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Linear distortion of Hausdorff dimension and Cantor's function

O. Dovgoshey and V. Ryazanov

Institute of Applied Mathematics and Mechanics, NAS of Ukraine
74 Roze Luxemburg str., Donetsk, 83114, UKRAINE

E-mail: aleksdov@mail.ru
E-mail: ryaz@iamm.ac.donetsk.ua

O. Martio and M. Vuorinen

Department of Mathematics and Statistics, P.O. Box 68
FIN – 00014 University of Helsinki, FINLAND

E-mail: martio@cc.helsinki.fi
E-mail: vuorinen@csc.fi

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Abstract

Let f be a mapping from a metric space X to a metric space Y, and let α be a positive real number. Write $\dim(E)$ and $\mathcal{H}^s(E)$ for the Hausdorff dimension and the s-dimensional Hausdorff measure of a set E. We give sufficient conditions that the equality $\dim(f(E)) = \alpha \dim(E)$ holds for each $E \subseteq X$. The problem is studied also for the Cantor ternary function G. It is shown that there is a subset M of the Cantor ternary set such that $\mathcal{H}^s(M) = 1$, with $s = \log 2/\log 3$ and $\dim(G(E)) = (\log 3/\log 2)\dim(E)$, for every $E \subseteq M$.

1. Statements of main results

Let f be a mapping from a metric space (X,ρ) to a metric space (Y,d). It is a simple fact that if the double inequality

$$c_1(\rho(x,y))^\alpha \le d(f(x),f(y)) \le c_2(\rho(x,y))^\alpha \tag{1.0}$$

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holds for all x, y from X where $\alpha \in (0, \infty)$ and c_1 and c_2 are some positive constants, then every set $A \subseteq X$ satisfies

$$\dim(f(A)) = \frac{1}{\alpha} \dim(A).$$

We are interested in necessary and sufficient local conditions under which this equality holds for every $A \subseteq X$. The following theorem provides conditions for this.

Write $\mathrm{ac}(A)$ for the set of all accumulation points of a set A. Let f be a mapping from a metric space (X, ρ) to a metric space (Y, d). If x and a are the points of X and $x \neq a$ we put

$$K_f(x, a) := \begin{cases} \dfrac{\log(d(f(x), f(a)))}{\log(\rho(x, a))} & \text{if} \quad f(x) \neq f(a) \\ \\ +\infty & \text{if} \quad f(x) = f(a) \end{cases}$$

Theorem 1.1

Let $f : (X, \rho) \to (Y, d)$ be a homeomorphism. Suppose that the limit

$$\lim_{x \to a} K_f(x, a) = \alpha(a) \in (0, \infty) \tag{1.1}$$

exists for every $a \in \mathrm{ac}(X)$. Then the following statements are equivalent.

(i) *There exists a set $X_0 \subseteq \mathrm{ac}X$ such that*

$$\alpha(a) = \alpha_0 \tag{1.2}$$

for all a in $\mathrm{ac}(X) \backslash X_0$ and either $\dim(Z) = 0$ or $\dim(Z) = \infty$ for every $Z \subseteq X_0$.

(ii) *For every $A \subseteq X$ the equality*

$$\dim(f(A)) = \frac{1}{\alpha_0} \dim(A) \tag{1.3}$$

holds.

Corollary 1.2

Let $f : X \to Y$ be a homeomorphism and let $\dim(X) < \infty$. Suppose that the limit (1.1) exists for every $a \in \mathrm{ac}(X)$. Then

$$\dim(f(A)) = \dim(A), \quad \forall\ A \subseteq X \tag{1.4}$$

if and only if

$$\alpha(a) = 1 \tag{1.5}$$

for every $a \in \mathrm{ac}(X) \backslash X_0$ where $X_0 \subset X$ is zero-dimensional.

Remark 1.3 Note that, as it follows from the proof of Theorem 1.1, the equivalence (i) $\iff$ (ii) is also valid if (1.1) holds only in $X \setminus X_*$ where

$$\dim(X_*) = \dim(f(X_*)) = 0. \tag{1.6}$$

See further consequences of Theorem 1.1 in the end of Section 2. Note also that, if $f : X \to Y$ is a continuous bijection and (1.1) holds, then it does not follow that f is a homeomorphism. On the other hand, if $f : (X, \rho) \to (Y, d)$ is a homeomorphism and for each $a \in \mathrm{ac}(X)$ we have $\alpha(a) = 1$, then there need not exist positive constants α and c such that the inequality $d(f(x), f(y)) \le c(\rho(x, y))^\alpha$ holds for all x and y in some ball $B(a, r) \subseteq X$; see Example 3.3.

In the third part, we investigate the following problem: Let $C \subset [0, 1]$ be the standard Cantor ternary set and let G be the Cantor function. Characterize the set of points $x \in C$ such that

$$\lim_{\substack{y\to x\\ y\in C}} \frac{\log|G(x) - G(y)|}{\log|x - y|} = \frac{\log 2}{\log 3}.$$

In Theorem 1.4 these points are characterized in terms of the spacing of $0's$ and $2's$ in ternary expansions.

Let x be a point of the Cantor ternary set C. Then x has a triadic representation

$$x = \sum_{m=1}^{\infty} \frac{2\alpha_m}{3^m}$$

where $\alpha_m \in \{0, 1\}$. Define a sequence $\{\mathcal{R}_x(n)\}_{n=1}^{\infty}$ by the rule

$$\mathcal{R}_x(n) := \begin{cases} \inf\{m - n : \alpha_m \ne \alpha_n, \quad m > n\} & \text{if} \quad \exists\ m > n : \alpha_m \ne \alpha_n \\ \\ 0 & \text{if} \quad \forall\ m > n : \alpha_m = \alpha_n, \end{cases} \tag{1.7}$$

$$\mathcal{R}_x(n) = 1 \iff (\alpha_n \ne \alpha_{n+1}),$$

i.e.

$$\mathcal{R}_x(n) = 2 \iff (\alpha_n = \alpha_{n+1})\ \&\ (\alpha_{n+1} \ne \alpha_{n+2})$$

and so on.

Theorem 1.4

Let x be a point of C. Then

$$\lim_{\substack{y\to x\\ y\in C}} \frac{\log|G(x) - G(y)|}{\log|x - y|} = \frac{\log 2}{\log 3} \tag{1.8}$$

if and only if

$$\lim_{n\to\infty} \frac{\mathcal{R}_x(n)}{n} = 0. \tag{1.9}$$

This theorem and Theorem 1.1 imply the following result.

Theorem 1.5

There exists a set $M \subseteq C$ *such that* $\mathcal{H}^s(M) = 1$ *with* $s = \log 2/\log 3$ *and* $\dim(G(A)) = \frac{\log 3}{\log 2}\dim(A)$, *for every* $A \subseteq M$.

Remark 1.6 It is well-known that the Cantor function G satisfies the inequality

$$|G(x) - G(y)| \leq 2|x - y|^{\log 2/\log 3} \tag{1.10}$$

for all x and y in $[0, 1]$. The proof can be found in [4], see also [3]. The Hausdorff dimension of the Cantor ternary set equals $\log 2/\log 3$ and, moreover, $\mathcal{H}^s(C) = 1$ for $s = \log 2/\log 3$.

2. Linear distortion of Hausdorff dimension under mappings of metric spaces

We recall the definitions of the Hausdorff dimension and the s-dimensional Hausdorff measure. Let (X, ρ) be a metric space and let

$$\operatorname{diam} A := \sup\{\rho(x, y) : x, y \in A\}$$

be the **diameter** of $A \subseteq X$ if $A \neq \emptyset$, $\operatorname{diam} \emptyset = 0$. If $A \subseteq \bigcup_{i\in I} E_i$ with $0 < \operatorname{diam} E_i \leq \delta$ for each index $i \in I$, then $\{E_i\}_{i\in I}$ is called *a* δ-cover of A. If all δ-covers of A are uncountable, then

$$\mathcal{H}^s_\delta(A) := \infty$$

for each $s \geq 0$ and, in the opposite case,

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_{i\in\mathbb{N}} (\operatorname{diam} E_i)^s : \{E_i\}_{i\in\mathbb{N}} \text{ is a countable } \delta\text{-cover of } A\right\} \tag{2.1}$$

for $s \geq 0$. The **s-dimensional Hausdorff measure** of A is defined by

$$\mathcal{H}^s(A) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(A). \tag{2.2}$$

Note that the limit exists because $\mathcal{H}^s_\delta(A)$ is nonincreasing function of δ and that the Hausdorff measure $\mathcal{H}^s$ is a regular Borel measure, see e.g. [7]. The **Hausdorff dimension** of A is the number $\dim(A)$ such that

$$\mathcal{H}^s(A) = \begin{cases} +\infty & \text{if } \quad s < \dim(A), \\ 0 & \text{if } \quad s > \dim(A). \end{cases}$$

DEFINITION 2.1 Let (X, ρ) be a metric space. A family $\mathcal{B}$ of closed balls $B(a, \delta)$ in (X, ρ) is said to fulfil the **condition (V)** if $B(a, \delta) \in \mathcal{B}$ whenever $a \in X$ and $\delta \in (0, \Delta(a)]$ for some $\Delta(a) > 0$.

Some results closely related to the next lemma can be found in [2, §2.8].

Lemma 2.2

Let A be a subset of the metric space (X, ρ) such that $\mathcal{H}^s(A) = 0$ for some $s > 0$ and let a family $\mathcal{B}$ of closed balls in (X, ρ) fulfil the (V) condition. Then, for all positive numbers δ and η, there is a countable δ-cover $\{B_i\}_{i\in\mathbb{N}} \subseteq \mathcal{B}$ of the set A with

$$\sum_{i\in\mathbb{N}} (\text{diam } B_i)^s \leq \eta. \tag{2.3}$$

Proof. It follows from (2.1), (2.2) and $\mathcal{H}^s(A) = 0$ that for every $\gamma > 0$ there is a $\frac{\delta}{2}$-cover $\{E_i^{(1)}\}_{i\in\mathbb{N}}$ of the set A such that

$$\sum_{i\in\mathbb{N}} (d_i^{(1)})^s \leq \frac{\gamma}{2}, \tag{2.4}$$

where $d_i^{(1)} := \text{diam } E_i^{(1)}$.

Suppose $\mathcal{B}$ fulfils the (V) condition. We shall say that $i \in \mathbb{N}$ is a **marked index** if there is a point $a_{i_0}^{(1)} \in E_{i_0}^{(1)} \cap A$ for which

$$d_{i_0}^{(1)} < \Delta(a_{i_0}^{(1)})$$

where Δ is the function from Definition 2.1. Let $\mathcal{I}^{(1)}$ be the set of all marked indices. It is obvious that

$$E_i^{(1)} \subseteq B(a_i^{(1)}, d_i^{(1)}) \in \mathcal{B}$$

for all $i \in \mathcal{I}^{(1)}$. Using (2.4) and

$$\text{diam } B(a_i^{(1)}, d_i^{(1)}) \leq 2d_i^{(1)},$$

we have

$$\sum_{i\in\mathcal{I}^{(1)}} (\text{diam } B(a_i^{(1)}, d_i^{(1)}))^s \leq 2^s \frac{\gamma}{2}. \tag{2.5}$$

It should be observed that $\{B(a_i^{(1)}, d_i^{(1)})\}_{i\in\mathcal{I}^{(1)}}$ is a δ-cover of the set

$$\left\{ a \in A : \Delta(a) > \left(\frac{\gamma}{2}\right)^{1/s} \right\}.$$

Really, if $a_0 \in A \backslash (\bigcup_{i\in\mathcal{I}^{(1)}} B(a_i^{(1)}, d_i^{(1)}))$, then there is $E_{i_0}^{(1)} \ni a_0$ with $d_{i_0}^{(1)} \geq \Delta(a_0)$. It follows from (2.4) that

$$\left(\frac{\gamma}{2}\right)^{1/s} \geq d_{i_0}^{(1)}.$$

Hence,

$$\Delta(a_0) \leq \left(\frac{\gamma}{2}\right)^{1/s}.$$

Reasoning similarly we can define the sequence $\{\mathcal{I}^{(n)}\}_{n=1}^{\infty}$ such that for each positive integer n:

$$\sum_{i\in\mathcal{I}^{(n)}} (\text{diam } B(a_i^{(n)}, d_i^{(n)}))^s \le 2^s \frac{\gamma}{2^n}, \tag{2.6.1}$$

$$B(a_i^{(n)}, d_i^{(n)}) \in \mathcal{B} \text{ and } \text{diam } B(a_i^{(n)}, d_i^{(n)}) \le \delta \text{ for each } i \in \mathcal{I}^{(n)}, \tag{2.6.2}$$

$$\left\{a \in A : \Delta(a) > \left(\frac{\gamma}{2^n}\right)^{1/s}\right\} \subseteq \bigcup_{i\in\mathcal{I}^{(n)}} B(a_i^{(n)}, d_i^{(n)}). \tag{2.6.3}$$

For this purpose we take a $\frac{\delta}{2}$-cover $\{E_i^{(n)}\}_{i\in\mathbb{N}}$ of A such that

$$\sum_{i\in\mathbb{N}} (\text{diam } E_i^{(n)})^s \le \frac{\gamma}{2^n}.$$

Now set $\gamma := \frac{1}{2^s}\eta$. Then (2.6.1) implies that

$$\sum_{n=1}^{\infty} \sum_{i\in\mathcal{I}^{(n)}} (\text{diam } B_i^{(n)}, d_i^{(n)})^s \le \eta.$$

Hence, by (2.6.2) and (2.6.3), the family

$$\left\{B(a_i^{(n)}, d_i^{(n)}) :\ n = 1, 2, ...;\ i \in \mathcal{I}^{(n)}\right\}$$

is a desired δ-cover of A. □

Proposition 2.3

Suppose that (X, ρ) and (Y, d) be metric spaces. Let $\beta \in (0, \infty)$ and let $f : X \to Y$ be a mapping such that

$$\liminf_{x\to a} K_f(x, a) \ge \beta \tag{2.7}$$

for each $a \in \text{ac}(X)$. Then we have

$$\dim(A) \ge \beta \dim(f(A)) \tag{2.8}$$

for every $A \subseteq X$.

Proof. If $\dim(A) = \infty$, then the inequality (2.8) is trivial. Suppose that $0 \le \dim(A) < s < \infty$. Then by the definition of the Hausdorff dimension we have $\mathcal{H}^s(A) = 0$.

For each $\varepsilon \in (0, \beta)$, define a family $\mathcal{B}_\varepsilon$ of the closed balls $B(a, \delta)$ in (X, ρ) by the rule

$$(B(a, \delta) \in \mathcal{B}_\varepsilon) \iff (\forall\, x \in B(a, \delta) : d(f(x), f(a)) \le (\rho(x, a))^{\beta-\varepsilon}). \tag{2.9}$$

It follows immediately from (2.7) that $\mathcal{B}_\varepsilon$ fulfils the condition (V). Hence by Lemma 2.2, for every $\eta > 0$, there is δ-cover $\{B_i(a_i, \delta_i)\}_{i\in\mathbb{N}} \subseteq \mathcal{B}_\varepsilon$ of A such that

$$\sum_{i\in\mathbb{N}} (\text{diam } B(a_i, \delta_i))^s \le \eta. \tag{2.10}$$

It follows from (2.9) that

$$\text{diam }(f(B(a_i,\delta_i)) \le 2(\text{diam }(B(a_i,\delta_i)))^{\beta-\varepsilon}.$$

The last inequality and (2.10) imply that $\{f(B(a_i,\delta_i))\}_{i\in\mathbb{N}}$ is a $2\delta^{\beta-\varepsilon}$-cover of $f(A)$ and

$$\sum_{i\in\mathbb{N}}(\text{diam } f(B(a_i,\delta_i)))^{s/(\beta-\varepsilon)} \le 2^{s/(\beta-\varepsilon)}\eta.$$

Consequently

$$\mathcal{H}^{s/(\beta-\varepsilon)}(f(A)) = 0,$$

that is

$$\dim(f(A)) \le \frac{s}{\beta-\varepsilon}$$

for all $\varepsilon \in (0,\beta)$ and every $s > \dim(A)$. Letting $\varepsilon \to 0$ and $s \to \dim(A)$ we have (2.8). □

Corollary 2.4

Suppose that (X,ρ) and (Y,d) are a metric spaces. Let $0 < \beta \le \alpha < \infty$ and let $f : X \to Y$ be a homeomorphism such that

$$\beta \le \liminf_{x\to a} K_f(x,a) \le \limsup_{x\to a} K_f(x,a) \le \alpha \tag{2.11}$$

for every $a \in \text{ac}(X)$. Then the inequalities

$$\frac{1}{\alpha}\dim(A) \le \dim(f(A)) \le \frac{1}{\beta}\dim(A) \tag{2.12}$$

hold for every $A \subseteq X$.

Proof. By Proposition 2.3 it suffices to prove the first inequality in (2.12).

Since f is a homeomorphism, we have

$$\text{ac}(Y) = f(\text{ac}(X)).$$

Let f^{-1} be the inverse map of f and let $a \in \text{ac }(X)$. Applying inequality (2.11) we obtain

$$\frac{1}{\alpha} \le \{\limsup_{x\to a} K_f(x,a)\}^{-1} = \liminf_{y\to b} K_{f^{-1}}(y,b)$$

where $b = f(a) \in \text{ac}(Y)$. Now the desired inequality follows from Proposition 2.3. □

2.5. *Proof of Theorem 1.1.* (i) $\Rightarrow$ (ii) Suppose that there is $X_0 \subseteq \text{ac}(X)$ such that $\alpha(a) = \alpha_0$ for every $a \in \text{ac}(X)\backslash X_0$, and for every $Z \subseteq X_0$ we have either $\dim(Z) = 0$ or $\dim(Z) = \infty$. Let A be a subset of X. Then

$$\dim(A) = \max\{\dim(A\backslash X_0),\ \dim(A\cap X_0)\}$$

and

$$\dim(f(A)) = \max\{\dim(f(A\backslash X_0)),\ \dim(f(A\cap X_0))\}.$$

Thus, by Corollary 2.4 it remains to prove that either

$$\dim(A \cap X_0) = \dim(f(A \cap X_0)) = 0$$

or

$$\dim(A \cap X_0) = \dim(f(A \cap X_0)) = +\infty.$$

To prove this we represent $A \cap X_0$ in the form

$$A \cap X_0 = \bigcup_{n=1}^{\infty} A_n$$

where

$$A_n := \left\{ a \in A \cap X_0 : \frac{1}{n} \leq \alpha(a) \leq n \right\}$$

for $n = 1, 2, \ldots$. From this representation we get the equalities

$$f(A \cap X_0) = \bigcup_{n=1}^{\infty} f(A_n),$$

$$\dim(A \cap X_0) = \sup_{1 \leq n < \infty} (\dim A_n),$$

$$\dim(f(A \cap X_0)) = \sup_{1 \leq n < \infty} \dim(f(A_n)).$$

Hence $\dim(A \cap X_0) = 0$ iff

$$\forall\, n \in \mathbb{N} : \dim(A_n) = 0.$$

It follows from the alternative

$$\dim(A_n) = 0 \ \text{ or } \ \dim(A_n) = \infty$$

that we have $\dim(A \cap X_0) = \infty$ iff there exists $n_0 \in \mathbb{N}$ such that $\dim(A_{n_0}) = \infty$.

Consequently, the conclusion follows by Corollary 2.4.

(ii) $\Rightarrow$ (i) Suppose that (1.3) holds for every $A \subseteq X$. Put

$$\begin{aligned} X_0^+ &:= \{a \in \mathrm{ac}X : \alpha(a) > \alpha_0\}, \\ X_0^- &:= \{a \in \mathrm{ac}X : \alpha(a) < \alpha_0\}, \quad \text{and} \\ X_0 &:= X_0^+ \cup X_0^-. \end{aligned}$$

By definition $\alpha(a) = \alpha_0$ for each $a \in (\mathrm{ac}X) \backslash X_0$. It remains to prove that

$$\dim(Z) = 0 \ \text{ or } \ \dim(Z) = +\infty$$

for each $Z \subseteq X_0$.

Suppose that for some $Z \subseteq X_0$

$$0 < \dim(Z) < \infty.$$

Then we have

$$0 < \dim(Z \cap X_0^+) < \infty \tag{2.13}$$

or

$$0 < \dim(Z \cap X_0^-) < \infty. \tag{2.14}$$

Consider the case (2.13) first. Set for each positive integer n

$$X_n^+ := \Big\{a \in \mathrm{ac}(X) : \alpha(a) \in \Big(\alpha_0 + \frac{1}{n}, \alpha_0 + n\Big)\Big\}. \tag{2.15}$$

Then

$$X_0^+ = \bigcup_{n=1}^{\infty} X_n^+$$

and (2.13) implies that for some $n_0 \in \mathbb{N}$

$$0 < \dim(Z \cap X_{n_0}^+) < \infty.$$

Hence, by Corollary 2.4 and (2.15), we have

$$\dim(f(Z \cap X_{n_0}^+)) \leq \frac{1}{\alpha_0 + \frac{1}{n_0}} \dim(Z \cap X_{n_0}^+) < \frac{1}{\alpha_0} \dim(Z \cap X_{n_0}^+).$$

This contradicts (1.3) with $A = Z \cap X_{n_0}^+$. The case (2.14) can be proved analogously. □

2.6. *Proof of Corollary 1.2.* In order to prove this corollary, it suffices to take $\alpha_0 = 1$. It should be observed here that the inequality $\dim(X) < \infty$ implies the equality $\dim(Z) = 0$ for each $Z \subseteq X_0$. □

Corollary 2.7

Suppose that (X, ρ) and (Y, d) are metric spaces, and

$$X = X^\circ \cup X^1, \quad X^\circ \cap X^1 = \emptyset.$$

Let $\alpha \in (0, \infty)$ and let $\varphi : X \to Y$ be a mapping such that:

$$\dim(X^\circ) = \dim(\varphi(X^\circ)) = 0. \tag{2.16}$$

2.8. *For every $a \in \mathrm{ac}(X^1) \cap X^1$,*

$$\lim_{\substack{x \to a \\ x \in X^1}} K_\varphi(x, a) = \alpha.$$

2.9. *The restriction $\varphi|_{X^1} : X^1 \to \varphi(X^1)$ is a homeomorphism.*
Then

$$\dim(\varphi(A)) = \frac{1}{\alpha} \dim(A) \tag{2.17}$$

for each $A \subseteq X$.

Proof. It follows from (2.16) that $\dim(A) = \dim(A \cap X^1)$ and $\dim(\varphi(A)) = \dim(\varphi(A \cap X^1))$ for every $A \subseteq X$. Consequently, it suffices to prove (2.17) for $A \subseteq X^1$.

For this purpose we can use Theorem 1.1 with $X = X^1$, $Y = \varphi(X^1)$, $X_0 = \emptyset$ and f equals $\varphi|_{X^1} : X^1 \to \varphi(X^1)$. □

Corollary 2.10

Suppose that all the conditions of Corollary 2.7 hold with an exception of 2.9. If X^1 can be represented in the form $X^1 = \bigcup_{j \in \mathbb{N}} X^1_j$ such that $\varphi|_{X^1_j} : X^1_j \to \varphi(X^1_j)$ is a homeomorphism for every $j \in \mathbb{N}$, then equality (2.17) holds for every $A \subseteq X$.

Proof. Reasoning as in the proof of Corollary 2.7 we can easily show that

$$\dim(\varphi(A)) = \sup_{j \in \mathbb{N}}(\dim(\varphi(A \cap X^1_j))) = \frac{1}{\alpha} \sup_{j \in \mathbb{N}}(\dim(A \cap X^1_j)) = \frac{1}{\alpha} \dim(A). \qquad \square$$

Corollary 2.11

Suppose that all the conditions of Corollary 2.7 hold except 2.9. If X^1 is separable and $\varphi|_{X^1} : X^1 \to \varphi(X^1)$ is a local homeomorphism, then equality (2.17) holds for each $A \subseteq X$.

Proof. Every separable metric space has a countable base, see e.g. [5, §21, II, Theorem 2]. Hence we can use Corollary 2.10. □

Let $cp(X)$ denote the set of all **condensation points** of a metric space X, i.e. points whose neighborhoods are not countable sets.

Corollary 2.12

Suppose that X and Y are metric spaces, $f : X \to Y$ is a local homeomorphism and X is separable. Let $\alpha \in (0, \infty)$ and let

$$\lim_{x \to a} K_f(x, a) = \alpha$$

for every $a \in cp(X)$. Then (2.17) holds for every $A \subseteq X$.

Proof. The set $X \backslash cp\ (X)$ is a countable set in every separable metric space X, [5, §23, III]. □

3. The Cantor ternary function

We recall the definitions of the Cantor ternary set C and ternary Cantor function G. Let $x \in [0, 1]$, then x belongs C if and only if x has a base 3 expansion using only the digits 0 and 2, i.e.

$$x = \sum_{m=1}^{\infty} \frac{2\alpha_m}{3^m}, \quad \alpha_m \in \{0,1\}. \tag{3.1}$$

The Cantor function G may be defined on C by the following rule. If $x \in C$ has the ternary expansion (3.1), then

$$G(x) := \sum_{m=1}^{\infty} \frac{\alpha_m}{2^m}.$$

In this section we will denote by C^1 the set of all endpoints of complementary intervals of C. We also write $C^\circ := C \backslash C^1$.

The proof of Theorem 1.4 needs the following lemma.

Lemma 3.1

Let $x \in C$ be a point with the representation (3.1) and let $\mathcal{R}_x(n)$ be the sequence from (1.7). Then

$$\mathcal{R}_x(n) = \mathcal{R}_{1-x}(n). \tag{3.2}$$

If x is not a right endpoint of a complementary interval of C, then

$$\sum_{m=n+1}^{\infty} \alpha_m 2^{-(m-n)} \geq 2^{-(1+\mathcal{R}_x(n+1))}, \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

Proof. Since $1-x = \sum\limits_{m=1}^{\infty} \frac{2(1-\alpha_m)}{3^m}$ we have (3.2). It remains to prove (3.3). If $\alpha_{n+1} = 1$, then (3.3) is obvious. In the opposite case, $2^{-(1+\mathcal{R}_x(n+1))}$ is the first positive element of the series $\sum\limits_{m=n+1}^{\infty} \alpha_m 2^{-(m-n)}$. $\square$

3.2. *Proof of Theorem 1.4.* Consider first the case where x is not an endpoint of some complementary interval of C. Suppose that x has representation (3.1), y tends to x, and

$$y = \sum_{m=1}^{\infty} \frac{2\beta_m}{3^m}$$

where $\beta_m = \beta_m(y) \in \{0,1\}$. Let $n_0 = n_0(x,y)$ be the smallest index m with $|\beta_m - \alpha_m| \neq 0$. Then using the definition of the Cantor function we have

$$\lim_{\substack{y\to x\\ y\in C}} \frac{\log|G(x)-G(y)|}{\log|x-y|} = \lim_{\substack{y\to x\\ y\in C}} \frac{\log\left|\sum\limits_{m=n_0}^{\infty}(\alpha_m-\beta_m)2^{-(m-n_0)}\right| - n_0\log 2}{\log\left|2\sum\limits_{m=n_0}^{\infty}(\alpha_m-\beta_m)3^{-(m-n_0)}\right| - n_0\log 3}.$$

It is easy to make sure that

$$1 \leq \left|2\sum_{m=n_0}^{\infty}(\alpha_m-\beta_m)3^{-(m-n_0)}\right| \leq 3$$

for all $y \in C$ and that $n_0(x,y)$ tends to infinity if $y \to x$, $y \in C$. Hence,

$$\frac{\log 3}{\log 2} \lim_{\substack{y\to x\\ y\in C}} \frac{\log|G(x)-G(y)|}{\log|x-y|} = \lim_{\substack{y\to x\\ y\in C}} \left(1 - \frac{\log\left|(\alpha_{n_0}-\beta_{n_0}) + \sum\limits_{m=n_0+1}^{\infty}(\alpha_m-\beta_m)2^{-(m-n_0)}\right|}{n_0 \log 2}\right).$$

Writing

$$z(x,y) := \left|(\alpha_{n_0}-\beta_{n_0}) + \sum_{m=n_0+1}^{\infty}(\alpha_m-\beta_m)2^{-(m-n_0)}\right|$$

we see that the limit relation (1.8) is equivalent to

$$\lim_{\substack{y\to x\\ y\in C}} \frac{\log z(x,y)}{n_0(x,y)} = 0. \tag{3.4}$$

Next we obtain bounds for $z(x,y)$. If

$$(\alpha_{n_0}-\beta_{n_0}) = 1,$$

then

$$z(x,y) = 1 + \sum_{m=n_0+1}^{\infty}(\alpha_m-\beta_m)2^{-(m-n_0)}$$

and hence

$$\sum_{m=n_0+1}^{\infty}\alpha_m 2^{-(m-n_0)} \le z(x,y) \le 2.$$

Consequently, by (3.3) we obtain

$$2^{-(1+\mathcal{R}_x(n_0+1))} \le z(x,y) \le 2. \tag{3.5}$$

If

$$(\alpha_{n_0}-\beta_{n_0}) = -1,$$

then

$$z(x,y) = 1 - \sum_{m=n_0+1}^{\infty}(\alpha_m-\beta_m)2^{-(m-n_0)},$$

and hence

$$1 - \sum_{m=n_0+1}^{\infty}\alpha_m 2^{-(m-n_0)} \le z(x,y) \le 2. \tag{3.6}$$

Since

$$1 - \sum_{m=n_0+1}^{\infty}\alpha_m 2^{-(m-n_0)} = \sum_{m=n_0+1}^{\infty}(1-\alpha_m)2^{-(m-n_0)},$$

relations (3.6), (3.3) and (3.2) imply that

$$z(x,y) \ge 2^{-(1+\mathcal{R}_{1-x}(n_0+1))} = 2^{-(1+\mathcal{R}_x(n_0+1))}.$$

Consequently, as in the first case, we have (3.5). Now, (3.4) follows from (3.5) and (1.9). Thus, the implication (1.9) $\Rightarrow$ (1.8) follows.

Suppose now that (1.9) does not hold. Then there is a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \frac{\mathcal{R}_x(1+n_j)}{1+n_j} = \limsup_{n\to\infty} \frac{\mathcal{R}_x(n)}{n} = a \in (0,\infty] \tag{3.7}$$

and

$$\alpha_{n_j} \neq \alpha_{n_j+1}, \quad \forall\; n_j \tag{3.8}$$

where α_{n_j} and α_{n_j+1} are digits in the representation (3.1).

Let $\{\alpha_m\}_{m=1}^{\infty}$ be a sequence of digits in (3.1). Putting

$$\beta_m^{(j)} := \begin{cases} 1-\alpha_m & \text{if } \; n_j \le m \le n_j + \mathcal{R}_x(1+n_j) \\ \\ \alpha_m & \text{otherwise} \end{cases} \tag{3.9}$$

and

$$y_j := \sum_{m=1}^{\infty} \frac{2\beta_m^{(j)}}{3^m},$$

we claim that

$$\lim_{j\to\infty} \frac{\log|G(x)-G(y_j)|}{\log|x-y_j|} = \frac{\log 2}{\log 3}\,(1+a).$$

Indeed, (3.8) and (3.9) imply the equalities

$$\frac{1}{2}|x-y_j| = \left(\frac{1}{3}\right)^{n_j} - \left(\frac{1}{3}\right)^{n_j+1} - \ldots - \left(\frac{1}{3}\right)^{n_j+\mathcal{R}_x(n_j+1)},$$

$$|G(x)-G(y_j)| = \left(\frac{1}{2}\right)^{n_j} - \left(\frac{1}{2}\right)^{n_j+1} - \ldots - \left(\frac{1}{2}\right)^{n_j+\mathcal{R}_x(n_j+1)}.$$

Hence, by (3.7), we have

$$\lim_{j\to\infty} \frac{\log|G(x)-G(y_j)|}{\log|x-y_j|} = \frac{\log 2}{\log 3} \lim_{j\to\infty} \frac{n_j+\mathcal{R}_x(1+n_j)}{n_j} = (1+a)\frac{\log 2}{\log 3}.$$

Consider now the case where $x \in C^1$. In this case there is $n_0 \in \mathbb{N}$ such that $\mathcal{R}_x(n) = 0$ for every $n \ge n_0$. It remains only to show that

$$\lim_{\substack{y\to x\\ y\in C}} \frac{\log|G(x)-G(y)|}{\log|x-y|} = \frac{\log 2}{\log 3}.$$

Suppose that x is a right endpoint of a complementary interval, (i.e. $\alpha_n = 0$ for all large enough n), y tends to x, and

$$y = \sum_{m=1}^{\infty} \frac{2\beta_m(y)}{3^m}, \quad \beta_m(y) \in \{0,1\}.$$

Then there are positive integers m_1 and $m_2 = m_2(y)$ such that

$$x = \sum_{m=1}^{m_1} \frac{2\alpha_m}{3^m},\ \beta_m(y) = \alpha_m$$

if $1 \le m \le m_1$, $m_2(y) > m_1$, $\beta_{m_2}(y) = 1$, and $\beta_m(y) = 0$ if $m_1 < m < m_2(y)$.

Hence

$$\lim_{\substack{y\to x\\ y\in C}} \frac{\log|G(x)-G(y)|}{\log|x-y|} = \lim_{\substack{y\to x\\ y\in C}} \frac{\log\left(\sum\limits_{m=m_2(y)}^{\infty} \beta_m 2^{-m}\right)}{\log\left(2\sum\limits_{m=m_2(y)}^{\infty} \beta_m 3^{-m}\right)} = \lim_{\substack{y\to x\\ y\in C}} \frac{-m_2(y)\log 2}{-m_2(y)\log 3} = \frac{\log 2}{\log 3}.$$

The case where x is a left endpoint of a complementary interval is similar. $\square$

EXAMPLE 3.3 Here we give an example of a homeomorphism $f : (X,\rho) \to (Y,d)$ such that:

(i) For every $a \in \mathrm{ac}(X)$

$$\lim_{x\to a} K_f(x,a) = 1.$$

(ii) For arbitrary positive α, c and for every ball $B(a,\delta) \subseteq X$ there exist $x, y \in B(a,\delta)$ for which

$$d(f(x),f(y)) \ge c(\rho(x,y))^\alpha.$$

The example is constructed with aid of the Cantor ternary set C and the Cantor function G.

Set

$$M := \left\{ x \in C : \lim_{\substack{y\to x\\ y\in C}} \frac{\log|G(x)-G(y)|}{\log|x-y|} = \frac{\log 2}{\log 3} \right\}$$

and

$$I^\circ := G(M \cap C^\circ). \tag{3.10}$$

Let $F_1 : I^\circ \to M \cap C^\circ$ be a function such that $F_1(G(y)) = y$ for every $y \in M \cap C^\circ$. It is easy to see that F_1 is a homeomorphism and by the definition of M we see that

$$\lim_{y\to a} K_{F_1}(a,y) = \frac{\log 3}{\log 2} \tag{3.11}$$

for every $a \in I^\circ$.

Let $\mathcal{E}$ be the space of infinite strings from the two-letters alphabet $\{0,1\}$. We may define a metric d on $\mathcal{E}$ by setting

$$d(\alpha,\beta) = \max_{1\le n<\infty} \frac{1}{2^n}|\alpha_n - \beta_n|$$

if $\alpha = \{\alpha_n\}_{n=1}^\infty$, $\beta = \{\beta_n\}_{n=1}^\infty$ are elements of $\mathcal{E}$.

Evidently, if $d(\alpha,\beta) \ne 0$, then there is a positive integer n such that $d(\alpha,\beta) = 2^{-n}$.

The space $(\mathcal{E}, d)$ is an ultrametric space and the map

$$C \ni \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} \xrightarrow{\Phi} \{\varepsilon_n\} \in \mathcal{E}$$

is a homeomorphism. (The proof can be found in [1, Chapter 2].)

We claim that the inequality

$$\frac{1}{2}|x-y|^{\log 2/\log 3} \leq d(\Phi(x), \Phi(y)) \leq |x-y|^{\log 2/\log 3} \tag{3.12}$$

holds for all $x, y \in C$. Indeed, if x and y have the representations

$$x = \sum_{i=1}^{\infty} 2\alpha_i 3^{-i}, \quad y = \sum_{i=1}^{\infty} 2\beta_i 3^{-i}, \quad \alpha_i,\ \beta_i \in \{0, 1\},$$

and if $d(\Phi(x), \Phi(y)) = 2^{-n}$, then we get

$$\frac{1}{2}|x-y| = \left|\sum_{i=n}^{\infty}(\alpha_i - \beta_i)3^{-i}\right| \geq 3^{-n}\left(1 - \sum_{i=1}^{\infty} 3^{-i}\right) = \frac{1}{2}(d(\Phi(x), \Phi(y)))^{\log 3/\log 2}.$$

A similar argument yields

$$|x-y| \leq 3(d(\Phi(x), \Phi(y))^{\log 3/\log 2}.$$

These inequalities imply (3.12).

Let $F_2 : M \cap C^{\circ} \to \Phi(M \cap C^{\circ})$ be the restriction of the map Φ on the set $M \cap C^{\circ}$. Obviously, F_2 is a homeomorphism and it follows from (3.12) that

$$\lim_{z \to a} K_{F_2}(a, z) = \frac{\log 2}{\log 3} \tag{3.13}$$

for each $a \in (M \cap C^{\circ})$. Now set, for every $x \in I^{\circ}$,

$$f(x) := F_2(F_1(x)).$$

The function f is a homeomorphism from I° to $\Phi(M \cap C^{\circ})$ and the limits (3.11), (3.13) imply that

$$\lim_{x \to a} K_f(x, a) = 1$$

for each $a \in I^{\circ}$, i.e., we get (i).

Suppose α, c are arbitrary positive constants and O is an open interval in $[0, 1]$. To prove (ii) we can choose $x \in C$ such that (3.7) holds and

$$\alpha > \frac{2}{1+a}, \quad a \in (0, \infty), \quad G(x) \in O.$$

Since M is a dense subset of a perfect set C, there are sequences $\{x_j\}_{j \in \mathbb{N}}$ and $\{y_j\}_{j \in \mathbb{N}}$ in M such that

$$\lim_{j \to \infty} x_j = \lim_{j \to \infty} y_j = x, \quad x_j \neq y_j \quad \forall\ j \in \mathbb{N},$$

and

$$\lim_{j\to\infty} \frac{\log|F_1(x_j) - F_1(y_j)|}{\log|x_j - y_j|} = \frac{\log 3}{\log 2}\frac{1}{(1+a)},$$

see the proof of Theorem 1.4. The last relation and (3.12) imply that

$$\lim_{j\to\infty} \frac{\log d(f(x_j), f(y_j))}{\log|x_j - y_j|} = \frac{1}{(1+a)}.$$

Hence there is $N_0 \in \mathbb{N}$ such that

$$d(f(x_j), f(y_j)) \geq |x_j - y_j|^{2/(1+a)}$$

for $j \geq N_0$. Since $\frac{2}{1+a} < \alpha$ and $\lim\limits_{j\to 0} |x_j - y_j| = 0$, we have (ii). □

To prove Theorem 1.5 we use the following two propositions.

Proposition 3.4

Let I° be the subset of the unit interval $[0,1]$ from (3.10). Then each number, simply normal to base 2, belongs to I°.

We recall the definition. Suppose x belongs to $[0,1]$ and has the following base 2 representation

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \quad a_n \in \{0,1\}.$$

This number x is called a **simply normal** to base 2 if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = \frac{1}{2}. \tag{3.14}$$

3.5. *Proof of Proposition 3.4.* Let $x_0 \in C$ be a point with the ternary expansion

$$x_0 = \sum_{n=1}^{\infty} \frac{2\alpha_n}{3^n}, \quad \alpha_n \in \{0,1\}.$$

Suppose $G(x_0)$ is a number simply normal to base 2. Since $G(x_0) = \sum\limits_{n=1}^{\infty} \frac{\alpha_n}{2^n}$, formula (3.14) implies that $x_0 \in C^\circ$. (If $x_0 \in C^1$, i.e. x is an endpoint of a complementary interval of C, then $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^{N} \alpha_n = 0$ or $\lim\limits_{N\to\infty} \frac{1}{N} \sum\limits_{n=1}^{N} \alpha_n = 1$.) Hence, it suffices to show that

$$\lim_{n\to\infty} \frac{\mathcal{R}_{x_0}(n)}{n} = 0. \tag{3.15}$$

Put $\mathbb{N}_0 := \{n : \alpha_n = 0\}$ and $\mathbb{N}_1 := \{n : \alpha_n = 1\}$. Since $x_0 \in C^\circ$ we have $\operatorname{card}(\mathbb{N}_0) = \operatorname{card}(\mathbb{N}_1) = \infty$. It follows from (3.14) that

$$\begin{aligned}\frac{1}{2} &= \lim_{\substack{m\to\infty\\ m\in\mathbb{N}_1}} \frac{1}{m+\mathcal{R}_{x_0}(m)} \sum_{n=1}^{m+\mathcal{R}_{x_0}(m)} \alpha_n \\ &= \lim_{\substack{m\to\infty\\ m\in\mathbb{N}_1}} \frac{m}{m+\mathcal{R}_{x_0}(m)} \frac{1}{m}\left(\mathcal{R}_{x_0}(m) - 1 + \sum_{n=1}^{m} \alpha_m\right) \\ &= \lim_{\substack{m\to\infty\\ m\in\mathbb{N}_1}} \frac{m}{m+\mathcal{R}_{x_0}(m)} \left(\frac{\mathcal{R}_{x_0}(m)}{m} + \frac{1}{2}\right).\end{aligned}$$

Hence we get

$$\lim_{\substack{m\to\infty\\ m\in\mathbb{N}_1}} \frac{\mathcal{R}_{x_0}(m)}{m} = 0.$$

A similar calculation yields

$$\lim_{\substack{m\to\infty\\ m\in\mathbb{N}_0}} \frac{\mathcal{R}_{x_0}(m)}{m} = 0.$$

Since $\mathbb{N} = \mathbb{N}_0 \cup \mathbb{N}_1$ we have (3.15). $\square$

The proof of the next lemma is well-known.

Lemma 3.6

Let m_1 be the Lebesgue measure on $\mathbb{R}$, and let $s = \log 2/\log 3$. Then

$$m_1(G(A)) = \mathcal{H}^s(A)$$

for every $A \subseteq C$.

3.7. *Proof of Theorem 1.5.* As in Example 3.3 set

$$M = \left\{ x \in C : \lim_{\substack{y\to x\\ y\in C}} K_G(y,x) = \frac{\log 2}{\log 3} \right\}.$$

We claim that

$$\dim(G(A)) = \frac{\log 3}{\log 2} \dim(A)) \tag{3.16}$$

for every $A \subseteq M$, and

$$\mathcal{H}^s(M) = 1 \tag{3.17}$$

for $s = \log 2/\log 3$.

In order to prove (3.16), we can apply Corollary 2.7 with $X = M$, $X^\circ = C^1$, $Y = [0,1]$. Observe that C^1 is countable and hence we have (2.16). The restriction

$$G|_{M\cap C^\circ} : M \cap C^\circ \to G(M \cap C^\circ)$$

is strictly increasing and continuous, so that it is a homeomorphism.

It remains to verify (3.17). Let N^- be the set of numbers which are not simply normal to base 2. It is known [6, p.103] that $m_1(N^-) = 0$. By Proposition 3.4 I° is the superset of the set of all simply normal to base 2 numbers. Consequently, $m_1(I^\circ) = 1$, and by Lemma 3.6 we have (3.17). □

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