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Weighted inequalities for Hardy-type operators involving suprema

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Abstract

Let u, b be two weight functions on $(0, \infty)$. Assume that u is continuous on $(0, \infty)$ and that b is such that the function $B(t) := \int_0^t b(s)\, ds$ satisfies $0 < B(t) < \infty$ for every $t \in (0, \infty)$. Let the operator $T_{u,b}$ be given at a measurable non-negative function g on $(0, \infty)$ by

$$(T_{u,b}g)(t) = \sup_{t \le \tau < \infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(s)b(s)\, ds.$$

We give necessary and sufficient conditions on weights v, w on $(0, \infty)$ for which there exists a positive constant C such that the inequality

$$\left(\int_0^\infty [(T_{u,b}g)\,(t)]^q\, w(t)\, dt \right)^{1/q} \le C \left(\int_0^\infty [g(t)]^p\, v(t)\, dt \right)^{1/p}$$

holds for every measurable non-negative function g on $(0, \infty)$, where $p, q \in (0, \infty)$ satisfy certain restrictions. We also characterize weights v, w on $(0, \infty)$ for which there exists a positive constant C such that the inequality

$$\left(\int_0^\infty [(T_{u,b}\varphi)\,(t)]^q\, w(t)\, dt \right)^{1/q} \le C \left(\int_0^\infty [\varphi(t)]^p\, v(t)\, dt \right)^{1/p}$$

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holds for every non-negative and non-increasing function φ on $(0,\infty)$. The crucial tool in our approach to the latter problem is the reduction of the given inequality to the pair of analogous inequalities involving more manageable operators, namely the classical Hardy-type integral operator and the operator

$$(R_u\varphi)(t) = \sup_{t\le\tau<\infty} u(\tau)\varphi(\tau).$$

Such estimates have recently been found indispensable in the study of problems involving fractional maximal operators and optimal Sobolev embeddings.

1. Introduction

Our aim is to study weighted one-dimensional inequalities for Hardy-type operators involving supremum. We do this using methods of discretization and antidiscretization.

Suppose f is a locally-integrable function on $\mathbb{R}^n$. Then its *non-increasing rearrangement* f^* is given by

$$f^*(t) = \inf\left\{\lambda>0;\, |\{x\in\mathbb{R}^n;\, |f(x)|>\lambda\}| \le t\right\}, \quad t\in(0,\infty).$$

Quite many familiar function spaces can be defined using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called *classical Lorentz spaces*.

Let $p\in(0,\infty)$ and let w be a *weight*, that is, a non-negative measurable and a.e. finite function on $(0,\infty)$. Then the classical Lorentz space $\Lambda^p(w)$ is defined as the set of all measurable functions f whose non-increasing rearrangement satisfies

$$\|f\|_{\Lambda^p(w)} := \left(\int_0^\infty [f^*(t)]^p\, w(t)\,dt\right)^{1/p} < \infty.$$

In their pioneering paper [1], Ariño and Muckenhoupt characterized when the Hardy–Littlewood maximal operator M, defined at $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$(Mf)(x) = \sup_{Q\ni x} |Q|^{-1}\int_Q |f(y)|\,dy, \qquad x\in\mathbb{R}^n,$$

(where the supremum is extended over all cubes $Q\subset\mathbb{R}^n$ with sides parallel to the coordinate axes and $|E|$ denotes the n-dimensional Lebesgue measure of $E\subset\mathbb{R}^n$) is bounded on the classical Lorentz space $\Lambda^p(w)$ when $p\in[1,\infty)$. In other words, they characterized the class of weights w for which the inequality

$$\int_0^\infty [(Mf)^*(t)]^p\, w(t)\,dt \lesssim \int_0^\infty [f^*(t)]^p\, w(t)\,dt$$

holds for every locally integrable f on $\mathbb{R}^n$.

(As usual, here and below, by $A \lesssim B$ and $A \gtrsim B$ we mean that $A \le CB$ and $CA \ge B$, respectively, where C is a positive constant independent of appropriate quantities involved in the expressions A and B. We will also write $A \approx B$ when both $A \lesssim B$ and $A \gtrsim B$ are satisfied.)

Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\downarrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-increasing on $(0,\infty)$.

The approach of [1] has two main ingredients. The first of them is the well-known two-sided estimate (cf. [2, Chapter 3, Theorem 3.8])

$$(Mf)^*(t) \approx (Pf^*)(t), \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ t \in (0,\infty), \tag{1.1}$$

where P is the *averaging operator* defined by

$$(Pg)(t) = \frac{1}{t}\int_0^t g(s)\,ds, \quad g \in \mathfrak{M}^+(0,\infty),\ t \in (0,\infty).$$

The second key ingredient is a characterization of the boundedness of P on the cone $\mathfrak{M}^+(0,\infty;\downarrow)$ in the *weighted Lebesgue space* $L^p(w)$ over $(0,\infty)$, whose norm is given by

$$\|g\|_{p,w} = \left(\int_0^\infty |g(t)|^p w(t)\,dt\right)^{1/p},$$

in other words, a characterization of the inequality

$$\left(\int_0^\infty [(P\varphi)(t)]^p\,w(t)\,dt\right)^{1/p} \lesssim \left(\int_0^\infty [\varphi(t)]^p\,w(t)\,dt\right)^{1/p}, \quad \varphi \in \mathfrak{M}^+(0,\infty;\downarrow). \tag{1.2}$$

The point is that (1.2) (which is restricted to non-increasing functions) is true for a substantially larger class of weights than the analogous, unrestricted, classical Hardy inequality

$$\left(\int_0^\infty [(Pg)(t)]^p\,w(t)\,dt\right)^{1/p} \lesssim \left(\int_0^\infty [g(t)]^p\,w(t)\,dt\right)^{1/p}, \quad g \in \mathfrak{M}^+(0,\infty).$$

This groundbreaking observation was made in [1].

In [22], Sawyer extended the result of [1] to the case of two weights and two exponents p, q. He characterized the validity of the inequality

$$\left(\int_0^\infty [(Mf)^*(t)]^q\,w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [f^*(t)]^p\,v(t)\,dt\right)^{1/p}, \quad L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

provided that $1 < p, q < \infty$. Further extensions were obtained for example in [23]. For more references, see [4].

An analogous problem for the *fractional maximal operator* in place of the Hardy–Littlewood maximal operator was studied in [6]. The fractional maximal operator, M_γ, $\gamma \in (0,n)$, is defined at $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$(M_\gamma f)(x) = \sup_{Q \ni x} |Q|^{\gamma/n-1} \int_Q |f(y)|\,dy, \qquad x \in \mathbb{R}^n,$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. It was shown in [6, Theorem 1.1] that

$$(M_\gamma f)^*(t) \lesssim \sup_{t\le\tau<\infty} \tau^{\gamma/n}(Pf^*)(\tau) \tag{1.3}$$

for every $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $t \in (0,\infty)$. This estimate is not two-sided as (1.1), but it is sharp in the following sense: for every $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$ there exists a function f on $\mathbb{R}^n$ such that $f^* = \varphi$ a.e. on $(0,\infty)$ and

$$(M_\gamma f)^*(t) \gtrsim \sup_{t\le\tau<\infty} \tau^{\gamma/n}(Pf^*)(\tau), \quad t \in (0,\infty). \tag{1.4}$$

Consequently, the role of Pf^* in the argument of Ariño and Muckenhoupt [1] is in the case of fractional maximal operator taken over by the expression on the right-hand sides of (1.3) and (1.4). Thus, in order to characterize boundedness of the fractional maximal operator M_γ between classical Lorentz spaces it is necessary and sufficient to characterize the validity of the weighted inequality

$$\left(\int_0^\infty \left[\sup_{t\le\tau<\infty} \tau^{\gamma/n-1}\int_0^\tau \varphi(s)\,ds\right]^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [\varphi(t)]^p\,v(t)\,dt\right)^{1/p}$$

for all $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$. This last estimate can be interpreted as a restricted weighted inequality for the operator T_γ, defined by

$$(T_\gamma g)(t) = \sup_{t\le\tau<\infty} \tau^{\gamma/n-1}\int_0^\tau g(s)\,ds, \qquad g \in \mathfrak{M}^+(0,\infty), \quad t \in (0,\infty). \tag{1.5}$$

Such a characterization was obtained in [6] for the particular case when $1 < p \le q < \infty$ and in [16, Theorem 2.10] in the case of more general operators and for extended range of p and q. Full proofs and some further extensions and applications can be found in [9].

The operator T_γ is a typical example of what we call *a Hardy-type operator involving suprema*. Rather interestingly, such operators have been recently encountered in various research projects. They have been found indispensable in the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds ([12]). They constitute a very useful tool for characterization of the associate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding ([18, 19]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance for example in [8, 10, 7, 20].

In this paper we investigate the behaviour of general two-weight Hardy-type operators involving supremum on weighted Lebesgue spaces and on classical Lorentz spaces. The operator T_γ, defined in (1.5), is a particular example of such operators. We cover all cases of parameters p, q, including the case $q < p$ which has not been treated so far and which requires a new method.

The paper is structured as follows. In Section 2 we define several auxiliary operators and prove certain relations between them. In Section 3 we characterize the validity of weighted inequalities for supremum operators on monotone functions, and in Section 4 on nonnegative functions.

2. Preliminaries

Definition 2.1 Let u be a continuous weight on $(0,\infty)$. We define the operator R_u at $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$ by

$$(R_u\varphi)(t) = \sup_{t\le\tau<\infty} u(\tau)\varphi(\tau), \quad t\in(0,\infty).$$

Let b be a weight and let $B(t) = \int_0^t b(s)\,ds$, $t\in(0,\infty)$. Assume that b is such that $0 < B(t) < \infty$ for every $t\in(0,\infty)$. The operator $T_{u,b}$ is defined at $g\in\mathfrak{M}^+(0,\infty)$ by

$$(T_{u,b}g)(t) = \sup_{t\le\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(s)b(s)\,ds, \quad t\in(0,\infty).$$

We also make use of the weighted Hardy operator

$$(P_{u,b}g)(t) = \frac{u(t)}{B(t)} \int_0^t g(s)b(s)\,ds, \quad t\in(0,\infty).$$

We start with a simple lemma which enables us to restrict our considerations to special weights u, namely to those for which $\frac{u}{B}$ is non-increasing. For this purpose, we put

$$\bar{u}(t) = B(t) \sup_{t\le\tau<\infty} \frac{u(\tau)}{B(\tau)}, \quad t\in(0,\infty).$$

Note that $\frac{\bar{u}}{B}$ is non-increasing on $(0,\infty)$. In the second part of the lemma we develop a principle which (under certain assumptions) reduces the inequalities involving $T_{u,b}$ to those involving R_u and $P_{u,b}$, which are considerably more manageable operators. This idea was first used in [6].

Lemma 2.2

Let u and b be as in Definition 2.1.

(i) *For every $g\in\mathfrak{M}^+(0,\infty)$ and $t\in(0,\infty)$,*

$$(T_{u,b}g)(t) = (T_{\bar{u},b}g)(t).$$

(ii) *Assume that*

$$\sup_{0<t<\infty} \frac{u(t)}{B(t)} \int_0^t \frac{b(s)}{u(s)}\,ds < \infty. \tag{2.1}$$

Then, for all $\varphi\in\mathfrak{M}^+(0,\infty;\downarrow)$,

$$(T_{u,b}\varphi)(t) \approx (R_u\varphi)(t) + (P_{\bar{u},b}\varphi)(t), \quad t\in(0,\infty).$$

Proof. (i) Fix $g\in\mathfrak{M}^+(0,\infty)$ and $t\in(0,\infty)$. Then, interchanging the suprema, we get

$$\begin{aligned}(T_{\bar{u},b}g)(t) &= \sup_{t\le\tau<\infty} \sup_{\tau\le y<\infty} \frac{u(y)}{B(y)} \int_0^\tau g(s)b(s)\,ds \\ &= \sup_{t\le y<\infty} \frac{u(y)}{B(y)} \sup_{t\le\tau\le y} \int_0^\tau g(s)b(s)\,ds = (T_{u,b}g)(t).\end{aligned}$$

(ii) If $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$, then

$$\frac{1}{B(t)}\int_0^t \varphi(s)b(s)\,ds \geq \varphi(t) \quad \text{for all} \quad t\in(0,\infty),$$

whence

$$(T_{u,b}\varphi)(t) \geq (R_u\varphi)(t) \quad \text{for every } t\in(0,\infty).$$

Moreover, by (i),

$$(T_{u,b}\varphi)(t) = (T_{\bar{u},b}\varphi)(t) \geq (P_{\bar{u},b}\varphi)(t) \quad \text{for every } t\in(0,\infty).$$

Conversely, using the monotonicity of $R_u\varphi$ and (2.1), we have

$$\begin{aligned}
(T_{u,b}\varphi)(t) &= \sup_{t\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\left(\int_0^t \varphi(s)b(s)\,ds + \int_t^\tau \varphi(s)b(s)\,ds\right)\\
&\leq \frac{\bar{u}(t)}{B(t)}\int_0^t \varphi(s)b(s)\,ds + \sup_{t\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\int_t^\tau \varphi(s)b(s)\,ds\\
&= (P_{\bar{u},b}\varphi)(t) + \sup_{t\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\int_t^\tau u(s)\varphi(s)\frac{b(s)}{u(s)}\,ds\\
&\leq (P_{\bar{u},b}\varphi)(t) + (R_u\varphi)(t)\cdot\sup_{t\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\int_t^\tau \frac{b(s)}{u(s)}\,ds\\
&\lesssim (P_{\bar{u},b}\varphi)(t) + (R_u\varphi)(t),
\end{aligned}$$

and the proof is complete. □

We will be particularly interested in the situation when (2.1) is valid with u replaced by $\bar{u}$. It is easy to show that this is equivalent to

$$\sup_{0<t<\infty}\frac{u(t)}{B(t)}\int_0^t \frac{b(s)}{\bar{u}(s)}\,ds < \infty. \tag{2.2}$$

3. Inequalities for non-increasing functions

Let $0<p,q<\infty$ and let u be a continuous weight. The first main aim of this section is to give a characterization of weights v and w such that the inequality

$$\left(\int_0^\infty [(R_u\varphi)(t)]^q\,w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [\varphi(t)]^p\,v(t)\,dt\right)^{1/p} \tag{3.1}$$

holds for all $\varphi\in\mathfrak{M}^+(0,\infty;\downarrow)$. The result is most innovative when $0<q<p<\infty$. We also present a simpler characterization in this case provided that u is equivalent to a non-decreasing function on $(0,\infty)$. We are led to considering such a special case by the important examples when $u(t)=t^\alpha$ with $\alpha\in(0,1)$ (cf. [6, 12]) or $u(t)=t^\alpha|\log t|^\beta$ with $\alpha\in(0,1)$ and $\beta\in\mathbb{R}$ (cf. [16, 9]).

DEFINITION 3.1 Let $\{x_k\}_{k=-\infty}^{\infty}$ be an increasing sequence in $(0,\infty)$ such that $\lim_{k\to-\infty} x_k = 0$ and $\lim_{k\to\infty} x_k = \infty$. Then we say that $\{x_k\}$ is a *covering sequence*. We also admit increasing sequences $\{x_k\}_{k=J}^{k=K}$, where either $J \in \mathbb{Z}$ and $x_J = 0$, or $K \in \mathbb{Z}$ and $x_K = \infty$, or both.

Theorem 3.2

Let $0 < p, q < \infty$ and let u be a continuous weight. Let v and w be weights such that $0 < \int_0^x v(t)\,dt < \infty$ and $0 < \int_0^x w(t)\,dt < \infty$ for every $x \in (0,\infty)$. When $q < p$, we define r by

$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p}. \tag{3.2}$$

(i) *Let $0 < p \le q < \infty$. Then (3.1) is satisfied for all $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$ if and only if*

$$\left(\int_0^x \Big[\sup_{t\le\tau\le x} u(\tau)\Big]^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^x v(t)\,dt\right)^{1/p} \quad \textit{for every } x \in (0,\infty). \tag{3.3}$$

(ii) *Let $0 < q < p < \infty$. Then (3.1) is satisfied for all $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$ if and only if*

$$\sup_{\{x_k\}} \sum_k \left(\int_{x_k}^{x_{k+1}} \Big[\sup_{t\le\tau\le x_{k+1}} u(\tau)\Big]^q w(t)\,dt\right)^{r/q} \left(\int_0^{x_{k+1}} v(t)\,dt\right)^{-r/p} < \infty, \tag{3.4}$$

where the supremum is taken over all covering sequences $\{x_k\}$.

Proof. ***Sufficiency:*** We will restrict ourselves to the case when $\int_0^\infty w(s)\,ds = \infty$ and $\int_0^\infty v(s)\,ds = \infty$. In the other cases the proof needs only simple modifications. We may choose covering sequences $\{x_k\}_{k\in\mathbb{Z}}$ and $\{y'_s\}_{s\in\mathbb{Z}}$ such that $\int_0^{x_k} w(t)\,dt = 2^k$ and $\int_0^{y'_s} v(t)\,dt = 2^s$. Then, since $R_u\varphi$ is nonincreasing on $(0,\infty)$,

$$\begin{aligned}
\int_0^\infty [(R_u\varphi)(t)]^q\, w(t)\,dt &\le \sum_{k=-\infty}^{\infty} [(R_u\varphi)(x_k)]^q \int_{x_k}^{x_{k+1}} w(t)\,dt \\
&= \sum_{k=-\infty}^{\infty} 2^k \sup_{i\ge k} \sup_{x_i\le\tau<x_{i+1}} [u(\tau)\varphi(\tau)]^q \\
&\le \sum_{k=-\infty}^{\infty} 2^k \sum_{i=k}^{\infty} \sup_{x_i\le\tau<x_{i+1}} [u(\tau)\varphi(\tau)]^q \\
&= \sum_{i=-\infty}^{\infty} \sup_{x_i\le\tau<x_{i+1}} [u(\tau)\varphi(\tau)]^q \sum_{k=-\infty}^{i} 2^k \\
&= \sum_{i=-\infty}^{\infty} 2^{i+1} \sup_{x_i\le\tau<x_{i+1}} [u(\tau)\varphi(\tau)]^q \\
&= 4 \sum_{i=-\infty}^{\infty} \left(\int_{x_{i-1}}^{x_i} w(t)\,dt\right) \sup_{x_i\le\tau<x_{i+1}} [u(\tau)\varphi(\tau)]^q \\
&\lesssim \sum_{i=-\infty}^{\infty} \left(\int_{x_{i-1}}^{x_i} w(t)\,dt\right) [u(z_i)\varphi(z_i)]^q\,,
\end{aligned}$$

where z_i is some point in $[x_i, x_{i+1}]$ such that

$$\sup_{x_i \le \tau < x_{i+1}} [u(\tau)\varphi(\tau)]^q \le 2\,[u(z_i)\varphi(z_i)]^q \,.$$

Thus,

$$\int_0^\infty [(R_u\varphi)(t)]^q\, w(t)\,dt \lesssim \sum_{i=-\infty}^{\infty} \left(\int_{z_{i-2}}^{z_i} w(t)\,dt \right) [u(z_i)\varphi(z_i)]^q \,.$$

For a technical reason, which will be apparent soon, it is convenient to write the last sum as

$$\sum_{k=-\infty}^{\infty} \left(\int_{z_{2k-2}}^{z_{2k}} w(t)\,dt \right) [u(z_{2k})\varphi(z_{2k})]^q + \sum_{k=-\infty}^{\infty} \left(\int_{z_{2k-1}}^{z_{2k+1}} w(t)\,dt \right) [u(z_{2k+1})\varphi(z_{2k+1})]^q$$
$$=: S_{\text{even}} + S_{\text{odd}}.$$

We shall estimate S_{even}. First, we reduce the sequence $\{y'_s\}$ according to a principle similar to that which was introduced by Q. Lai in [13]: Fix $k \in \mathbb{Z}$. If the interval $[z_{2k-2}, z_{2k})$ contains more than one element of the sequence $\{y'_s\}$, we delete from this sequence all such elements but the one which lies nearest to z_{2k-2}. Thus, every interval $[z_{2k-2}, z_{2k})$, $k \in \mathbb{Z}$, now contains at most one element of the reduced sequence, which we denote by $\{y_n\}_{n\in\mathbb{Z}}$. Formally, we denote

$$\begin{aligned} Y_k &= \left\{ s \in \mathbb{Z};\ y'_s \in [z_{2k-2}, z_{2k}) \right\}, \quad k \in \mathbb{Z}, \\ J &= \{ k \in \mathbb{Z};\ Y_k \ne \emptyset \}, \\ \theta_k &= \min \left\{ y'_s;\ s \in Y_k \right\}, \quad k \in J, \end{aligned}$$

and

$$Y = \{\theta_k\}_{k\in J} \,.$$

Then Y is a subsequence of $\{y'_s\}_{s\in\mathbb{Z}}$, which we enumerate as $\{y_n\}_{n\in\mathbb{Z}}$. Clearly, $y_n < y_{n+1}$ for all $n \in \mathbb{Z}$. Now, $\{y_n\}$ is a covering sequence having the following properties: Suppose that for some $n, k, s \in \mathbb{Z}$ we have

$$y_n < z_{2k} \le y_{n+1} = y'_s. \tag{3.5}$$

We claim that then, necessarily,

$$y_{n-1} \le y'_{s-2}, \tag{3.6}$$
$$y'_{s-1} < z_{2k}, \tag{3.7}$$

and

$$y_{n-1} < z_{2k-2}. \tag{3.8}$$

Indeed, as $\{y_n\}_{n\in\mathbb{Z}}$ is a subsequence of $\{y'_s\}_{s\in\mathbb{Z}}$ and $y_{n+1} = y'_s$, it follows that $y_n \le y'_{s-1}$ and $y_{n-1} \le y'_{s-2}$. This yields (3.6). Next, since $y'_s = \theta_{k+1}$, we have $\{y'_j\}_{j\in\mathbb{Z}} \cap [z_{2k}, y'_s) = \emptyset$, which yields (3.7). Finally, the set $Y \cap [z_{2k-2}, y'_s)$ is either empty (in which case (3.8) is obvious) or has just one element, y_n, in which case (3.8) follows again. By (3.6) and (3.7), for all $n, k, s \in \mathbb{Z}$ satisfying (3.5),

$$\int_0^{y_{n+1}} v(t)\,dt = 4 \int_{y'_{s-2}}^{y'_{s-1}} v(t)\,dt \le 4 \int_{y_{n-1}}^{z_{2k}} v(t)\,dt.$$

Together with the assumptions on v, this yields that

$$\int_{y_{n-1}}^{z_{2k}} v(t)\,dt > 0.$$

Using this and the monotonicity of φ, we get

$$\begin{aligned}[\varphi(z_{2k})]^p &\le \left(\int_{y_{n-1}}^{z_{2k}} v(t)\,dt\right)^{-1} \int_{y_{n-1}}^{z_{2k}} [\varphi(t)]^p\,v(t)\,dt \\ &\le 4\left(\int_0^{y_{n+1}} v(t)\,dt\right)^{-1} \int_{y_{n-1}}^{y_{n+1}} [\varphi(t)]^p\,v(t)\,dt.\end{aligned}$$

Hence,

$$[\varphi(z_{2k})]^q \lesssim \left(\int_0^{y_{n+1}} v(t)\,dt\right)^{-q/p} \left(\int_{y_{n-1}}^{y_{n+1}} [\varphi(t)]^p\,v(t)\,dt\right)^{q/p} \tag{3.9}$$

for every $n, k \in \mathbb{Z}$ such that $y_n < z_{2k} \le y_{n+1}$.

Denote $A_n = \{k \in \mathbb{Z};\ y_n < z_{2k} \le y_{n+1}\}$, $n \in \mathbb{Z}$. Then, by (3.9), (3.8) and (3.5),

$$\begin{aligned}S_{\text{even}} &= \sum_{n=-\infty}^{\infty} \sum_{k\in A_n} \left(\int_{z_{2k-2}}^{z_{2k}} w(t)\,dt\right) [u(z_{2k})\varphi(z_{2k})]^q \\ &\lesssim \sum_{n=-\infty}^{\infty} \sum_{k\in A_n} \int_{z_{2k-2}}^{z_{2k}} \big[\sup_{t\le x\le y_{n+1}} u(x)\big]^q w(t)\,dt \\ &\quad\times \left(\int_0^{y_{n+1}} v(t)\,dt\right)^{-q/p} \left(\int_{y_{n-1}}^{y_{n+1}} [\varphi(t)]^p\,v(t)\,dt\right)^{q/p} \\ &\le \sum_{n=-\infty}^{\infty} \int_{y_{n-1}}^{y_{n+1}} \big[\sup_{t\le x\le y_{n+1}} u(x)\big]^q w(t)\,dt \\ &\quad\times \left(\int_0^{y_{n+1}} v(t)\,dt\right)^{-q/p} \left(\int_{y_{n-1}}^{y_{n+1}} [\varphi(t)]^p\,v(t)\,dt\right)^{q/p}.\end{aligned} \tag{3.10}$$

Assume first that $0 < p \le q < \infty$. Then (3.10), (3.3) and the convexity of the function $t^{q/p}$ yield

$$\begin{aligned}S_{\text{even}} &\lesssim \sum_{n=-\infty}^{\infty} \left(\int_{y_{n-1}}^{y_{n+1}} [\varphi(t)]^p\,v(t)\,dt\right)^{q/p} \\ &\lesssim \left(\sum_{n=-\infty}^{\infty} \int_{y_{n-1}}^{y_{n+1}} [\varphi(t)]^p\,v(t)\,dt\right)^{q/p} \lesssim \left(\int_0^{\infty} [\varphi(t)]^p\,v(t)\,dt\right)^{q/p}.\end{aligned}$$

When $0 < q < p < \infty$, we apply Hölder's inequality for sums with exponents $\frac{r}{q}$ and $\frac{p}{q}$ to (3.10), and then use (3.4) to get

$$
\begin{aligned}
S_{\text{even}} &\lesssim \left[\sum_{n=-\infty}^{\infty} \left(\int_{y_{n-1}}^{y_{n+1}} [\sup_{t\le\tau\le y_{n+1}} u(\tau)]^q w(t)\,dt \right)^{r/q} \left(\int_0^{y_{n+1}} v(t)\,dt \right)^{-r/p} \right]^{q/r} \\
&\quad \times \left(\sum_{n=-\infty}^{\infty} \int_{y_{n-1}}^{y_{n+1}} [\varphi(t)]^p\, v(t)\,dt \right)^{q/p} \\
&= \Bigg[\sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-2}}^{y_{2m}} [\sup_{t\le\tau\le y_{2m}} u(\tau)]^q\, w(t)\,dt \right)^{r/q} \left(\int_0^{y_{2m}} v(t)\,dt \right)^{-r/p} \\
&\quad + \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-1}}^{y_{2m+1}} [\sup_{t\le\tau\le y_{2m+1}} u(\tau)]^q\, w(t)\,dt \right)^{r/q} \left(\int_0^{y_{2m+1}} v(t)\,dt \right)^{-r/p} \Bigg]^{q/r} \\
&\quad \times \left(\sum_{n=-\infty}^{\infty} \int_{y_{n-1}}^{y_{n+1}} [\varphi(t)]^p\, v(t)\,dt \right)^{q/p} \\
&\lesssim \left(\int_0^{\infty} [\varphi(t)]^p\, v(t)\,dt \right)^{q/p}
\end{aligned}
$$

since both $\{y_{2m}\}_{m\in\mathbb{Z}}$ and $\{y_{2m-1}\}_{m\in\mathbb{Z}}$ are covering sequences.

It is clear from the argument that we can estimate S_{odd} in the same way, with possibly different sequence $\{y_n\}_{n\in\mathbb{Z}}$. This completes the proof of the sufficiency part.

Necessity: We first observe that

$$
\left(R_u \chi_{(0,x]}\right)(t) = \chi_{(0,x]}(t) \sup_{t\le\tau\le x} u(\tau), \qquad x \in (0,\infty).
$$

Thus, testing the inequality (3.1) with functions $\varphi(t) = \chi_{(0,x]}(t)$, $x \in (0,\infty)$, we get (3.3) for any $p, q \in (0,\infty)$. When $0 < p \le q < \infty$, this proves the necessity part of the theorem.

Let $0 < q < p < \infty$ and let $\{x_k\}$ be a fixed covering sequence. For $N \in \mathbb{N}$ and $t \in (0,\infty)$, put

$$
\varphi_N(t) = \chi_{(0,x_{-N}]}(t) \left(\sum_{i=-N}^{N} \alpha_i \right)^{1/p} + \sum_{k=-N}^{N} \chi_{(x_k,x_{k+1}]}(t) \left(\sum_{i=k}^{N} \alpha_i \right)^{1/p},
$$

where

$$
\alpha_i = \left(\int_{x_i}^{x_{i+1}} [\sup_{t\le\tau\le x_{i+1}} u(\tau)]^q\, w(t)\,dt \right)^{r/q} \left(\int_0^{x_{i+1}} v(t)\,dt \right)^{-r/q}.
$$

Then φ_N is non-increasing on $(0,\infty)$ and

$$\begin{aligned}
&\int_0^\infty [(R_u\varphi_N)(t)]^q\, w(t)\, dt \\
&\geq \sum_{k=-N}^{N} \int_{x_k}^{x_{k+1}} \big[\sup_{t\leq\tau\leq x_{k+1}} u(\tau)\big]^q w(t)\, dt \left(\sum_{i=k}^{N} \alpha_i\right)^{q/p} \\
&\geq \sum_{k=-N}^{N} \left(\int_{x_k}^{x_{k+1}} \big[\sup_{t\leq\tau\leq x_{k+1}} u(\tau)\big]^q w(t)\, dt\right) \alpha_k^{q/p} \\
&= \sum_{k=-N}^{N} \left(\int_{x_k}^{x_{k+1}} \big[\sup_{t\leq\tau\leq x_{k+1}} u(\tau)\big]^q w(t)\, dt\right)^{r/q} \left(\int_0^{x_{k+1}} v(t)\, dt\right)^{-r/p} =: A_N
\end{aligned}$$

since $1+\frac{r}{p}=\frac{r}{q}$. On the other hand,

$$\begin{aligned}
\int_0^\infty [\varphi_N(t)]^p\, v(t)\, dt &= \int_0^{x_{-N}} [\varphi_N(t)]^p\, v(t)\, dt + \sum_{k=-N}^{N} \int_{x_k}^{x_{k+1}} [\varphi_N(t)]^p\, v(t)\, dt \\
&= \int_0^{x_{-N}} \left(\sum_{i=-N}^{N} \alpha_i\right) v(t)\, dt + \sum_{k=-N}^{N} \int_{x_k}^{x_{k+1}} \left(\sum_{i=k}^{N} \alpha_i\right) v(t)\, dt \\
&= \int_0^{x_{-N}} \left(\sum_{i=-N}^{N} \alpha_i\right) v(t)\, dt + \sum_{i=-N}^{N} \alpha_i \sum_{k=-N}^{i} \int_{x_k}^{x_{k+1}} v(t)\, dt \\
&= \sum_{i=-N}^{N} \alpha_i \int_0^{x_{-N}} v(t)\, dt + \sum_{i=-N}^{N} \alpha_i \int_{x_{-N}}^{x_{i+1}} v(t)\, dt \\
&= \sum_{i=-N}^{N} \alpha_i \int_0^{x_{i+1}} v(t)\, dt \\
&= \sum_{i=-N}^{N} \left(\int_{x_i}^{x_{i+1}} \big[\sup_{t\leq\tau\leq x_{i+1}} u(\tau)\big]^q w(t)\, dt\right)^{r/q} \left(\int_0^{x_{i+1}} v(t)\, dt\right)^{-r/q} \int_0^{x_{i+1}} v(t)\, dt \\
&= \sum_{i=-N}^{N} \left(\int_{x_i}^{x_{i+1}} \big[\sup_{t\leq\tau\leq x_{i+1}} u(\tau)\big]^q w(t)\, dt\right)^{r/q} \left(\int_0^{x_{i+1}} v(t)\, dt\right)^{-r/p} = A_N.
\end{aligned}$$

Consequently, (3.1) implies

$$A_N^{1/q} \lesssim A_N^{1/p}. \tag{3.11}$$

As mentioned above, (3.3) is necessary for the validity of (3.1) also when $0<q<p$. Therefore, for every $k\in\mathbb{Z}$,

$$\left(\int_{x_k}^{x_{k+1}} \big[\sup_{t\leq\tau\leq x_{k+1}} u(\tau)\big]^q w(t)\, dt\right)^{1/q} \left(\int_0^{x_{k+1}} v(t)\, dt\right)^{-1/p} < \infty.$$

Hence also $A_N<\infty$ since it is a finite sum of finite numbers. Thus, by (3.11), $A_N\leq C$ for some $C>0$ independent of $N\in\mathbb{N}$. On letting $N\to\infty$, the assertion follows. □

Remark 3.3 Suppose that u is equivalent to a non-decreasing function on $(0,\infty)$. Then (3.3) reduces to

$$u(x)\left(\int_0^x w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^x v(t)\,dt\right)^{1/p} \quad \text{for every } x\in(0,\infty).$$

As a particular case of this result, we recover [6, Lemma 3.1]. Analogously, (3.4) is simplified to

$$\sup_{\{x_k\}}\sum_k \left[u(x_{k+1})\right]^r \left(\int_{x_k}^{x_{k+1}} w(t)\,dt\right)^{r/q}\left(\int_0^{x_{k+1}} v(t)\,dt\right)^{-r/p} < \infty. \tag{3.12}$$

Moreover, the next theorem shows that in this case (3.4) can be replaced by an integral condition in the spirit of [17, Theorem 1.15].

Theorem 3.4

Let $0<q<p<\infty$ and let u, v, w and r be as in Theorem 3.2. *Moreover, assume that u is equivalent to a non-decreasing function on* $(0,\infty)$. *Then the inequality* (3.1) *holds if and only if*

$$\int_0^\infty \left(\int_0^t w(s)\,ds\right)^{r/p}\left[\sup_{t\le\tau<\infty} u(\tau)\left(\int_0^\tau v(s)\,ds\right)^{-1/p}\right]^r w(t)\,dt<\infty. \tag{3.13}$$

Proof. Again, we will prove the assertion only in the case when $\int_0^\infty w(s)\,ds=\infty$ and $\int_0^\infty v(s)\,ds=\infty$.

As u is equivalent to a non-decreasing function on $(0,\infty)$, Theorem 3.2 and Remark 3.3 imply that it is sufficient to show that (3.13) is equivalent to (3.12). Assume that (3.13) holds. Let $\{x_k\}$ be a covering sequence. Then

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left(\int_{x_k}^{x_{k+1}} w(t)\,dt\right)^{r/q}\left[u(x_{k+1})\right]^r\left(\int_0^{x_{k+1}} v(t)\,dt\right)^{-r/p}\\
&\approx \sum_{k=-\infty}^{\infty}\left(\int_{x_k}^{x_{k+1}}\left(\int_{x_k}^t w(s)\,ds\right)^{r/p} w(t)\,dt\right)\left[u(x_{k+1})\right]^r\left(\int_0^{x_{k+1}} v(t)\,dt\right)^{-r/p}\\
&\le \sum_{k=-\infty}^{\infty}\int_{x_k}^{x_{k+1}}\left(\int_0^t w(s)\,ds\right)^{r/p}\left[\sup_{t\le\tau<\infty} u(\tau)\left(\int_0^\tau v(s)\,ds\right)^{-1/p}\right]^r w(t)\,dt\\
&= \int_0^\infty\left(\int_0^t w(s)\,dt\right)^{r/p}\left[\sup_{t\le\tau<\infty} u(\tau)\left(\int_0^\tau v(s)\,ds\right)^{-1/p}\right]^r w(t)\,dt<\infty,
\end{aligned}$$

and (3.12) follows.

Now, assume that (3.12) holds. Let $\{x_k\}$ be such that $\int_0^{x_k} w(t)\,dt = 2^k$. Then

$$\begin{aligned}
&\int_0^\infty \left(\int_0^t w(s)\,ds\right)^{r/p} \left[\sup_{t\le\tau<\infty} u(\tau)\left(\int_0^\tau v(y)\,dy\right)^{-1/p}\right]^r w(t)\,dt \\
&\le \sum_{k=-\infty}^{\infty} \left(\int_{x_k}^{x_{k+1}} \left(\int_0^t w(s)\,ds\right)^{r/p} w(t)\,dt\right) \left[\sup_{x_k\le\tau<\infty} u(\tau)\left(\int_0^\tau v(y)\,dy\right)^{-1/p}\right]^r \\
&\lesssim \sum_{k=-\infty}^{\infty} 2^{rk/q} \sum_{i=k}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} u(\tau)\left(\int_0^\tau v(y)\,dy\right)^{-1/p}\right]^r \\
&= \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} u(\tau)\left(\int_0^\tau v(y)\,dy\right)^{-1/p}\right]^r \sum_{k=-\infty}^{i} 2^{kr/q} \\
&\approx \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} u(\tau)\left(\int_0^\tau v(t)\,dt\right)^{-1/p}\right]^r 2^{ir/q} \\
&\approx \sum_{i=-\infty}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} u(\tau)\left(\int_0^\tau v(t)\,dt\right)^{-1/p}\right]^r \left(\int_{x_{i-1}}^{x_i} w(t)\,dt\right)^{r/q} \\
&\lesssim \sum_{i=-\infty}^{\infty} \left(\int_{y_{i-1}}^{y_{i+1}} w(t)\,dt\right)^{r/q} [u(y_{i+1})]^r \left(\int_0^{y_{i+1}} v(t)\,dt\right)^{-r/p},
\end{aligned}$$

where $y_{i+1} \in [x_i, x_{i+1})$ is such that

$$\left[\sup_{x_i\le\tau<x_{i+1}} u(\tau)\left(\int_0^\tau v(t)\,dt\right)^{-1/p}\right]^r \le 2\left[u(y_{i+1})\left(\int_0^{y_{i+1}} v(t)\,dt\right)^{-1/p}\right]^r.$$

Therefore,

$$\begin{aligned}
&\int_0^\infty \left(\int_0^t w(s)\,ds\right)^{r/p} \left[\sup_{t\le\tau<\infty} u(\tau)\left(\int_0^\tau v(y)\,dy\right)^{-1/p}\right]^r w(t)\,dt \\
&\quad\lesssim \sum_{i=-\infty}^{\infty} \left(\int_{y_{i-1}}^{y_{i+1}} [\sup_{t\le\tau\le y_{i+1}} u(\tau)]^q\, w(t)\,dt\right)^{r/q} \left(\int_0^{y_{i+1}} v(t)\,dt\right)^{-r/p} \\
&\quad= \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-2}}^{y_{2m}} [\sup_{t\le\tau\le y_{2m}} u(\tau)]^q\, w(t)\,dt\right)^{r/q} \left(\int_0^{y_{2m}} v(t)\,dt\right)^{-r/p} \\
&\qquad+ \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-1}}^{y_{2m+1}} [\sup_{t\le\tau\le y_{2m+1}} u(\tau)]^q\, w(t)\,dt\right)^{r/q} \left(\int_0^{y_{2m+1}} v(t)\,dt\right)^{-r/p} < \infty.
\end{aligned}$$

We note that the monotonicity of u was needed only in the proof of the first part. □

Our next aim is to characterize the validity of the inequality

$$\left(\int_0^\infty [(T_{u,b}\varphi)(t)]^q\, w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [\varphi(t)]^p\, v(t)\,dt\right)^{1/p} \tag{3.14}$$

for all $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$. To this end one can combine Lemma 2.2, Theorem 3.2 and the known results on weighted inequalities for monotone functions due to Sawyer [22], Stepanov [23] and others. For example, when (2.1) is true with u replaced by $\bar{u}$ (that is, when (2.2) is satisfied), then we obtain the following theorem.

As usual, for $1 \le p < \infty$ we put $p' = \frac{p}{p-1}$ when $p > 1$ and $p' = \infty$ when $p = 1$. Moreover, we denote

$$V(t) = \int_0^t v(s)\,ds, \quad t \in (0,\infty).$$

Theorem 3.5

Let $0 < p, q < \infty$. When $q < p$, define r by (3.2). Let b be a weight such that $0 < B(t) < \infty$ for every $t \in (0,\infty)$, where $B(t) = \int_0^t b(s)\,ds$. Let u, v and w be as in Theorem 3.2 *and assume that* (2.2) *is satisfied.*

(i) *Let $1 < p \le q < \infty$. Then* (3.14) *holds on $\mathfrak{M}^+(0,\infty;\downarrow)$ if and only if*

$$\left(\int_0^x [\sup_{t\le\tau\le x} \bar{u}(\tau)]^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^x v(t)\,dt\right)^{1/p} \quad \text{for all } x \in (0,\infty), \tag{3.15}$$

and

$$\sup_{x>0} \left(\int_x^\infty \left(\frac{\bar{u}(t)}{B(t)}\right)^q w(t)\,dt\right)^{1/q} \left(\int_0^x \left(\frac{B(t)}{V(t)}\right)^{p'} v(t)\,dt\right)^{1/p'} < \infty. \tag{3.16}$$

(ii) *Let $0 < p \le q < \infty$ and $0 < p \le 1$. Then* (3.14) *holds on $\mathfrak{M}^+(0,\infty;\downarrow)$ if and only if* (3.15) *is satisfied and*

$$B(x)\left(\int_x^\infty \left(\frac{\bar{u}(t)}{B(t)}\right)^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^x v(t)\,dt\right)^{1/p} \quad \text{for all } x \in (0,\infty). \tag{3.17}$$

(iii) *Let $1 < p < \infty$, $0 < q < p < \infty$ and $q \ne 1$. Then* (3.14) *holds on $\mathfrak{M}^+(0,\infty;\downarrow)$ if and only if*

$$\sup_{\{x_k\}} \sum_k \left(\int_{x_k}^{x_{k+1}} [\sup_{t\le\tau\le x_{k+1}} \bar{u}(\tau)]^q w(t)\,dt\right)^{r/q} \left(\int_0^{x_{k+1}} v(t)\,dt\right)^{-r/p} < \infty, \tag{3.18}$$

where the supremum is taken over all covering sequences, and

$$\int_0^\infty \left(\int_t^\infty \left(\frac{\bar{u}(s)}{B(s)}\right)^q w(s)\,ds\right)^{r/q} \left(\int_0^t \left(\frac{B(s)}{V(s)}\right)^{p'} v(s)\,ds\right)^{r/q'} \times \left(\frac{B(t)}{V(t)}\right)^{p'} v(t)\,dt < \infty. \tag{3.19}$$

(iv) *Let $1 < p < \infty$ and $q = 1$. Then* (3.14) *holds on $\mathfrak{M}^+(0,\infty;\downarrow)$ if and only if* (3.18) *is satisfied and*

$$\left(\int_0^\infty \left(\int_0^t \bar{u}(s)w(s)\,ds + B(t)\int_t^\infty \frac{\bar{u}(s)}{B(s)} w(s)\,ds\right)^{p'} \frac{v(t)}{[V(t)]^{p'}}\,dt\right)^{1/p'} < \infty. \tag{3.20}$$

(v) *Let* $0 < q < p \le 1$. *Then* (3.14) *holds on* $\mathfrak{M}^+(0,\infty;\downarrow)$ *if and only if* (3.18) *is satisfied and*

$$\left(\int_0^\infty \left(\operatorname*{ess\,sup}_{0<s\le t} \frac{B(s)}{[V(s)]^{1/p}}\right)^r \left(\int_t^\infty \left(\frac{\bar{u}(s)}{B(s)}\right)^q w(s)\,ds\right)^{r/p} \tag{3.21}$$

$$\times \left(\frac{\bar{u}(t)}{B(t)}\right)^q w(t)\,dt\right)^{1/r} < \infty. \tag{3.22}$$

Proof. All the four statements of the theorem are proved along the same line of argument: first, by Lemma 2.2 (i), $T_{\bar{u},b} = T_{u,b}$. Moreover, $\bar{\bar{u}} = \bar{u}$. Hence, by Lemma 2.2 (ii), the inequality (3.14) holds on $\mathfrak{M}^+(0,\infty;\downarrow)$ if and only if both inequalities

$$\left(\int_0^\infty [(R_{\bar{u}}\varphi)(t)]^q\, w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [\varphi(t)]^p\, v(t)\,dt\right)^{1/p} \tag{3.23}$$

and

$$\left(\int_0^\infty [(P_{\bar{u},b}\varphi)(t)]^q\, w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [\varphi(t)]^p\, v(t)\,dt\right)^{1/p} \tag{3.24}$$

are satisfied on $\mathfrak{M}^+(0,\infty;\downarrow)$. Now, necessary and sufficient conditions for (3.23) were given in Theorem 3.2. On the other hand, $P_{\bar{u},b}$ is just a weighted Hardy-type operator. Thus, necessary and sufficient conditions for (3.24) follow from known general criteria. These are in all cases formulated in form of a pair of conditions, one of which is always covered by the condition characterizing (3.23). Let us be more precise.

In the case (i), it follows from Theorem 3.2 that (3.23) holds if and only if (3.15) is satisfied. Next, by [22, Theorem 1] and criteria for the validity of Hardy-type inequalities on $\mathfrak{M}^+(0,\infty)$, cf. e.g. [17], (3.24) holds if and only if both (3.16) and

$$\left(\int_0^x [\bar{u}(t)]^q\, w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^x v(t)\,dt\right)^{1/p}, \quad x \in (0,\infty), \tag{3.25}$$

are satisfied. However, (3.25) clearly follows from (3.15).

In the case (ii), (3.23) holds if and only if (3.15) is satisfied, while (3.24) holds if and only if both (3.17) and (3.25) hold (this can be found in [5, Theorem 2.4] and a particular case in [14, Theorem 2.2]). Again, (3.25) follows from (3.15).

In the case (iii), (3.23) holds if and only if (3.18) is satisfied, while (3.24) holds if and only if both (3.19) and

$$\left(\int_0^\infty \left(\int_0^t [\bar{u}(s)]^q\, w(s)\,ds\right)^{r/p} [V(t)]^{-r/p}\, [\bar{u}(t)]^q\, w(t)\,dt\right)^{1/r} < \infty \tag{3.26}$$

are satisfied. (This can be proved as in [11, Corollary 4.7], for a particular case see also [23, Theorem 3a]). We now claim that (3.18) implies (3.26). Indeed, by Theorem 3.2, (3.18) is equivalent to (3.23), which in turn yields

$$\left(\int_0^\infty [\varphi(t)]^q\, [\bar{u}(t)]^q\; w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [\varphi(t)]^p\, v(t)\,dt\right)^{1/p} \tag{3.27}$$

for all $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$. The last inequality is just an embedding of classical Lorentz spaces, and the criteria for such embedding, given in [22, Remark on p. 148], and in [23, Proposition 1] (see also [4, Theorem 3.1]) show that (3.26) is equivalent to (3.27).

In the case (iv), (3.23) holds if and only if (3.18) is satisfied. Since $q = 1$, we have $r = \frac{r}{q} = p'$ and $\frac{r}{p} = p' - 1$. Moreover, by the Fubini theorem, (3.24) is equivalent to

$$\int_0^\infty \varphi(t)b(t)\int_t^\infty \frac{\bar{u}(s)w(s)}{B(s)}\,ds\,dt \lesssim \left(\int_0^\infty [\varphi(t)]^p v(t)\,dt\right)^{1/p}.$$

This is a particular case of an embedding between classical Lorentz spaces. By [23, Proposition 1], this is true if and only if (3.20) is true and

$$\left(\int_0^\infty \bar{u}(t)w(t)\,dt\right)^{p'}\left(\int_0^\infty v(s)\,ds\right)^{1-p'} < \infty.$$

However, the latter estimate is a special case of (3.18).

In the case (v), (3.23) holds if and only if (3.18) is satisfied. By [3, Theorem 3.1], (3.24) holds if and only if both (3.21) and (3.26) are true. However, as shown above in the proof of the case (iii), (3.26) follows from (3.18). □

Remark 3.6 As in Theorem 3.4, in the case when $\bar{u}$ is equivalent to a non-decreasing function on $(0,\infty)$, the condition (3.18) can be replaced by the integral condition

$$\int_0^\infty \left(\int_0^t w(s)\,ds\right)^{r/p}\left[\sup_{t\le\tau<\infty}\bar{u}(\tau)\left(\int_0^\tau v(s)\,ds\right)^{-1/p}\right]^r w(t)\,dt < \infty.$$

Remark 3.7 Theorem 3.5 with slightly modified assumptions can also be proved using a reduction theorem from [11]. Note that Theorem 6.2 in [11] states that, for $1 < p < \infty$ and $0 < q < \infty$, the inequality (3.14) holds for all $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$ if and only if (3.1) holds for all $\varphi \in \mathfrak{M}^+(0,\infty;\downarrow)$ and simultaneously the estimate

$$\left(\int_0^\infty [(T_{u,b}g)(t)]^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [g(t)]^p \left(\frac{V(t)}{B(t)}\right)^p [v(t)]^{1-p}\,dt\right)^{1/p}$$

is satisfied for all $g \in \mathfrak{M}^+(0,\infty)$. This motivates us to study when the inequality

$$\left(\int_0^\infty [(T_{u,b}g)(t)]^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [g(t)]^p v(t)\,dt\right)^{1/p} \tag{3.28}$$

holds for all $g \in \mathfrak{M}^+(0,\infty)$. This will be done in the next section.

4. Inequalities for nonnegative functions

In this section we study the same weighted inequalities as above but without the restriction to non-increasing functions. In particular, we wish to establish when the inequality (3.28) holds for all $g \in \mathfrak{M}^+(0,\infty)$.

Recall that if $p \in (1,\infty)$, then the conjugate number p' is given by $p' = \frac{p}{p-1}$. We shall also use the following notation.

Notation. For a given weight v, $0 \le \alpha < \beta \le \infty$ and $1 \le p < \infty$, we denote

$$\sigma_p(\alpha,\beta) = \begin{cases} \left(\int_\alpha^\beta [v(t)]^{1-p'}\,dt\right)^{1/p'} & \text{when } 1 < p < \infty \\ \operatorname*{ess\,sup}_{\alpha<t<\beta} \frac{1}{v(t)} & \text{when } p = 1. \end{cases}$$

We first consider the particular case when $b \equiv 1$. Then we write $T_u := T_{u,b}$ and since $B(t) = t$, $t \in (0,\infty)$, we have

$$T_u g(t) = \sup_{t \le s < \infty} \frac{u(s)}{s} \int_0^s g(y)\,dy.$$

We will characterize the validity of the inequality

$$\left(\int_0^\infty [(T_u g)(t)]^q\,w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty [g(t)]^p\,v(t)\,dt\right)^{1/p} \tag{4.1}$$

on the set $\mathfrak{M}^+(0,\infty)$.

Theorem 4.1

Assume that $1 \le p < \infty$ and $0 < q < \infty$. When $q < p$, we define r by (3.2). *Let u, v and w be as in* Theorem 3.2.

(i) *Let $p \le q$. Then* (4.1) *holds on $\mathfrak{M}^+(0,\infty)$ if and only if*

$$\sup_{x>0} \left(\left(\frac{\bar{u}(x)}{x}\right)^q \int_0^x w(t)\,dt + \int_x^\infty \left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt\right)^{1/q} \sigma_p(0,x) < \infty. \tag{4.2}$$

(ii) *Let $q < p$. Then* (4.1) *holds on $\mathfrak{M}^+(0,\infty)$ if and only if*

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k-1}}^{x_{k+1}} \min\left\{\frac{\bar{u}(x_k)}{x_k}, \frac{\bar{u}(t)}{t}\right\}^q w(t)\,dt\right)^{r/q} [\sigma_p(x_{k-1},x_k)]^r\right)^{1/r} < \infty, \tag{4.3}$$

where the supremum is taken over all covering sequences $\{x_k\}$.

Proof. Sufficiency: By (4.2), $\int_0^x w(t)\,dt < \infty$ for every $x \in (0,\infty)$. As in the proofs above, we may assume with no loss of generality that $\int_0^\infty w(t)\,dt = \infty$. Thus, there is a covering sequence $\{x_k\}_{k\in\mathbb{Z}}$ such that $\int_0^{x_k} w(t)\,dt = 2^k$. Then

$$
\begin{aligned}
\int_0^\infty [(T_u g)(t)]^q\, w(t)\, dt &\le \sum_{k=-\infty}^{\infty} \left(\int_{x_k}^{x_{k+1}} w(t)\, dt \right) \left(\sup_{x_k \le \tau < \infty} \frac{u(\tau)}{\tau} \int_0^\tau g(s)\, ds \right)^q \\
&= \sum_{k=-\infty}^{\infty} 2^k \left(\sup_{k \le i < \infty} \sup_{x_i \le \tau < x_{i+1}} \frac{u(\tau)}{\tau} \int_0^\tau g(s)\, ds \right)^q \\
&\le \sum_{k=-\infty}^{\infty} 2^k \sum_{i=k}^{\infty} \sup_{x_i \le \tau < x_{i+1}} \left(\frac{u(\tau)}{\tau} \int_0^\tau g(s)\, ds \right)^q \\
&= \sum_{i=-\infty}^{\infty} \sup_{x_i \le \tau < x_{i+1}} \left(\frac{u(\tau)}{\tau} \int_0^\tau g(s)\, ds \right)^q \sum_{k=-\infty}^{i} 2^k \\
&= \sum_{i=-\infty}^{\infty} \sup_{x_i \le \tau < x_{i+1}} \left(\frac{u(\tau)}{\tau} \int_0^\tau g(s)\, ds \right)^q 2^{i+1} \\
&\approx \sum_{i=-\infty}^{\infty} \left(\int_{x_{i-1}}^{x_i} w(t)\, dt \right) \sup_{x_i \le \tau < x_{i+1}} \left(\frac{u(\tau)}{\tau} \int_0^\tau g(s)\, ds \right)^q \\
&\lesssim \sum_{i=-\infty}^{\infty} \left(\int_{x_{i-1}}^{x_i} w(t)\, dt \right) \left(\frac{u(z_i)}{z_i} \int_0^{z_i} g(s)\, ds \right)^q,
\end{aligned}
$$

where z_i is some point in $[x_i, x_{i+1})$ such that

$$
\sup_{x_i \le \tau < x_{i+1}} \left(\frac{u(\tau)}{\tau} \int_0^\tau g(s)\, ds \right)^q \le 2 \left(\frac{u(z_i)}{z_i} \int_0^{z_i} g(s)\, ds \right)^q.
$$

Together with the inclusion $(x_{i-1}, x_i) \subset (z_{i-2}, z_i)$, this implies that

$$
\begin{aligned}
\int_0^\infty [(T_u g)(t)]^q\, w(t)\, dt &\lesssim \sum_{i=-\infty}^{\infty} \left(\int_{z_{i-2}}^{z_i} w(t)\, dt \right) \left(\frac{u(z_i)}{z_i} \int_0^{z_i} g(s)\, ds \right)^q \\
&\approx \sum_{i=-\infty}^{\infty} \left(\int_{z_{i-2}}^{z_i} w(t)\, dt \right) \left(\frac{u(z_i)}{z_i} \int_{z_{i-2}}^{z_i} g(s)\, ds \right)^q \\
&\quad + \sum_{i=-\infty}^{\infty} \left(\int_{z_{i-2}}^{z_i} w(t)\, dt \right) \left(\frac{u(z_i)}{z_i} \int_0^{z_{i-2}} g(s)\, ds \right)^q \\
&=: I_1 + I_2.
\end{aligned}
$$

Now, by the Hölder inequality,

$$
I_1 \le \sum_{i=-\infty}^{\infty} \left(\int_{z_{i-2}}^{z_i} w(t)\, dt \right) \left(\frac{u(z_i)}{z_i} \sigma_p(z_{i-2}, z_i) \right)^q \left(\int_{z_{i-2}}^{z_i} [g(y)]^p v(y)\, dy \right)^{q/p}. \qquad (4.4)
$$

Thus, when $p \le q$, we get by (4.2) that

$$
I_1 \lesssim \sum_{i=-\infty}^{\infty} \left(\int_{z_{i-2}}^{z_i} [g(y)]^p v(y)\, dy \right)^{q/p} \lesssim \left(\int_0^\infty [g(y)]^p v(y)\, dy \right)^{q/p}.
$$

When $q < p$, we use the Hölder inequality for sums with exponents $\frac{p}{q}$ and $\frac{r}{q}$ in (4.4) to get

$$I_1 \lesssim \left(\sum_{i=-\infty}^{\infty} \left(\int_{z_{i-2}}^{z_i} w(t)\, dt \right)^{r/q} \left(\frac{u(z_i)}{z_i} \right)^r [\sigma_p(z_{i-2}, z_i)]^r \right)^{(p-q)/p} \times \left(\sum_{i=-\infty}^{\infty} \int_{z_{i-2}}^{z_i} [g(y)]^p v(y)\, dy \right)^{q/p}.$$

Now, we make use of the facts that $\{z_{2m}\}_{m\in\mathbb{Z}}$ and $\{z_{2m+1}\}_{m\in\mathbb{Z}}$ are covering sequences and that $u \le \bar{u}$ and $\frac{\bar{u}(t)}{t}$ is non-increasing on $(0, \infty)$. Therefore, writing the first sum in the last formula in the form

$$\sum_{i=-\infty}^{\infty} = \sum_{i=2m} + \sum_{i=2m+1},$$

we obtain by (4.3),

$$I_1 \lesssim \left(\int_0^\infty [g(y)]^p v(y)\, dy \right)^{q/p}.$$

Let us estimate I_2. Since $u \le \bar{u}$ and $\frac{\bar{u}(t)}{t}$ is non-increasing,

$$\begin{aligned} I_2 &\le \sum_{i=-\infty}^{\infty} \int_{z_{i-2}}^{z_i} \left(\frac{\bar{u}(t)}{t} \right)^q \left(\int_0^t g(s)\, ds \right)^q w(t)\, dt \\ &\lesssim \int_0^\infty \left(\int_0^t g(s)\, ds \right)^q \left(\frac{\bar{u}(t)}{t} \right)^q w(t)\, dt. \end{aligned}$$

We now claim that

$$\left(\int_0^\infty \left(\int_0^t g(s)\, ds \right)^q \left(\frac{\bar{u}(t)}{t} \right)^q w(t)\, dt \right)^{1/q} \lesssim \left(\int_0^\infty [g(t)]^p v(t)\, dt \right)^{1/p}. \tag{4.5}$$

Indeed, when $p \le q$, a necessary and sufficient condition for (4.5) is (cf. e.g. [17])

$$\sup_{x>0} \left(\int_x^\infty \left(\frac{\bar{u}(t)}{t} \right)^q w(t)\, dt \right)^{1/q} \sigma_p(0, x) < \infty,$$

which clearly follows from (4.2). When $q < p$, a necessary and sufficient condition for (4.5) is, by [21, Theorem 3],

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_k}^{x_{k+1}} \left(\frac{\bar{u}(t)}{t} \right)^q w(t)\, dt \right)^{r/q} [\sigma_p(x_{k-1}, x_k)]^r \right)^{1/r} < \infty. \tag{4.6}$$

Since $\frac{\bar{u}(t)}{t}$ is non-increasing, this condition follows from (4.3). Hence, (4.5) holds, and we get

$$I_2 \lesssim \left(\int_0^\infty [g(s)]^p v(s)\, ds \right)^{q/p}.$$

This finishes the proof of the sufficiency part.

Necessity: Fix $x \in (0,\infty)$ and let $E_n = \{t \in (0,x];\ v(t) > \frac{1}{n}\}$, $n \in \mathbb{N}$. Let $p > 1$. If $g \in \mathfrak{M}^+(0,\infty)$ and $\operatorname{supp} g \subset [0,x]$, then, by Lemma 2.2 (i) and the monotonicity of $\frac{\bar{u}(t)}{t}$,

$$(T_u g)(t) \ge \min\left\{\frac{\bar{u}(t)}{t}, \frac{\bar{u}(x)}{x}\right\} \int_0^\infty g(s)\, ds.$$

So, testing (4.1) with functions $g = \chi_{E_n} v^{1-p'}$ and then varying $x \in (0,\infty)$, we get

$$\left(\int_0^\infty \min\left\{\frac{\bar{u}(t)}{t}, \frac{\bar{u}(x)}{x}\right\}^q w(t)\, dt\right)^{1/q} \int_{E_n} [v(t)]^{1-p'}\, dt \lesssim \left(\int_{E_n} [v(t)]^{1-p'}\, dt\right)^{1/p}.$$

Since the right hand side is finite and non-zero, this yields

$$\left(\int_0^\infty \min\left\{\frac{\bar{u}(t)}{t}, \frac{\bar{u}(x)}{x}\right\}^q w(t)\, dt\right)^{1/q} \left(\int_{E_n} [v(t)]^{1-p'}\, dt\right)^{1/p'} \le C,$$

with $C > 0$ independent of $n \in \mathbb{N}$ and $x \in (0,\infty)$. Taking supremum first over n and then over x and using the fact that $\frac{\bar{u}(t)}{t}$ is non-increasing, we obtain (4.2).

When $p = 1$, we test (4.1) on functions $g = \frac{\chi_E}{v}$, where $E \subset (0,x]$ is measurable and $x \in (0,\infty)$. We get analogously

$$\left(\int_0^\infty \min\left\{\frac{\bar{u}(t)}{t}, \frac{\bar{u}(x)}{x}\right\}^q w(t)\, dt\right)^{1/q} |E|^{-1} \int_E \frac{1}{v(t)}\, dt \le C$$

with some $C > 0$ independent of $x \in (0,\infty)$ and $E \subset (0,x)$. For every fixed $x \in (0,\infty)$,

$$\sup_{E \subset (0,x)} |E|^{-1} \int_E \frac{1}{v(t)}\, dt = \operatorname*{ess\,sup}_{0<t\le x} \frac{1}{v(t)}.$$

Therefore, (4.2) follows on taking supremum first over E and then over x. This finishes the proof of the necessity part when $p \le q$. In fact, we have shown that (4.1) implies (4.2) for *any* $p \in [1,\infty)$ *and* $q \in (0,\infty)$.

Now assume $q < p$ and $p > 1$, and let $\{x_k\}$ be a fixed covering sequence. For $N \in \mathbb{N}$ and $t \in (0,\infty)$, put

$$g_N(t) := \sum_{k=-N}^{N} \chi_{[x_{k-1},x_k)}(t) \alpha_k^{r/(qp)} \beta_k^{r/(q'p)} [v(t)]^{1-p'},$$

where

$$\alpha_k := \int_{x_k}^{x_{k+1}} \left(\frac{\bar{u}(s)}{s}\right)^q w(s)\, ds, \qquad \beta_k := \int_{x_{k-1}}^{x_k} [v(s)]^{1-p'}\, ds, \quad k \in \mathbb{Z}.$$

Then

$$\int_0^\infty [g_N(t)]^p v(t)\, dt = \sum_{k=-N}^{N} \alpha_k^{r/q} \beta_k^{r/p'} =: \gamma_N.$$

Since

$$(T_u g_N)(t) \geq \frac{\bar{u}(t)}{t} \alpha_k^{r/(qp)} \beta_k^{r/(q'p)} \int_{x_{k-1}}^{x_k} [v(s)]^{1-p'} \, ds = \alpha_k^{r/(qp)} \beta_k^{r/(qp')} \frac{\bar{u}(t)}{t}$$

for all $k \in \mathbb{Z} \cap [-N, N]$ and $t \in [x_k, x_{k+1})$, we get

$$\begin{aligned} \int_0^\infty [(T_u g_N)(t)]^q \, w(t) \, dt &\geq \sum_{k=-N}^{N} \int_{x_k}^{x_{k+1}} [(T_u g_N)(t)]^q \, w(t) \, dt \\ &\geq \sum_{k=-N}^{N} \alpha_k^{r/p} \beta_k^{r/p'} \int_{x_k}^{x_{k+1}} \left(\frac{\bar{u}(t)}{t} \right)^q w(t) \, dt = \gamma_N. \end{aligned}$$

As we have noticed above, (4.2) is a necessary condition for (4.1). This yields that $\alpha_k^{1/q} \beta_k^{1/p'} < \infty$ for every $k \in \mathbb{Z}$. Hence also $\gamma_N < \infty$ for every $N \in \mathbb{N}$. Consequently, (4.2) and our estimates imply that $\gamma_N^{1/r} \leq C$ with $C > 0$ independent of $N \in \mathbb{N}$. On letting $N \to \infty$, we obtain (cf. (4.6))

$$\left[\sum_{k=-\infty}^{\infty} \left(\int_{x_k}^{x_{k+1}} \left(\frac{\bar{u}(s)}{s} \right)^q w(s) \, ds \right)^{r/q} \left(\int_{x_{k-1}}^{x_k} [v(s)]^{1-p'} \, ds \right)^{r/p'} \right]^{1/r} < \infty. \tag{4.7}$$

Now, for $N \in \mathbb{N}$ and $t \in (0, \infty)$, we denote

$$\tilde{g}_N(t) := \sum_{k=-N}^{N} \chi_{[x_{k-1}, x_k)}(t) \tilde{\alpha}_k^{r/(qp)} \beta_k^{r/(q'p)} [v(t)]^{1-p'},$$

where, for $k \in \mathbb{Z}$,

$$\tilde{\alpha}_k := \left(\frac{\bar{u}(x_k)}{x_k} \right)^q \int_{x_{k-1}}^{x_k} w(s) \, ds,$$

and β_k are given as above. Then

$$\int_0^\infty [\tilde{g}_N(t)]^p \, v(t) \, dt = \sum_{k=-N}^{N} \tilde{\alpha}_k^{r/q} \beta_k^{r/p'} =: \tilde{\gamma}_N.$$

Since (recall that, by Lemma 2.2 (i), $T_{\bar{u}} = T_u$)

$$(T_u \tilde{g}_N)(t) \geq \sup_{x_k \leq \tau < \infty} \frac{\bar{u}(\tau)}{\tau} \tilde{\alpha}_k^{r/(qp)} \beta_k^{r/(q'p)} \int_0^\tau v(s)^{1-p'} \, ds \geq \tilde{\alpha}_k^{r/(qp)} \beta_k^{r/(qp')} \frac{\bar{u}(x_k)}{x_k}$$

for all $k \in \mathbb{Z} \cap [-N, N]$ and $t \in [x_{k-1}, x_k)$, we get

$$\begin{aligned} \int_0^\infty [(T_u \tilde{g}_N)(t)]^q \, w(t) \, dt &\geq \sum_{k=-N}^{N} \int_{x_{k-1}}^{x_k} [(T_u g_N)(t)]^q \, w(t) \, dt \\ &\geq \sum_{k=-N}^{N} \tilde{\alpha}_k^{r/q} \beta_k^{r/p'} = \tilde{\gamma}_N. \end{aligned}$$

Similar argument as above yields

$$\left[\sum_{k=-\infty}^{\infty}\left(\left(\frac{\bar{u}(x_k)}{x_k}\right)^q\int_{x_{k-1}}^{x_k} w(t)\,dt\right)^{r/q}\left(\int_{x_{k-1}}^{x_k}[v(s)]^{1-p'}\,ds\right)^{r/p'}\right]^{1/r}<\infty. \tag{4.8}$$

Now, (4.7) and (4.8) imply (4.3) for $p>1$.

Assume that $p=1$, $0<q<1$, and let $\{x_k\}$ be a fixed covering sequence. For $k\in\mathbb{Z}$, let E_k be a measurable subset of the interval $[x_{k-1},x_k)$. If we denote

$$c_k:=\int_{x_k}^{x_{k+1}}\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt,\quad d_k:=\int_{E_k}\frac{ds}{v(s)}\quad\text{and}\quad A_k:=\left(\frac{c_k}{|E_k|}\right)^{r/q}d_k^r,$$

then (recall that $r=\frac{q}{1-q}$ and so $r+1=\frac{r}{q}$)

$$A_k|E_k|=c_k^{r/q}|E_k|^{-r}d_k^r=(A_kd_k)^q\,c_k. \tag{4.9}$$

Putting, for $N\in\mathbb{N}$ and $t\in(0,\infty)$,

$$h_N(t):=\sum_{k=-N}^{N}A_k\frac{\chi_{E_k}(t)}{v(t)},$$

we get

$$\int_0^\infty h_N(t)v(t)\,dt=\sum_{k=-N}^{N}A_k|E_k|=:\varrho_N. \tag{4.10}$$

Moreover, for all $k\in\mathbb{Z}\cap[-N,N]$ and $t\in[x_k,x_{k+1})$,

$$\begin{aligned}(T_uh_N)(t)&\ge\sup_{t\le\tau<\infty}\frac{\bar{u}(\tau)}{\tau}\sum_{j=-N}^{N}A_j\int_0^\tau\frac{\chi_{E_j}(s)}{v(s)}\,ds\\&\ge\frac{\bar{u}(t)}{t}\sum_{j=-N}^{N}A_j\int_0^t\frac{\chi_{E_j}(s)}{v(s)}\,ds\\&\ge A_k\frac{\bar{u}(t)}{t}\int_{E_k}\frac{ds}{v(s)}\\&=A_kd_k\frac{\bar{u}(t)}{t}.\end{aligned}$$

Together with (4.9), this implies that

$$\begin{aligned}\int_0^\infty[(T_uh_N)(t)]^q\,w(t)\,dt&\ge\sum_{k=-N}^{N}\int_{x_k}^{x_{k+1}}[(T_uh_N)(t)]^q\,w(t)\,dt\\&\ge\sum_{k=-N}^{N}(A_kd_k)^q\int_{x_k}^{x_{k+1}}\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt\\&=\sum_{k=-N}^{N}(A_kd_k)^qc_k=\varrho_N.\end{aligned} \tag{4.11}$$

As above, (4.2) shows that

$$c_k^{1/q} d_k < \infty \qquad \text{for every} \quad k \in \mathbb{Z}.$$

Hence also $\varrho_N < \infty$ for every $N \in \mathbb{N}$. Thus, (4.10), (4.11) and (4.1) yield

$$\varrho_N^{1/r} \leq C$$

with $C > 0$ independent of $N \in \mathbb{N}$ and of the choice of E_k. Taking supremum over all choices of $\{E_k\}_{k \in \mathbb{Z}}$ and then letting $N \to \infty$, we obtain

$$\sum_{k=-\infty}^{\infty} \left(\int_{x_k}^{x_{k+1}} \left(\frac{\bar{u}(s)}{s} \right)^q w(s)\, ds \right)^{r/q} \left(\operatorname*{ess\,sup}_{x_{k-1} \leq t < x_k} \frac{1}{v(t)} \right)^r < \infty. \tag{4.12}$$

Finally, if we take d_k as above and put, for $k \in \mathbb{Z}$,

$$\tilde{c}_k := \left(\frac{\bar{u}(x_k)}{x_k} \right)^q \int_{x_{k-1}}^{x_k} w(t)\, dt \quad \text{and} \quad \tilde{A}_k := \left(\frac{\tilde{c}_k}{|E_k|} \right)^{r/q} d_k^r,$$

then

$$\tilde{A}_k |E_k| = \tilde{c}_k^{r/q} |E_k|^{-r} d_k^r = \left(\tilde{A}_k d_k \right)^q \tilde{c}_k. \tag{4.13}$$

Defining

$$\tilde{h}_N(t) = \sum_{k=-N}^{N} \tilde{A}_k \frac{\chi_{E_k}(t)}{v(t)}, \qquad N \in \mathbb{N}, \quad t \in (0, \infty),$$

we get

$$\int_0^\infty \tilde{h}_N(t) v(t)\, dt = \sum_{k=-N}^{N} \tilde{A}_k |E_k| =: \tilde{\varrho}_N. \tag{4.14}$$

Moreover, for all $k \in \mathbb{Z} \cap [-N, N]$ and $t \in [x_{k-1}, x_k)$,

$$\begin{aligned}
(T_u \tilde{h}_N)(t) &\geq \sup_{x_k \leq \tau < \infty} \frac{\bar{u}(\tau)}{\tau} \sum_{j=-N}^{N} \tilde{A}_j \int_0^\tau \frac{\chi_{E_j}(s)}{v(s)}\, ds \\
&\geq \frac{\bar{u}(x_k)}{x_k} \sum_{j=-N}^{N} \tilde{A}_j \int_{x_{k-1}}^{x_k} \frac{\chi_{E_j}(s)}{v(s)}\, ds \\
&\geq \frac{\bar{u}(x_k)}{x_k} \tilde{A}_k d_k.
\end{aligned}$$

Together with (4.13), this implies that

$$\begin{aligned}
\int_0^\infty \left[(T_u \tilde{h}_N)(t) \right]^q w(t)\, dt &\geq \sum_{k=-N}^{N} \int_{x_{k-1}}^{x_k} \left[(T_u \tilde{h}_N)(t) \right]^q w(t)\, dt \\
&\geq \sum_{k=-N}^{N} (\tilde{A}_k d_k)^q \left(\frac{\bar{u}(x_k)}{x_k} \right)^q \int_{x_{k-1}}^{x_k} w(t)\, dt \\
&= \sum_{k=-N}^{N} (\tilde{A}_k d_k)^q \tilde{c}_k = \tilde{\varrho}_N.
\end{aligned} \tag{4.15}$$

Again, (4.2) shows that

$$\tilde{c}_k^{1/q} d_k < \infty \qquad \text{for every} \quad k \in \mathbb{Z}.$$

Hence also $\tilde{\varrho}_N < \infty$ for every $N \in \mathbb{N}$. Thus, (4.14), (4.15) and (4.1) yield

$$\tilde{\varrho}_N^{1/r} \le C$$

with $C > 0$ independent of $N \in \mathbb{N}$ and of the choice of E_k. Taking supremum over all choices of $\{E_k\}_{k\in\mathbb{Z}}$ and then letting $N \to \infty$, we obtain

$$\sum_{k=-\infty}^{\infty} \left(\frac{\bar{u}(x_k)}{x_k}\right)^r \left(\int_{x_{k-1}}^{x_k} w(s)\,ds\right)^{r/q} \left(\operatorname*{ess\,sup}_{x_{k-1}\le t<x_k} \frac{1}{v(t)}\right)^r < \infty. \tag{4.16}$$

Now, (4.3) follows from (4.16) and (4.12). □

Our next aim is to obtain an analogous result involving a general operator $T_{u,b}$.

Theorem 4.2

Assume that $1 \le p < \infty$ and $0 < q < \infty$. When $q < p$, we define r by (3.2)*. Let b be a weight such that $b(t) > 0$ for a.e. $t \in (0,\infty)$ and $B(t) := \int_0^t b(s)\,ds < \infty$ for all $t \in (0,\infty)$. Let u, v and w be as in* Theorem 3.2*. For $0 \le \alpha < \beta \le \infty$, we denote*

$$\sigma_{p,b}(\alpha,\beta) = \begin{cases} \left(\int_\alpha^\beta [v(t)]^{1-p'}\,[b(t)]^{p'}\,dt\right)^{1/p'} & \text{when } 1 < p < \infty \\ \operatorname*{ess\,sup}_{\alpha<t<\beta} \frac{b(t)}{v(t)} & \text{when } p = 1. \end{cases}$$

(i) *Let $p \le q$. Then* (3.28) *holds on $\mathfrak{M}^+(0,\infty)$ if and only if*

$$\sup_{x>0} \left(\left(\frac{\bar{u}(x)}{B(x)}\right)^q \int_0^x w(t)\,dt + \int_x^\infty \left(\frac{\bar{u}(t)}{B(t)}\right)^q w(t)\,dt\right)^{1/q} \sigma_{p,b}(0,x) < \infty.$$

(ii) *Let $q < p$. Then* (3.28) *holds on $\mathfrak{M}^+(0,\infty)$ if and only if*

$$\sup_{\{x_k\}} \left(\sum_k \left(\int_{x_{k-1}}^{x_{k+1}} \min\left\{\frac{\bar{u}(x_k)}{B(x_k)}, \frac{\bar{u}(t)}{B(t)}\right\}^q w(t)\,dt\right)^{r/q} [\sigma_{p,b}(x_{k-1},x_k)]^r\right)^{1/r} < \infty,$$

where the supremum is taken over all covering sequences $\{x_k\}$.

Proof. The assertion follows immediately from Theorem 4.1 on changing the parameters

$$g(t) \to g(t)b(t), \quad v(t) \to v(t)\,[b(t)]^{-p} \quad \text{and} \quad u(t) \to u(t)\frac{t}{B(t)}.$$ □

We have also an alternative integral criterion that characterizes (4.1). This fact resembles the similar situation involving classical Hardy integral operator, for which a “sum criterion” was found by Sawyer in [21] and an “integral criterion” was established by Maz’ya in [15].

We shall need the following elementary lemma.

Lemma 4.3

Let $c > 1$ and $0 < q < \infty$. Let either $J \in \mathbb{Z}$ or $J = -\infty$. Then

$$\sum_{k=J}^{\infty} c^{-k} \left(\sum_{i=J}^{k} a_i \right)^q \approx \sum_{k=J}^{\infty} c^{-k} a_k^q.$$

for any sequence $\{a_i\}_{i\in\mathbb{Z}}$ of non-negative numbers.

Proof. When $0 < q \le 1$, we have

$$\sum_{k=J}^{\infty} c^{-k} a_k^q \le \sum_{k=J}^{\infty} c^{-k} \left(\sum_{i=J}^{k} a_i \right)^q \le \sum_{k=J}^{\infty} c^{-k} \sum_{i=J}^{k} a_i^q = \sum_{i=J}^{\infty} a_i^q \sum_{k=i}^{\infty} c^{-k} \approx \sum_{i=J}^{\infty} a_i^q c^{-i}.$$

When $1 < q < \infty$, let $\alpha \in (0, \frac{1}{q})$. Then we obtain by the Hölder inequality that

$$\begin{aligned} \sum_{k=J}^{\infty} c^{-k} a_k^q &\le \sum_{k=J}^{\infty} c^{-k} \left(\sum_{i=J}^{k} a_i \right)^q \\ &= \sum_{k=J}^{\infty} c^{-k} \left(\sum_{i=J}^{k} a_i c^{-\alpha i} c^{\alpha i} \right)^q \\ &\le \sum_{k=J}^{\infty} c^{-k} \sum_{i=J}^{k} a_i^q c^{-\alpha q i} \left(\sum_{i=J}^{k} c^{\alpha q' i} \right)^{q/q'} \\ &\lesssim \sum_{k=J}^{\infty} c^{-k} \sum_{i=J}^{k} a_i^q c^{\alpha q (k-i)} \\ &= \sum_{i=J}^{\infty} a_i^q c^{-\alpha q i} \sum_{k=i}^{\infty} c^{(\alpha q - 1)k} \\ &\approx \sum_{i=J}^{\infty} a_i^q c^{-i}. \end{aligned}$$

□

Theorem 4.4

Let u, v and w be as in Theorem 4.1. *Let $1 \le p < \infty$ and $0 < q < p$ and let r be defined by* (3.2). *Then the inequality* (4.1) *holds for all $g \in \mathfrak{M}^+(0,\infty)$ if and only if*

$$\left(\int_0^\infty \left(\int_t^\infty \left(\frac{\bar{u}(s)}{s} \right)^q w(s)\, ds \right)^{r/p} \left(\frac{\bar{u}(t)}{t} \right)^q [\sigma_p(0,t)]^r\, w(t)\, dt \right)^{1/r} < \infty \tag{4.17}$$

and

$$\left(\int_0^\infty \left(\int_0^t w(s)\, ds \right)^{r/p} \left[\sup_{t \le \tau < \infty} \frac{\bar{u}(\tau)}{\tau} \sigma_p(0,\tau) \right]^r w(t)\, dt \right)^{1/r} < \infty. \tag{4.18}$$

Proof. As usual, we restrict ourselves to the case when $\int_0^\infty w(s)\, ds = \infty$ and $\int_0^\infty v(s)\, ds = \infty$.

By Theorem 4.1, it is sufficient to show that (4.17) and (4.18) are equivalent to (4.3). Assume first that (4.17) and (4.18) hold. Let $\{x_k\}$ be a covering sequence. Since

$$\left(\int_{x_k}^{x_{k+1}}\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt\right)^{r/q} \approx \int_{x_k}^{x_{k+1}}\left(\int_t^{x_{k+1}}\left(\frac{\bar{u}(s)}{s}\right)^q w(s)\,ds\right)^{r/p}\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt,$$

we get from (4.17) that

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left(\int_{x_k}^{x_{k+1}}\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt\right)^{r/q}[\sigma_p(x_{k-1},x_k)]^r\\ &\quad\lesssim \int_0^\infty\left(\int_t^\infty\left(\frac{\bar{u}(s)}{s}\right)^q w(s)\,ds\right)^{r/p}\left(\frac{\bar{u}(t)}{t}\right)^q[\sigma_p(0,t)]^r\,w(t)\,dt<\infty.\end{aligned}$$

Similarly, since

$$\left(\int_{x_{k-1}}^{x_k} w(t)\,dt\right)^{r/q} \approx \int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t} w(s)\,ds\right)^{r/p} w(t)\,dt,$$

we obtain from (4.18),

$$\sum_{k=-\infty}^{\infty}\left(\frac{\bar{u}(x_k)}{x_k}\right)^r\left(\int_{x_{k-1}}^{x_k} w(t)\,dt\right)^{r/q}[\sigma_p(x_{k-1},x_k)]^r<\infty,$$

and (4.3) follows.

As for the converse, assume that (4.3) holds. If

$$\int_0^\infty\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt<\infty,$$

then there exists $J\in\mathbb{Z}$ such that

$$2^{-J-1}<\int_0^\infty\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt\le 2^{-J}.$$

If

$$\int_0^\infty\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt=\infty,$$

we set $J:=-\infty$. Next, for $k\in\mathbb{Z}$, $k>J$, we choose $x_k\in(0,\infty)$ so that

$$\int_{x_k}^\infty\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt=2^{-k}.$$

If $J>-\infty$, then we further set $x_J:=0$. Then

$$\begin{aligned}&\int_0^\infty\left(\int_t^\infty\left(\frac{\bar{u}(s)}{s}\right)^q w(s)\,ds\right)^{r/p}\left(\frac{\bar{u}(t)}{t}\right)^q[\sigma_p(0,t)]^r\,w(t)\,dt\\ &\quad\lesssim\sum_{k=J+1}^{\infty}\int_{x_{k-1}}^{x_k}\left(\int_t^\infty\left(\frac{\bar{u}(s)}{s}\right)^q w(s)\,ds\right)^{r/p}\left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt\cdot[\sigma_p(0,x_k)]^r\\ &\quad\lesssim\sum_{k=J}^{\infty}2^{-(rk)/q}\,[\sigma_p(0,x_k)]^r\,.\end{aligned}$$

Now, it is easy to see that Lemma 4.3 implies

$$\sum_{k=J}^{\infty} 2^{-(rk)/q} \left[\sigma_p(0,x_k)\right]^r \approx \sum_{k=J}^{\infty} \left[\sigma_p(x_{k-1},x_k)\right]^r 2^{-(kr)/q}.$$

Therefore, we finally obtain

$$\begin{aligned}\int_0^\infty \left(\int_t^\infty \left(\frac{\bar{u}(s)}{s}\right)^q w(s)\,ds\right)^{r/p} \left(\frac{\bar{u}(t)}{t}\right)^q \left[\sigma_p(0,t)\right]^r w(t)\,dt \\ \lesssim \sum_{k=J}^{\infty} \left[\sigma_p(x_{k-1},x_k)\right]^r \left(\int_{x_k}^{x_{k+1}} \left(\frac{\bar{u}(t)}{t}\right)^q w(t)\,dt\right)^{r/q} < \infty,\end{aligned}$$

which yields (4.17).

It just remains to verify (4.18). We take $\{x_k\}$ so that $\int_0^{x_k} w(t)\,dt = 2^k$. Then, working as in the proof of Theorem 3.4, implication (3.12)⇒(3.13), we get

$$\begin{aligned}&\int_0^\infty \left(\int_0^t w(s)\,ds\right)^{r/p} \left[\sup_{t\le\tau<\infty} \frac{\bar{u}(\tau)}{\tau}\sigma_p(0,\tau)\right]^r w(t)\,dt \\ &\le \sum_{k=-\infty}^{\infty} \left(\int_{x_k}^{x_{k+1}} \left(\int_0^t w(s)\,ds\right)^{r/p} w(t)\,dt\right) \left[\sup_{x_k\le\tau<\infty} \frac{\bar{u}(\tau)}{\tau}\sigma_p(0,\tau)\right]^r \\ &\approx \sum_{k=-\infty}^{\infty} 2^{(rk)/q} \left[\sup_{x_k\le\tau<\infty} \frac{\bar{u}(\tau)}{\tau}\sigma_p(0,\tau)\right]^r \\ &\le \sum_{k=-\infty}^{\infty} 2^{(rk)/q} \sum_{i=k}^{\infty} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{\bar{u}(\tau)}{\tau}\sigma_p(0,\tau)\right]^r \\ &\lesssim \sum_{i=-\infty}^{\infty} 2^{(ri)/q} \left[\sup_{x_i\le\tau<x_{i+1}} \frac{\bar{u}(\tau)}{\tau}\sigma_p(0,\tau)\right]^r \\ &\lesssim \sum_{i=-\infty}^{\infty} \left(\int_{y_{i-1}}^{y_{i+1}} w(t)\,dt\right)^{r/q} \left[\frac{\bar{u}(y_{i+1})}{y_{i+1}}\sigma_p(0,y_{i+1})\right]^r,\end{aligned}$$

where $y_{i+1} \in [x_i, x_{i+1})$ is such that

$$\left[\sup_{x_i\le\tau<x_{i+1}} \frac{\bar{u}(\tau)}{\tau}\sigma_p(0,\tau)\right]^r \le 2\left[\frac{\bar{u}(y_{i+1})}{y_{i+1}}\sigma_p(0,y_{i+1})\right]^r.$$

Splitting the last sum into two, we have

$$\begin{aligned}&\int_0^\infty \left(\int_0^t w(s)\,ds\right)^{r/p} \left[\sup_{t\le\tau<\infty} \frac{\bar{u}(\tau)}{\tau}\sigma_p(0,\tau)\right]^r w(t)\,dt \\ &\lesssim \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-2}}^{y_{2m}} w(t)\,dt\right)^{r/q} \left[\frac{\bar{u}(y_{2m})}{y_{2m}}\sigma_p(0,y_{2m})\right]^r \\ &\quad + \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-1}}^{y_{2m+1}} w(t)\,dt\right)^{r/q} \left[\frac{\bar{u}(y_{2m+1})}{y_{2m+1}}\sigma_p(0,y_{2m+1})\right]^r \\ &=: I + II.\end{aligned}$$

Now,

$$\begin{aligned} I &\approx \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-2}}^{y_{2m}} w(t)\,dt \right)^{r/q} \left[\frac{\bar{u}(y_{2m})}{y_{2m}} \sigma_p(0, y_{2m-2}) \right]^r \\ &\quad + \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-2}}^{y_{2m}} w(t)\,dt \right)^{r/q} \left[\frac{\bar{u}(y_{2m})}{y_{2m}} \sigma_p(y_{2m-2}, y_{2m}) \right]^r \\ &=: I_1 + I_2. \end{aligned}$$

We shall estimate I_1 using the finiteness of (4.17), which we have already proved. By the monotonicity of $\frac{\bar{u}(t)}{t}$,

$$\begin{aligned} I_1 &\le \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-2}}^{y_{2m}} \left(\frac{\bar{u}(y_{2m})}{y_{2m}} \right)^q w(t)\,dt \right)^{r/q} [\sigma_p(0, y_{2m-2})]^r \\ &\lesssim \sum_{m=-\infty}^{\infty} \left(\int_{y_{2m-2}}^{y_{2m}} \left(\int_t^{y_{2m}} \left(\frac{\bar{u}(s)}{s} \right)^q w(s)\,ds \right)^{r/p} \left(\frac{\bar{u}(t)}{t} \right)^q w(t)\,dt \right) [\sigma_p(0, y_{2m-2})]^r \\ &\le \sum_{m=-\infty}^{\infty} \int_{y_{2m-2}}^{y_{2m}} \left(\int_t^{\infty} \left(\frac{\bar{u}(s)}{s} \right)^q w(s)\,ds \right)^{r/p} \left(\frac{\bar{u}(t)}{t} \right)^q [\sigma_p(0,t)]^r\, w(t)\,dt \\ &= \int_0^{\infty} \left(\int_t^{\infty} \left(\frac{\bar{u}(s)}{s} \right)^q w(s)\,ds \right)^{r/p} \left(\frac{\bar{u}(t)}{t} \right)^q [\sigma_p(0,t)]^r\, w(t)\,dt < \infty \end{aligned}$$

by (4.17).

A straightforward application of Theorem 4.1 (ii) yields $I_2 < \infty$. The estimate of II is analogous. □

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