{L_1(0,2\pi)}$. Moreover, using trigonometric approximation of the function $h(t)$, similarly to Jackson's theorem, one can show that $a_{2n+2}(T) \le C\,\omega_{L_1}(h, \frac{1}{n})$, for any $n = 1, 2, \dots$, where

$$\omega_{L_1}(h,\delta) = \sup_{0\le s\le\delta} \int_0^{2\pi} |h(t+s) - h(t)|\,dt.$$

Let now some parameter function $\varphi(t)$ and parameter space E be given. By analogy to usual Besov spaces, let us define

$$B_{1,\varphi,E} = \Big\{h:\ \|h\|_{B_{1,\varphi,E}} = \|h\|_{L_1} + \Big\|\Big\{\varphi(n)\,\omega_{L_1}\Big(h, \frac{1}{n}\Big)\Big\}\Big\|_{\widetilde{E}} < \infty\Big\}.$$

Like (6.2) we can show that

$$\|\{\varphi(n)\,a_n(T)\}\|_{\widetilde{E}} \le C\big(\|T\|_{\mathcal{L}(L_1,L_1)} + \|\{\varphi(n)\,a_{2n+2}(T)\}\|_{\widetilde{E}}\big).$$

Then Theorem 6.3 gives immediately, that the eigenvalues $\lambda_n(T)$ of the operator (6.4), arranged in the descending order, satisfy the condition $\{\varphi(n)|\lambda_n(T)|\} \in \widetilde{E}$ whenever $h \in B_{1,\varphi,E}$.

And one more consequence can be stated here. Namely, we may use the result of T. Carleman (see [13, Proposition 6.5.5]), claiming that the eigenvalues of the convolution operator (6.4) coincide with the (complex) Fourier coefficients of the function $h(t)$, multiplied by 2π and arranged in the descending order. Hence these coefficients also form a sequence belonging to $\ell_{\varphi,E}$.

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