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Boundary value problems and duality between L^p Dirichlet and regularity problems for second order parabolic systems in non-cylindrical domains

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Abstract

In this paper we consider general second order, symmetric and strongly elliptic parabolic systems with real valued and constant coefficients in the setting of a class of time-varying, non-smooth infinite cylinders

$$\Omega = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > A(x, t)\}.$$

We prove solvability of Dirichlet, Neumann as well as regularity type problems with data in L^p and $L^p_{1,1/2}$ (the parabolic Sobolev space having tangential (spatial) gradients and half a time derivative in L^p) for $p \in (2-\epsilon, 2+\epsilon)$ assuming that $A(x,\cdot)$ is uniformly Lipschitz with respect to the time variable and that $\|D^t_{1/2}A\|_* \leq \epsilon_0 < \infty$ for ϵ_0 small enough ($\|D^t_{1/2}A\|_*$ is the parabolic BMO-norm of a half-derivative in time). We also prove a general structural theorem (duality theorem between Dirichlet and regularity problems) stating that if the Dirichlet problem is solvable in L^p with the relevant bound on the parabolic non-tangential maximal function then the regularity problem can be solved with data in $L^q_{1,1/2}(\partial\Omega)$ with $q^{-1} + p^{-1} = 1$. As a technical tool, which also is of independent interest, we prove certain square function estimates for solutions to the system.

Keywords: Second order parabolic systems, time-varying cylinder, Carleson measure, maximal function, parabolic singular integrals, square functions, potential theory.
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1. Introduction and statement of main results

In recent years considerable progress has been made in the study of linear second order parabolic systems and equations in non-smooth, time-varying domains. In particular in [9], [11] the L^2 solvability for Dirichlet, Neumann and regularity type problems were established for parabolic Lamé systems as well as a linearized system of non-stationary Navier-Stokes equations.

General second order symmetric parabolic systems with real valued and constant coefficients satisfying the Legendre-Hadamard ellipticity condition have the following form for relevant n, m and constant coefficient matrices A^{rs},

$$\frac{\partial u^r}{\partial t} = \frac{\partial}{\partial x_i} A_{ij}^{rs} \frac{\partial u^s}{\partial x_j} \quad 0 \le i, j \le n-1 \quad 1 \le r, s \le m, \tag{1}$$

$$A_{ij}^{rs} = A_{ji}^{sr}, \tag{2}$$

$$A_{ij}^{rs} \nu_i \nu_j \eta^r \eta^s \ge c|\nu|^2 |\eta|^2 \quad \text{for all} \quad \nu \in \mathbb{R}^n \quad \eta \in \mathbb{R}^m. \tag{3}$$

From the perspective of boundary value problems in a domain Ω the most natural boundary conditions for $\vec{u} = (u^1, ..., u^m)$ are Dirichlet conditions, $\vec{u} = \vec{f}$, and Neumann type conditions $(\frac{\partial \vec{u}}{\partial \nu})^r = N^i A_{ij}^{rs} \frac{\partial u^s}{\partial x_j} = f^r$. Here $N = (N^0, ..., N^{n-1})$ is the (in the space variables) outward directed normal define on the boundary of the time-slices of Ω (this is made more precise below). A prototype for the kind of systems considered in this paper is the parabolic Lamé system

$$\frac{\partial \vec{u}}{\partial t} = \mu \Delta \vec{u} + (\sigma + \mu) \nabla (\text{div } \vec{u}). \tag{4}$$

The stationary version of this system appears in linear elasticity and the constants μ and σ are referred to as Lamé moduli. The parabolic Lamé system can be represented as a second order, symmetric, constant coefficient system satisfying the Legendre-Hadamard ellipticity condition in an infinite number of ways. For example if $n = m = 2$ we can express this system in the following two ways,

$$A_{ij}^{rs} = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} = \begin{pmatrix} 2\mu + \sigma & 0 & 0 & \sigma \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \sigma & 0 & 0 & 2\mu + \sigma \end{pmatrix}, \tag{5}$$

$$A_{ij}^{rs} = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} = \begin{pmatrix} 2\mu + \sigma & 0 & 0 & \mu + \sigma \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ \mu + \sigma & 0 & 0 & 2\mu + \sigma \end{pmatrix}. \tag{6}$$

Using integration by parts each representation give rise to a conormal derivative,

$$\left(\frac{\partial \vec{u}}{\partial \nu}\right)^r = \left(\frac{\partial \vec{u}}{\partial \nu_A}\right)^r = N^i A_{ij}^{rs} \frac{\partial u^s}{\partial x_j}. \tag{7}$$

In particular the examples in (5) and (6) give rise to the following two conormals,

$$\frac{\partial \vec{u}}{\partial \nu_1} = \sigma(\text{div } \vec{u})N + \mu((\nabla \vec{u}) + (\nabla \vec{u})^T)N, \tag{8}$$

$$\frac{\partial \vec{u}}{\partial \nu_2} = (\mu + \sigma)(\text{div } \vec{u})N + \mu(\nabla \vec{u})N. \tag{9}$$

Here $(\nabla\vec{u})^T$ is the transpose of the matrix under consideration and $(\nabla\vec{u})+(\nabla\vec{u})^T$ is the matrix of symmetric gradients. A Neumann type problem with (8) being prescribed on the boundary of the domain is usually referred to as the traction boundary value problem and in the stationary case this condition is the most relevant one from the physical point of view. In this case the corresponding matrix (5) is only semi-positive definite while in (6) the Lamé system is represented in terms of a positive definite matrix.

In this paper we will assume, partially as we are interested in proving a structural theorem about the duality between the Dirichlet and regularity problems for systems, that our system satisfies a condition stronger than the Legendre-Hadamard condition. In fact we will assume that

$$A_{ij}^{rs}\eta_i^r\eta_j^s \geq c\sum_{(l,q)}|\eta_l^q|^2 \quad \text{for all} \quad \eta_i, \eta_j \in \mathbb{R}^m. \tag{10}$$

This condition implies that the matrix $\{A_{ij}^{rs}\}$ is positive definite. As described above many systems can be represented in an infinite number of ways and our results on Dirichlet problems and our duality result will apply to any system having at least one representation in terms of a positive definite matrix $\{A_{ij}^{rs}\}$.

Our geometric set up is that of time-varying non-smooth domains of the form

$$\Omega = \{(x_0, x, t) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R} : \ x_0 > A(x,t)\} \tag{11}$$

where $n \geq 2$ and where the function $A(x,t) : \mathbb{R}^n \to \mathbb{R}$ is compactly supported. In order to introduce our regularity assumptions on the function A, which give a clear connection to parabolic singular integrals, we have to introduce some more notation. Let $z = (x,t) \in \mathbb{R}^{n-1}\times\mathbb{R}$ and let $\|z\|$ be the unique positive solution ρ of the equation

$$\frac{t^2}{\rho^4} + \sum_{i=0}^{n-1}\frac{x_i^2}{\rho^2} = 1. \tag{12}$$

Note that $\|(\delta x, \delta^2 t)\| = \delta\|(x,t)\|$ and we will call $\|z\|$ the parabolic norm of z. By definition the parabolic BMO is the space of locally integrable functions modulo constants satisfying

$$\|b\|_* := \sup_B \frac{1}{|B|}\int_B |b(z) - m_B b|dz < \infty \tag{13}$$

where $z = (x,t)$, B denotes the parabolic ball $B = B_r(z_0) = \{z \in \mathbb{R} = \|z - z_0\| < r\}$ and $m_B b$ denotes the average of the function b over the ball B. Let $\hat{\ },\vee$ be the Fourier and the inverse Fourier transform on $\mathbb{R}^n$, and let ξ, τ denote the phase variables. Following Fabes-Riviere [3, 4] we define a parabolic half-order time derivative by

$$\mathbb{D}_n A(x,t) := \left(\frac{\tau}{\|(\xi,\tau)\|}\hat{A}(\xi,\tau)\right)^{\vee}(x,t). \tag{14}$$

We let $\|\cdot\|_\infty$ be the supremum norm and define $\|A\|_{\text{comm}} = \|\nabla_x A\|_\infty + \|\mathbb{D}_n A\|_*$ where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right)$, $\|\nabla_x A\|_\infty := \sup_t \|\nabla_x A(\cdot,t)\|_\infty$. The regularity condition we impose on our domains in (11) is that $\|A\|_{\text{comm}} < \infty$. As explained in [5, p. 213] the notation $\|A\|_{\text{comm}}$ reflects the fact that this quantity is equivalent to the operator norm of the commutator $[\sqrt{\Delta - \partial/\partial t}, A]$ and since this commutator is the parabolic analogue of the first Calderón commutator, the present condition is, at least from the point of singular integrals the appropriate parabolic analogue of the Lipschitz domains which have been considered in the elliptic theory. One can also prove that $\|A\|_{\text{comm}} \leq \beta < \infty$ implies that $A(x,t)$ is parabolically Lipschitz in the following sense,

$$|A(x,t) - A(y,s)| \leq \beta(|x-y| + |t-s|^{1/2}) \quad x, y \in \mathbb{R}^n \quad t, s \in \mathbb{R}. \tag{15}$$

We furthermore introduce for $0 < \alpha \leq 2$ and $g \in C_0^\infty(\mathbb{R})$ the fractional differentiation operators D_α by

$$(D_\alpha g)^\wedge(\tau) := |\tau|^\alpha \hat{g}(\tau). \tag{16}$$

It is well-known that if $0 < \alpha < 1$ then

$$D_\alpha g(s) = c \int_{\mathbb{R}} \frac{g(s) - g(\tau)}{|s-\tau|^{1+\alpha}} d\tau, \tag{17}$$

whenever $s \in \mathbb{R}$, i.e., $I_\alpha = cD_\alpha^{-1}$, where $I_\alpha(s) = |s|^{\alpha-1}$ for $s \in \mathbb{R}$ is the one-dimensional Riesz transform of order α and c is a universal constant. If $h \in C_0^\infty(\mathbb{R}^n)$ then by $D_\alpha^t h : \mathbb{R}^n \to \mathbb{R}$ we will mean $D_\alpha h(x, \cdot)$ defined a.e. for each $x \in \mathbb{R}^{n-1}$. In [6] it is proved that

$$\|A\|_{\text{comm}} := \|\nabla_x A\|_\infty + \|\mathbb{D}_n A\|_* \approx \|\nabla_x A\|_\infty + \|D_{1/2}^t A\|_* \tag{18}$$

and that given $\varepsilon > 0,\ 0 < \varepsilon < 1$ and $\gamma,\ 0 < \gamma < \infty$ there exists $\delta = \delta(\varepsilon, \gamma) > 0$ such that if $\|\nabla_x A\|_\infty \leq \gamma < \infty$ then

$$\min\{\|D_{1/2}^t A\|_*,\ \|\mathbb{D}_n A\|_*\} \leq \delta \Rightarrow \max\{\|D_{1/2}^t A\|_*,\ \|\mathbb{D}_n A\|_*\} \leq \varepsilon. \tag{19}$$

I.e. the smallness of $\|\mathbb{D}_n A\|_*$ could equivalently be stated as a smallness condition on $\|D_{1/2}^t A\|_*$.

The surface measure on $\partial\Omega$ is defined as $d\sigma_t dt$, where $d\sigma_t$ is the naturally defined surface measure on the Lipschitz graph $\partial\Omega_t$. Here $\Omega_t = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \{t\}; x_0 > A(x,t)\}$ and the unit outer normal to Ω_t is denoted by $N_t = (N_t^0, ..., N_t^{n-1})$. $L^p(\partial\Omega)$ denotes the L^p-spaces w.r.t. the measure $d\sigma_t dt$. Following Fabes and Jodeit [2] we define a parabolic Sobolev space in the following way. Let $\pi = \partial\Omega \to \mathbb{R}^n$ be the projection $\pi(A(x,t), x, t) = (x,t)$ and set $\tilde{f} = f \circ \pi^{-1}$. $L_{1,1/2}^p(\partial\Omega)$ is defined to consist of equivalence classes of functions f with distributional derivatives in x satisfying $\|f\|_{L_{1,1/2}^p(\partial\Omega)} < \infty$, where

$$\|f\|_{L_{1,1/2}^p(\partial\Omega)} := \|\tilde{f}\|_{L_{1,1/2}^p(\mathbb{R}^n)} := \|\mathbb{D}\tilde{f}\|_p. \tag{20}$$

Here

$$(\mathbb{D}\tilde{f})^\wedge(\xi, \tau) := \|(\xi, \tau)\| \hat{\tilde{f}}(\xi, \tau), \tag{21}$$

i.e., $\tilde{f} = \mathbb{D}^{-1}\phi, \quad \phi \in L^p(\mathbb{R}^n)$ where $\mathbb{D}^{-1}$ is a parabolic Riesz potential. By applying Plancherel's theorem, if $p = 2$, we have

$$\|\mathbb{D}\tilde{f}\|_2 \approx \|D^t_{1/2}\tilde{f}\|_2 + \|\nabla_x \tilde{f}\|_2, \tag{22}$$

where $D^t_{1/2}$ denotes the one-dimensional one half fractional derivative of f in the time variable introduced in (16-17).

We are now ready to formulate the first four main results proved in this paper. In the statement of these theorems H denotes the Hilbert transform in the t-variable and the symbol $\tilde{N}_*$ refers to a non tangential maximal function operator defined in the bulk of the paper.

Theorem 1 (The Dirichlet problem)

Let Ω be as in (11) and assume that $\{A^{rs}_{ij}\}$ are real constants satisfying (2) and (10). Let $\|A\|_{\text{comm}} \leq \beta < \infty$ and assume that $\|D^t_{1/2}A\|_* \leq \epsilon_0 < \infty$. If $\epsilon_0 = \epsilon_0(\|\nabla_x A\|_\infty)$ is small enough then the following is true: given $\vec{f} \in L^2(\partial\Omega)$ there exists a unique $\vec{u}$, $\|\tilde{N}_*(\vec{u})\|_2 < \infty$, satisfying the following conditions,

$$\begin{aligned} \frac{\partial u^r}{\partial t} &= \frac{\partial}{\partial x_i} A^{rs}_{ij} \frac{\partial u^s}{\partial x_j} \quad 0 \leq i,j \leq n-1 \quad 1 \leq r,s \leq m \quad \textit{in} \quad \Omega, \\ \vec{u} &= \vec{f} \quad \text{a.e } \textit{on} \quad \partial\Omega. \end{aligned}$$

There furthermore exists $\vec{g} \in L^2(\partial\Omega)$ such that $\vec{u}$ can be represented as a double layer potential, $\vec{u} = D\vec{g}$, and

$$\|\tilde{N}_*(D\vec{g})\|_2 \leq C_\beta \|\vec{f}\|_2.$$

Theorem 2 (The Neumann problem)

Under the same assumptions as in Theorem 1 the following is true: given $\vec{f} \in L^2(\partial\Omega)$ there exists a unique $\vec{u}$ (modulo a constant vector), $\|\tilde{N}_*(\nabla\vec{u})\|_2 < \infty$, satisfying the following conditions,

$$\begin{aligned} \frac{\partial u^r}{\partial t} &= \frac{\partial}{\partial x_i} A^{rs}_{ij} \frac{\partial u^s}{\partial x_j} \quad 0 \leq i,j \leq n-1 \quad 1 \leq r,s \leq m \quad \textit{in} \quad \Omega, \\ \left(\frac{\partial \vec{u}}{\partial \nu}\right)^r &= N^i A^{rs}_{ij} \frac{\partial u^s}{\partial x_j} = f^r \quad \text{a.e } \textit{on} \quad \partial\Omega. \end{aligned}$$

There furthermore exists $\vec{g} \in L^2(\partial\Omega)$ such that $\vec{u}$ can be represented as a single layer potential, $\vec{u} = S\vec{g}$, and

$$\|\tilde{N}_*(\nabla S\vec{g})\|_2 + \|\tilde{N}_*(HD^t_{1/2}S\vec{g})\|_2 \leq C_\beta \|\vec{f}\|_2.$$

Theorem 3 (The regularity problem)

Under the same assumptions as in Theorem 1 the following is true: given $\vec{f} \in L^2_{1,1/2}(\partial\Omega)$ there exists a unique $\vec{u}$ (modulo a constant vector), $\|\tilde{N}_*(\nabla\vec{u})\|_2 < \infty$, satisfying the following conditions,

$$\begin{aligned} \frac{\partial u^r}{\partial t} &= \frac{\partial}{\partial x_i} A^{rs}_{ij} \frac{\partial u^s}{\partial x_j} \quad 0 \leq i,j \leq n-1 \quad 1 \leq r,s \leq m \quad \textit{in} \quad \Omega, \\ \vec{u} &= \vec{f}. \end{aligned}$$

There furthermore exists $\vec{g} \in L^2(\partial\Omega)$ such that $\vec{u}$ can be represented as a single layer potential, $\vec{u} = S\vec{g}$, and

$$\|\tilde{N}_*(\nabla S\vec{g})\|_2 + \|\tilde{N}_*(HD^t_{1/2}S\vec{g})\|_2 \leq C_\beta \|\vec{f}\|_{L^2_{1,1/2}(\partial\Omega)}.$$

Theorem 4 (Duality between the adjoint Dirichlet problem and the regularity problem)

Let

$$\begin{aligned} \Omega = \Omega_1 &= \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : \ x_0 > A(x,t)\}, \\ \Omega_2 &= \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : \ x_0 < A(x,t)\} \end{aligned}$$

and assume that $\|A\|_{\text{comm}} \leq \beta < \infty$ and that $\{A^{rs}_{ij}\}$ are real constants satisfying (2) and (10). Assume furthermore that for some $p \in (1,\infty)$ and for $k \in \{1,2\}$ there exist, given $\vec{f} \in L^p(\partial\Omega)$, a solution $\vec{v}_k$, satisfying $\|\tilde{N}_{*,k}(\vec{v}_k)\|_p < \infty$ ($\tilde{N}_{*,k}$ is, for $k \in \{1,2\}$, a non tangential maximal function operator defined Ω_k as the domain of reference), to the problem

$$\begin{aligned} -\frac{\partial v^r_k}{\partial t} &= \frac{\partial}{\partial x_i} A^{rs}_{ij} \frac{\partial v^s_k}{\partial x_j} \quad 0 \leq i,j \leq n-1 \quad 1 \leq r,s \leq m \quad \textit{in} \quad \Omega_k, \\ \vec{v}_k &= \vec{f} \quad \text{a.e } \textit{on} \quad \partial\Omega. \end{aligned}$$

Then for q satisfying, $q^{-1} + p^{-1} = 1$ there exist, given $\vec{g} \in L^q_{1,1/2}(\partial\Omega)$, a unique solution $\vec{u}$ (modulo a constant vector), satisfying $\|\tilde{N}_*(\nabla\vec{u})\|_q < \infty$, to the problem

$$\begin{aligned} \frac{\partial u^r}{\partial t} &= \frac{\partial}{\partial x_i} A^{rs}_{ij} \frac{\partial u^s}{\partial x_j} \quad 0 \leq i,j \leq n-1 \quad 1 \leq r,s \leq m \quad \text{in} \quad \Omega, \\ \vec{u} &= \vec{g} \quad \text{a.e on} \quad \partial\Omega \quad \text{in} \quad L^q_{1,1/2}(\partial\Omega). \end{aligned}$$

We emphasize that Theorem 1, Theorem 2 and Theorem 3 were previously proven in [11] in the case of the parabolic Lamé system in (4). Also note that by standard real-variable arguments it follows that the L^2 result is self-improving in the sense that there exists some small $\epsilon = \epsilon(\Omega) > 0$ such that the theorems and the inequalities remain valid for $2 - \epsilon < p < 2 + \epsilon$.

Theorem 4 states that if the adjoint Dirichlet problem is solvable in Ω and its complement, with data in $L^p(\partial\Omega)$ and with an appropriate bound on the non tangential maximal function operator, then the regularity problem is solvable with data in $L^q_{1,1/2}(\partial\Omega)$ for q satisfying, $q^{-1} + p^{-1} = 1$. The reverse statement, i.e., that solvability of the regularity problem implies solvability of the Dirichlet problem follows immediately from the fact if the regularity problem is solvable then an appropriate matrix of Green functions can be constructed. Using this matrix the Dirichlet problem can be solved by standard arguments. The theorem generalizes to the setting of systems the result of [7] for the heat equation. For the Laplace equation in Lipschitz domains it is well known that there exist a duality between solvability of the Dirichlet problem and the regularity problem of the type considered in this paper. In fact in that case the same is true for the Dirichlet and Neumann problem (see [10] for an account of this and related issues). Also note that compared to the case of the Laplace operator questions concerning the duality between the Dirichlet problem and the Neumann and regularity problems can not be paralleled, in the setting of non-smooth time-varying domains, in the case of the heat equation. This is clear from the results in [7] and [8] due to Hofmann and Lewis.

Concerning the proof of our results one key component is the following square function estimate for solutions $\vec{u}$ of our system (for the definition of the change of variables $\rho(\lambda, x, t) : \mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \mathbb{R} \to \Omega$ we again refer to the bulk of the paper).

Theorem 5

Let Ω be as in (11) and assume that $\|A\|_{\text{comm}} \leq \beta < \infty$. Let $\vec{u}$ be a solution to the second order parabolic system in (1) and assume that the constants defining the system, $\{A^{rs}_{ij}\}$, satisfy (2) and (10). Then for $r \in \{1, 2, .., m\}$

$$(i) \qquad \int_0^\infty \int_{\mathbb{R}^n} |u^r_{x_i x_j} \circ \rho|^2 \lambda dz d\lambda \leq c_\beta \|\tilde{N}_*(\nabla \vec{u})\|_2^2, \quad 0 \leq i, j \leq n-1,$$

$$(ii) \qquad \int_0^\infty \int_{\mathbb{R}^n} |u^r_{x_i t} \circ \rho|^2 \lambda^3 dz d\lambda \leq c_\beta \|\tilde{N}_*(\nabla \vec{u})\|_2^2, \quad 0 \leq i \leq n-1,$$

$$(iii) \qquad \int_0^\infty \int_{\mathbb{R}^n} \left|D^t_{1/2}(u^r_{x_i} \circ \rho)\right|^2 \lambda dz d\lambda \leq c_\beta \|\tilde{N}_*(\nabla \vec{u})\|_2^2, \quad 0 \leq i \leq n-1,$$

$$(iv) \qquad \int_0^\infty \int_{\mathbb{R}^n} \left|D^t_{3/2}(u^r \circ \rho)\right|^2 \lambda^3 dz d\lambda \leq c_\beta \|\tilde{N}_*(\nabla \vec{u})\|_2^2.$$

In fact having established the square function estimate in Theorem 5, Theorem 1, Theorem 2 and Theorem 3 can be proven by copying the lengthy arguments in [11]. Apart from Theorem 5 the main technical estimates of [11], i.e., [11, Lemma 3.4] and [11, Theorem 5.1], follows from estimates on parabolic singular integrals, Theorem 5 and by using the symmetry of the system. In the case of the second part of [11, Theorem 5.1] this may initially not seem obvious to the reader but by reanalyzing the argument of proof based on integration by parts, the reader can verify that as a result of the symmetry of our system the appropriate version of the statement of that technical component can be proved. We omit the details. Based on these estimates and the fact that we are assuming the strong ellipticity condition in (10) all of the results in [11, Section 6] based on Rellich identities and inequalities can be verified to hold in our setting. Concerning the proof of Theorem 1, Theorem 2 and Theorem 3 we will not reproduce more of these arguments in this paper. Instead from the perspective of proofs we will focus on the proof of Theorem 4 and Theorem 5.

The difficult part of Theorem 5 is part (i). In [11] this theorem was proven under the assumption that $\vec{u}$ solves the parabolic Lamé system in Ω. The argument explored certain specific features of that particular system and in particular the argument used the fact that if $\vec{u}$ solves the system in (4) then div $\vec{u}$ solves a heat equation in Ω. Still based on the arguments in [1] the intuition is that the type of square function estimates considered here should be valid, in the time-varying case, for all second order parabolic systems of the type we consider. That this is the case is proven in this paper. Concerning the proof of the duality theorem, i.e., Theorem 4, our argument is a generalization of the argument in [7] for the heat equation. That argument is based on ideas originated in [15] for the Laplace equation and explored in [13] in the study of higher order elliptic equations and systems in Lipschitz domains. The generalization of the elliptic approach to the situation of time-varying cylinders is already technically highly complicated in the case of the heat operator but the reader will notice that equipped with the square function estimate in Theorem 5 part (i) (and several variation on the same theme) we are able to carry the argument through also in the case of our systems.

It is our belief that our work is an encouraging contribution to the study of parabolic systems in this genuinely parabolic setting and that the next step is to go for parabolic versions of the results in [13] in a relevant set of time-varying cylinders.

The rest of the paper is organized as follows. In Section 2 we state a key lemma (Lemma 6) on Carleson measures, recall some facts on singular integrals and prove Theorem 5. Section 3 is devoted to the proof of Theorem 4.

2. Carleson measures, singular integrals and square functions

In this section we prove the square function estimates of Theorem 5 as well as a number of variation of these. Our geometric set up is, as described in the introduction, time-varying domains as in (11) where the regularity condition on the function A can be summarized as $\|A\|_{\mathrm{comm}} = \|\nabla_x A\|_\infty + \|\mathbb{D}_n A\|_* < \infty$.

2.1 Carleson measures and the non-tangential maximal function

Let $P(z) \in C_0^\infty(\mathbb{R}^n)$. We furthermore assume that $P(z)$ is a non-negative even function and that $\int_{\mathbb{R}^n} P(z)dz = 1$. I.e. we assume that $P(z)$ is a parabolic approximation of the identity. Let $d = n+1$ and define

$$P_\lambda(z) = \lambda^{-d} P(\lambda^{-\alpha} z) = \lambda^{-d} P\Big(\frac{x}{\lambda}, \frac{t}{\lambda^2}\Big). \tag{23}$$

For a locally integrabel function f we denote by $P_\lambda f$ the naturally defined operation of convolution. Define a 'parabolic' lifting $\rho(\lambda, x, t)$ from $\mathbb{R}_+^{n+1} = \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \mathbb{R}$ onto $\Omega = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} :\ x_0 > A(x,t)\}$ in the following way,

$$\rho(\lambda, x, t) = (\lambda + P_{\gamma\lambda} A(x,t), x, t) \quad \rho(0, x, t) = (A(x,t), x, t). \tag{24}$$

Here γ is a small parameter and we can adjust γ, as $\|\nabla_x A\|_\infty < \infty$, so that,

$$\frac{1}{2} \le 1 + \frac{\partial P_{\gamma\lambda} A(z)}{\partial \lambda} \le 3/2.$$

The following lemma is crucial and incorporates the geometry information in an analytic and quantitative way (for a proof see [6, p. 365–366]).

Lemma 6

Let σ, θ be non-negative integers and let $\phi = (\phi_1, \ldots, \phi_{n-1})$ be a multiindex. Define $\ell = \sigma + |\phi| + \theta$. Assume that $\|A\|_{\mathrm{comm}} \leq \beta < \infty$ and let

$$dv = \left(\frac{\partial^\ell P_{\gamma\lambda} A(x,t)}{\partial \lambda^\sigma \partial x^\phi \partial t^\theta} \right)^2 \lambda^{2\ell + 2\theta - 3} dx dt d\lambda$$

define a measure on $\mathbb{R}^{n+1}_+$. Then this measure is a Carleson measure on $\mathbb{R}^{n+1}_+$ if either $\sigma + \theta \geq 1$ or $|\phi| \geq 2$ and

$$(i) \quad v(B_r(z) \times (0,r)) \leq C r^d \gamma^{(2-2\phi-4\theta)} b^2 (1+\beta)^2.$$

Here $b = \|\mathbb{D}_n A\|_*$ if $\theta \geq 1$ and $b = 1$, if $\theta = 0$. Moreover if $\ell \geq 1$ then

$$(ii) \quad \left\| \frac{\partial^\ell P_{\gamma\lambda} A}{\partial \lambda^\sigma \partial x^\phi \partial t^\theta} \right\|_\infty \leq C_1 \gamma^{(1-|\phi|-2\theta)} \lambda^{1-\ell-\theta} b(1+\beta).$$

Remark 1 Let us in this context give a short digression to Littlewood-Paley theory. Let in the following $g \in C_0^\infty(\mathbb{R}^n)$ and recall that $\mathbb{D}_n = \mathbb{D}^{-1} \circ \frac{\partial}{\partial t}$ where $\mathbb{D}_n$ are $\mathbb{D}$ are the parabolic differential operator defined in (14) and (21). Let furthermore P_λ be as (23). As P_λ is an even function and therefore has vanishing first order moments, it follows by standard arguments that $\tilde{Q}_\lambda g = \mathbb{D}^{-1} \frac{\partial}{\partial \lambda} P_\lambda g$ satisfies the following Littlewood-Paley estimate for $p \in (1, \infty)$

$$\left(\int_{\mathbb{R}^n} \left(\int_0^\infty |\tilde{Q}_\lambda g|^2 \lambda^{-1} d\lambda \right)^{p/2} dx dt \right)^{1/p} \leq c_p ||g||_p.$$

Let $a > 0$ and $(X,t) = (x_0, x, t) \in \partial\Omega$. We let $\tilde{\Gamma}_a(X,t)$ be the parabolic cone

$$\tilde{\Gamma}_a(X,t) = \{(y_0, y, s) \in \Omega : \|(y-x, t-s)\| < a|y_0 - x_0|\}. \tag{25}$$

If h is a function defined on Ω we define the non-tangential maximal function $\tilde{N}_*(h) = \tilde{N}_*^a(h) : \partial\Omega \to \mathbb{R}$ by

$$\tilde{N}_*(h)(X,t) = \tilde{N}_*^a(h)(X,t) = \sup_{(Y,s) \in \tilde{\Gamma}_a(X,t)} |h|(Y,s). \tag{26}$$

We also introduce appropriate truncated version of this in the following way. Let $r > 0$ and let the parabolic cone, truncated at height r and centered at $(X,t) = (x_0, x, t) \in \partial\Omega$, be defined as

$$\tilde{\Gamma}_{a,r}(X,t) = \{(y_0, y, s) \in \Omega : \|(y-x, t-s)\| < a|y_0 - x_0|, \quad y_0 - x_0 < r\}.$$

Similarly we define the truncated non-tangential maximal function $\tilde{N}_*^{a,r}(h) : \partial\Omega \to \mathbb{R}$ by

$$\tilde{N}_*^{a,r}(h)(X,t) = \sup_{(Y,s)\in\tilde{\Gamma}_{a,r}(X,t)} |h|(Y,s).$$

To continue we let $B_{a\lambda}(x,t) = \{(y,s) : ||(x-y,t-s)|| < a\lambda\}$ and define

$$\begin{aligned}\Gamma_a(x,t) &:= \{(\lambda,y,s), \lambda > 0, (y,s) \in B_{a\lambda}(x,t)\},\\ \Gamma_{a,r}(x,t) &:= \{(\lambda,y,s), r > \lambda > 0, (y,s) \in B_{a\lambda}(x,t)\}.\end{aligned}$$

For a function g defined on $\mathbb{R}_+^{n+1}$ and for $a \geq 1$ fixed we also introduce the following maximal function $N_*^a(g) : \mathbb{R}_+^{n+1} \to \mathbb{R}$,

$$N_*(g)(x,t) = N_*^a(g)(x,t) = \sup_{\Gamma_a(x,t)} |g|(\lambda,y,s).$$

Similarly we introduce the truncated maximal function as

$$N_*^{a,r}(g)(x,t) = \sup_{\Gamma_{a,r}(x,t)} |g|(\lambda,y,s).$$

Let $\rho(\lambda,x,t)$ be the parabolic lifting introduced in the previous subsection. Note that if a and r are fixed numbers then one can easily prove that

$$\tilde{\Gamma}_{\tilde{a},\tilde{r}}(\rho(0,x,t)) \subset \rho(\Gamma_{a,r}(x,t)),$$

provided that $\tilde{a}$ is sufficiently small depending on a and $\|A\|_{\text{comm}}$ and $\tilde{r} = r + P_{\gamma r}A(x,t)$. Hence choosing γ small we can make sure that $|\tilde{r} - r|$ is small.

If again $a > 0$ and $(X,t) = (x_0,x,t) \in \partial\Omega$ we define using the notation just introduced, for functions u defined in Ω, the associate square function as well as the associated truncated square function as

$$\begin{aligned}S^a(u)(X,t) &= \Bigg(\int_{\tilde{\Gamma}_a(X,t)} |\nabla u(Y,s)|^2 \delta(Y,s)^{-n} dY ds\Bigg)^{1/2},\\ S^{a,r}(u)(X,t) &= \Bigg(\int_{\tilde{\Gamma}_{a,r}(X,t)} |\nabla u(Y,s)|^2 \delta(Y,s)^{-n} dY ds\Bigg)^{1/2}.\end{aligned}$$

Here $\delta(Y,s)$ is the parabolic distance from $(Y,s) \in \Omega$ to $\partial\Omega$. Note that if $F \subset \partial\Omega$, then by the theorem of Fubini,

$$\begin{aligned}\int_F |S^a(u)(X,t)|^2 d\sigma(X,t) &\sim \int_{\cup_{(X,t)\in F}\tilde{\Gamma}_a(X,t)} |\nabla u(Y,s)|^2 \delta(Y,s) dY ds,\\ \int_F |S^{a,r}(u)(X,t)|^2 d\sigma(X,t) &\sim \int_{\cup_{(X,t)\in F}\tilde{\Gamma}_{a,r}(X,t)} |\nabla u(Y,s)|^2 \delta(Y,s) dY ds.\end{aligned}$$

2.2 Singular integrals

Let in the following $(X,t) = (x_0, x_1, ..., x_{n-1}, t)$, $(x,t) = (x_1, ..., x_{n-1}, t)$. Let

$$\Gamma(X,t) = \Big(\Gamma_{j,k}(X,t)\Big)_{m\times m}$$

be a fundamental solution to the parabolic system under consideration. For the explicit construction in case of the Lamé system we refer to [11]. For $\vec{f} \in L^2(\partial\Omega)$ we define

$$S\vec{f}(X,t) = \int_{-\infty}^{t} \int_{\partial\Omega_s} \Gamma(X-Q,t-s)\vec{f}(Q,s)d\sigma_s(Q)ds \tag{27}$$

$$D\vec{f}(X,t) = \int_{-\infty}^{t} \int_{\partial\Omega_s} \left\{\frac{\partial}{\partial\nu_s}\Gamma(X-Q,t-s)\right\}^T \vec{f}(Q,s)d\sigma_s(Q)ds \tag{28}$$

where $\frac{\partial}{\partial\nu_s}$ is the conormal derivative defined in (7) applied to each column of the matrix. $S\vec{f}$ and $D\vec{f}$ are the single respectively double layer potentials. In [11] the following is proved in the case of the Lamé system and the argument carries over to this more general setting.

Theorem 7

Let $\|A\|_{\text{comm}} \le \beta < \infty$ and let $\vec{f} \in L^p(\partial\Omega)$ with $1 < p < \infty$. Define for $(P,t) \in \partial\Omega$ and $j = 0, 1 \ldots, n-1$ the following operators

$$K^j\vec{f}(P,t) := p.v \int_{-\infty}^{t} \int_{\partial\Omega_s} \frac{\partial\Gamma}{\partial x_j}(P-Q,t-s)\vec{f}(Q,s)d\sigma_s(Q)ds.$$

Then

$$\|K^j\vec{f}\|_p \le C_{\beta,p}\|\vec{f}\|_p.$$

Note that in case of a vector $\vec{g}$, $\|\vec{g}\|_p$ is defined as the sum of $\|g^j\|_p$ where g^j is component j of $\vec{g}$. A consequence of the last theorem is that $K^j\vec{f}(P,t)$ exists for a.e. $(P,t) \in \partial\Omega$ w.r.t. $d\sigma_t dt$. We now consider continuity of $S_b\vec{f}$ in the regularity space $L^p_{1,1/2}(\partial\Omega)$. By definition

$$S_b\vec{f}(P,t) = \int_{-\infty}^{t} \int_{\partial\Omega_s} \Gamma(P-Q,t-s)\vec{f}(Q,s)d\sigma_s(Q)ds \tag{29}$$

for all $(P,t) \in \partial\Omega$ for which this expression make sense. Based on the result in [5], [6] the following can be proved (see [11])

Theorem 8

Let $\|A\|_{\text{comm}} \le \beta < \infty$ and let $\vec{f} \in L^p(\partial\Omega)$. Then

$$\|S_b\vec{f}\|_{L^p_{1,1/2}(\partial\Omega)} \le C_\beta\|\vec{f}\|_p.$$

2.3 Square function estimates

In this section we will prove Theorem 5. We also prove a number of variations and state a number of remarks which will be useful in the next section.

Proof. Let $\|A\|_{\text{comm}} \leq \beta < \infty$ and let $\vec{u}$ be a solution to the second order parabolic system in (1) and assume that the system fulfills (2) and (10). We will start by proving (i) of Theorem 5. In particular we will prove that

$$\sum_{r,i} \int_0^\infty \int_{\mathbb{R}^n} |u^r_{x_i} \circ \rho|^2 \lambda dz d\lambda \leq c_\beta \|\tilde{N}_*(\vec{u})\|_2^2, \tag{30}$$

where $\rho(\lambda, x, t) = (\lambda + P_{\gamma\lambda} A(x,t), x, t)$, $\rho(0, x, t) = (A(x,t), x, t)$ was introduced in (23)-(24). Part (i) of Theorem 5 then follows by applying (30) to the vector $\vec{u}_{x_j}$.

Define

$$Q^{rs} = A^{rs}_{00} + \sum_{(i,j), i\neq 0, j\neq 0} A^{rs}_{ij} \frac{\partial P_{\gamma\lambda} A}{\partial x_i} \frac{\partial P_{\gamma\lambda} A}{\partial x_j} - 2 \sum_{j, j\neq 0} A^{rs}_{0j} \frac{\partial P_{\gamma\lambda} A}{\partial x_j}. \tag{31}$$

$Q = \{Q^{rs}\}$ is a $m \times m$ matrix. By the Legendre-Hadamard condition Q is a positive definite matrix with eigenvalues bounded from below by $C(1 + |\nabla P_{\gamma\lambda} A|^2)$. To prove the theorem we will start by manipulating the expression

$$I = -\int_{\mathbb{R}^n} Q^{rs} (u^r \circ \rho)(u^s \circ \rho) dz. \tag{32}$$

Here $\{r, s\}$ are initially assumed fixed, but we will at certain instances also sum over these indices. Integrating, in I, once by parts in the λ-direction we have,

$$\begin{aligned}
I &= -\int_0^\infty \int_{\mathbb{R}^n} Q^{rs}_{\lambda\lambda} (u^r \circ \rho)(u^s \circ \rho) \lambda dz d\lambda - \int_0^\infty \int_{\mathbb{R}^n} Q^{rs} (u^r \circ \rho)_{\lambda\lambda} (u^s \circ \rho) \lambda dz d\lambda \\
&\quad - \int_0^\infty \int_{\mathbb{R}^n} Q^{rs} (u^r \circ \rho)(u^s \circ \rho)_{\lambda\lambda} \lambda dz d\lambda - 2 \int_0^\infty \int_{\mathbb{R}^n} Q^{rs}_\lambda (u^r \circ \rho)(u^s \circ \rho)_\lambda \lambda dz d\lambda \\
&\quad -2 \int_0^\infty \int_{\mathbb{R}^n} Q^{rs}_\lambda (u^r \circ \rho)_\lambda (u^s \circ \rho) \lambda dz d\lambda - 2 \int_0^\infty \int_{\mathbb{R}^n} Q^{rs} (u^r \circ \rho)_\lambda (u^s \circ \rho)_\lambda \lambda dz d\lambda \\
&= T_1 + T_2 + T_3 + T_4 + T_5 + T_6.
\end{aligned} \tag{33}$$

To continue we first analyze T_2 and T_3. By symmetry we only have to treat T_2. Let from now on $D = (1 + \frac{\partial P_{\gamma\lambda} A}{\partial \lambda})$. Then by carrying out the differentiation we get

$$\begin{aligned}
T_2 &= -\int_0^\infty \int_{\mathbb{R}^n} Q^{rs} (u^r_{x_0 x_0} \circ \rho)(u^s \circ \rho) D^2 \lambda dz d\lambda \\
&\quad - \int_0^\infty \int_{\mathbb{R}^n} Q^{rs} (u^r_{x_0} \circ \rho)(u^s \circ \rho) \frac{\partial^2 P_{\gamma\lambda} A}{\partial \lambda^2} \lambda dz d\lambda.
\end{aligned}$$

As $u_t^s = A_{ij}^{rs} u_{x_i x_j}^r$ we have

$$u_t^p = A_{00}^{rp} u_{x_0 x_0}^r + \sum_{(i,j,r),(i,j)\neq(0,0)} A_{ij}^{rp} u_{x_i x_j}^r. \tag{34}$$

We now change variables. Assume initially that $i \neq 0$, $j \neq 0$. Then

$$\begin{aligned}(u_{x_i x_j}^r \circ \rho) &= (u_{x_j}^r \circ \rho)_{x_i} - (u_{x_j x_0}^r \circ \rho)\frac{\partial P_{\gamma\lambda}A}{\partial x_i} \\ &= (u_{x_j}^r \circ \rho)_{x_i} - (u_{x_0}^r \circ \rho)_{x_j}\frac{\partial P_{\gamma\lambda}A}{\partial x_i} + (u_{x_0 x_0}^r \circ \rho)\frac{\partial P_{\gamma\lambda}A}{\partial x_i}\frac{\partial P_{\gamma\lambda}A}{\partial x_j}. \end{aligned} \tag{35}$$

Suppose that $i = 0$, $j \neq 0$. Then

$$(u_{x_0 x_j}^r \circ \rho) \;=\; (u_{x_0}^r \circ \rho)_{x_j} - (u_{x_0 x_0}^r \circ \rho)\frac{\partial P_{\gamma\lambda}A}{\partial x_j}. \tag{36}$$

Combining (34-36),

$$\begin{aligned} u_t^p \circ \rho \;=\; & A_{ij}^{rp} u_{x_i x_j}^r \circ \rho = A_{00}^{rp} u_{x_0 x_0}^r \circ \rho \\ & + \sum_{(i,j,r),i\neq 0,j\neq 0} A_{ij}^{rp}\Big[(u_{x_j}^r \circ \rho)_{x_i} - (u_{x_0}^r \circ \rho)_{x_j}\frac{\partial P_{\gamma\lambda}A}{\partial x_i} + (u_{x_0 x_0}^r \circ \rho)\frac{\partial P_{\gamma\lambda}A}{\partial x_i}\frac{\partial P_{\gamma\lambda}A}{\partial x_j}\Big] \\ & + 2\sum_{(j,r),j\neq 0} A_{0j}^{rp}\Big[(u_{x_0}^r \circ \rho)_{x_j} - (u_{x_0 x_0}^r \circ \rho)\frac{\partial P_{\gamma\lambda}A}{\partial x_j}\Big]. \end{aligned}$$

Grouping the terms properly,

$$\begin{aligned} u_t^p \circ \rho = & \sum_r Q^{rp} u_{x_0 x_0}^r \circ \rho + \sum_{(i,j,r),i\neq 0,j\neq 0} A_{ij}^{rp}[(u_{x_j}^r \circ \rho)_{x_i} - (u_{x_0}^r \circ \rho)_{x_j}\frac{\partial P_{\gamma\lambda}A}{\partial x_i}] \\ & + 2\sum_{(j,r),j\neq 0} A_{0j}^{rp}(u_{x_0}^r \circ \rho)_{x_j}. \end{aligned} \tag{37}$$

Importing (37) into the formula for T_2 and putting $p = s$ we have,

$$\begin{aligned} T_2 \;=\; & -\int_0^\infty \int_{\mathbb{R}^n} (u_t^s \circ \rho)(u^s \circ \rho) D^2 \lambda dz d\lambda \\ & + \sum_{(i,j,r),i\neq 0,j\neq 0} \int_0^\infty \int_{\mathbb{R}^n} A_{ij}^{rs}(u_{x_j}^r \circ \rho)_{x_i}(u^s \circ \rho) D^2 \lambda dz d\lambda \\ & - \sum_{(i,j,r),i\neq 0,j\neq 0} \int_0^\infty \int_{\mathbb{R}^n} A_{ij}^{rs}(u_{x_0}^r \circ \rho)_{x_j}\frac{\partial P_{\gamma\lambda}A}{\partial x_i}(u^s \circ \rho) D^2 \lambda dz d\lambda \\ & + 2\sum_{(j,r),j\neq 0} \int_0^\infty \int_{\mathbb{R}^n} A_{0j}^{rs}(u_{x_0}^r \circ \rho)_{x_j}(u^s \circ \rho) D^2 \lambda dz d\lambda \\ & - \int_0^\infty \int_{\mathbb{R}^n} Q^{rs}(u_{x_0}^r \circ \rho)(u^s \circ \rho)\frac{\partial^2 P_{\gamma\lambda}A}{\partial \lambda^2} \lambda dz d\lambda. \end{aligned}$$

By partial integration in the second, third and fourth terms of this expression we get

$$
\begin{aligned}
T_2 &+ \int_0^\infty \int_{\mathbb{R}^n} (u_t^s \circ \rho)(u^s \circ \rho) D^2 \lambda dz d\lambda \\
&+ \sum_{(i,j,r), i \neq 0, j \neq 0} \int_0^\infty \int_{\mathbb{R}^n} A_{ij}^{rs} (u_{x_j}^r \circ \rho)(u^s \circ \rho)_{x_i} D^2 \lambda dz d\lambda \\
&- \sum_{(i,j,r), i \neq 0, j \neq 0} \int_0^\infty \int_{\mathbb{R}^n} A_{ij}^{rs} (u_{x_0}^r \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial x_i} (u^s \circ \rho)_{x_j} D^2 \lambda dz d\lambda \\
&+ 2 \sum_{(j,r), j \neq 0} \int_0^\infty \int_{\mathbb{R}^n} A_{0j}^{rs} (u_{x_0}^r \circ \rho)(u^s \circ \rho)_{x_j} D^2 \lambda dz d\lambda = G.
\end{aligned}
$$

Here G denotes a sum of terms to which we can apply Lemma 6 and conclude that

$$
|G| \leq C \Big(\int_0^\infty \int_{\mathbb{R}^n} |u_{x_0}^r \circ \rho|^2 \lambda dz d\lambda \Big)^{1/2} \Big(\int_{\mathbb{R}^n} |N_*(u^s \circ \rho)|^2 \, dz \Big)^{1/2}. \tag{38}
$$

Defining Λ as the set of index $\{(i,j,r), i \neq 0, j \neq 0\}$ can continue and conclude that

$$
T_2 + \int_0^\infty \int_{\mathbb{R}^n} (u_t^s \circ \rho)(u^s \circ \rho) D^2 \lambda dz d\lambda - G
$$

equals

$$
\begin{aligned}
&- \sum_\Lambda \int_0^\infty \int_{\mathbb{R}^n} A_{ij}^{rs} (u_{x_j}^r \circ \rho)(u_{x_i}^s \circ \rho) D^2 \lambda dz d\lambda \\
&- \sum_\Lambda \int_0^\infty \int_{\mathbb{R}^n} A_{ij}^{rs} (u_{x_j}^r \circ \rho)(u_{x_0}^s \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial x_i} D^2 \lambda dz d\lambda \\
&+ \sum_\Lambda \int_0^\infty \int_{\mathbb{R}^n} A_{ij}^{rs} (u_{x_0}^r \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial x_i} (u_{x_j}^s \circ \rho) D^2 \lambda dz d\lambda \\
&+ \sum_\Lambda \int_0^\infty \int_{\mathbb{R}^n} A_{ij}^{rs} (u_{x_0}^r \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial x_i} \frac{\partial P_{\gamma\lambda} A}{\partial x_j} (u_{x_0}^s \circ \rho) D^2 \lambda dz d\lambda \\
&- 2 \sum_{(j,r), j \neq 0} \int_0^\infty \int_{\mathbb{R}^n} A_{0j}^{rs} (u_{x_0}^r \circ \rho)(u_{x_j}^s \circ \rho) D^2 \lambda dz d\lambda \\
&- 2 \sum_{(j,r), j \neq 0} \int_0^\infty \int_{\mathbb{R}^n} A_{0j}^{rs} (u_{x_0}^r \circ \rho)(u_{x_0}^s \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial x_j} D^2 \lambda dz d\lambda.
\end{aligned}
$$

In this expansion the second and third term will cancel if we sum over the indices (r,s). In the following we will neglect these two terms and assume that we are summing over all indices (i.e., we will use summation convention). By simple manipulations we can therefore conclude that the last set of expressions equals

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^n} Q^{rs}(u^r_{x_0}\circ\rho)(u^s_{x_0}\circ\rho)D^2\lambda dz d\lambda - \int_0^\infty\int_{\mathbb{R}^n} A^{rs}_{00}(u^r_{x_0}\circ\rho)(u^s_{x_0}\circ\rho)D^2\lambda dz d\lambda\\
-&\int_0^\infty\int_{\mathbb{R}^n} A^{rs}_{ij}(u^r_{x_j}\circ\rho)(u^s_{x_i}\circ\rho)D^2\lambda dz d\lambda + \int_0^\infty\int_{\mathbb{R}^n} A^{rs}_{00}(u^r_{x_0}\circ\rho)(u^s_{x_0}\circ\rho)D^2\lambda dz d\lambda\\
=&\int_0^\infty\int_{\mathbb{R}^n} Q^{rs}(u^r_{x_0}\circ\rho)(u^s_{x_0}\circ\rho)D^2\lambda dz d\lambda - \int_0^\infty\int_{\mathbb{R}^n} A^{rs}_{ij}(u^r_{x_j}\circ\rho)(u^s_{x_i}\circ\rho)D^2\lambda dz d\lambda.
\end{aligned}
$$

Hence importing this into (33) (assuming a similar derivation for T_3) we get,

$$
\begin{aligned}
I =\ & T_1+T_4+T_5+T_6+G\\
&+2\int_0^\infty\int_{\mathbb{R}^n} Q^{rs}(u^r_{x_0}\circ\rho)(u^s_{x_0}\circ\rho)D^2\lambda dz d\lambda - 2\int_0^\infty\int_{\mathbb{R}^n} A^{rs}_{ij}(u^r_{x_j}\circ\rho)(u^s_{x_i}\circ\rho)D^2\lambda dz d\lambda\\
&-\int_0^\infty\int_{\mathbb{R}^n}(u^s_t\circ\rho)(u^s\circ\rho)D^2\lambda dz d\lambda - \int_0^\infty\int_{\mathbb{R}^n}(u^r_t\circ\rho)(u^r\circ\rho)D^2\lambda dz d\lambda.
\end{aligned}
$$

In fact this can be simplified to

$$
\begin{aligned}
I =\ & T_1+T_4+T_5+G-2\int_0^\infty\int_{\mathbb{R}^n} A^{rs}_{ij}(u^r_{x_j}\circ\rho)(u^s_{x_i}\circ\rho)D^2\lambda dz d\lambda\\
&-\int_0^\infty\int_{\mathbb{R}^n}(u^s_t\circ\rho)(u^s\circ\rho)D^2\lambda dz d\lambda - \int_0^\infty\int_{\mathbb{R}^n}(u^r_t\circ\rho)(u^r\circ\rho)D^2\lambda dz d\lambda.
\end{aligned}
\tag{39}
$$

Here the last cancellation follows from the form of the expression in T_6. Using the Carleson measure conditions of Lemma 6 we easily see that,

$$
|T_4|+|T_5|\leq C\sum_{r,s,j}\left(\int_0^\infty\int_{\mathbb{R}^n}|u^r_{x_0}\circ\rho|^2\lambda dz d\lambda\right)^{1/2}\left(\int_{\mathbb{R}^n}|N_*(u^s\circ\rho)|^2\,dz\right)^{1/2}. \tag{40}
$$

The term T_1 in (33), (39) decouples into a linear combination of terms of the types

$$
\begin{aligned}
A &= \int_0^\infty\int_{\mathbb{R}^n}\frac{\partial^3 P_{\gamma\lambda}A}{\partial x_i\partial\lambda^2}(u^r\circ\rho)(u^s\circ\rho)\lambda dz d\lambda,\\
B &= \int_0^\infty\int_{\mathbb{R}^n}\frac{\partial^2 P_{\gamma\lambda}A}{\partial x_i\partial\lambda}\frac{\partial^2 P_{\gamma\lambda}A}{\partial x_j\partial\lambda}(u^r\circ\rho)(u^s\circ\rho)\lambda dz d\lambda,\\
C &= \int_0^\infty\int_{\mathbb{R}^n}\frac{\partial P_{\gamma\lambda}A}{\partial x_i}\frac{\partial^3 P_{\gamma\lambda}A}{\partial x_j\partial\lambda^2}(u^r\circ\rho)(u^s\circ\rho)\lambda dz d\lambda.
\end{aligned}
$$

In A, B and C we have that $i \neq 0$. We intend to prove that

$$|A|+|B|+|C| \leq C\sum_{r,s,j}\Big(\int_0^\infty\int_{\mathbb{R}^n}|u^r_{x_j}\circ\rho|^2\lambda dz d\lambda\Big)^{1/2}\Big(\int_{\mathbb{R}^n}|N_*(u^s\circ\rho)|^2\,dz\Big)^{1/2} + C\sum_r\int_{\mathbb{R}^n}|N_*(u^r\circ\rho)|^2\,dz. \tag{41}$$

For A and C this is proven by lifting the derivative w.r.t. x_i in $\frac{\partial^3 P_{\gamma\lambda}A}{\partial x_i\partial\lambda^2}$ using partial integration. The rest is a consequence of Lemma 6. Combining (38-41) we can therefore conclude that,

$$2\int_0^\infty\int_{\mathbb{R}^n}A^{rs}_{ij}(u^r_{x_j}\circ\rho)(u^s_{x_i}\circ\rho)D^2\lambda dz d\lambda = -\int_0^\infty\int_{\mathbb{R}^n}(u^s_t\circ\rho)(u^s\circ\rho)D^2\lambda dz d\lambda - \int_0^\infty\int_{\mathbb{R}^n}(u^r_t\circ\rho)(u^r\circ\rho)D^2\lambda dz d\lambda + E. \tag{42}$$

Here $|E|$ is bounded by the expression on the r.h.s. of (41). From this and (42) as well as our strong ellipticity condition in (10) we are left with terms of the type,

$$\int_0^\infty\int_{\mathbb{R}^n}(u^s_t\circ\rho)(u^s\circ\rho)D^2\lambda dz d\lambda. \tag{43}$$

But the expression in (43) equals,

$$\int_0^\infty\int_{\mathbb{R}^n}(u^s\circ\rho)_t(u^s\circ\rho)D^2\lambda dz d\lambda - \int_0^\infty\int_{\mathbb{R}^n}(u^s\circ\rho)_\lambda(u^s\circ\rho)\frac{\partial P_{\gamma\lambda}A}{\partial t}D^2\lambda dz d\lambda\,. \tag{44}$$

The second term in (44) satisfies the same estimate as the term E in (42). In the first term of (44) we integrate by parts w.r.t. λ and get (ignoring the minus sign),

$$\int_0^\infty\int_{\mathbb{R}^n}(u^s\circ\rho)_{\lambda t}(u^s\circ\rho)D^2\lambda^2 dz d\lambda + \int_0^\infty\int_{\mathbb{R}^n}(u^s\circ\rho)_t(u^s\circ\rho)_\lambda D^2\lambda^2 dz d\lambda + 2\int_0^\infty\int_{\mathbb{R}^n}(u^s\circ\rho)_t(u^s\circ\rho)D\frac{\partial^2 P_{\gamma\lambda}A}{\partial\lambda^2}\lambda^2 dz d\lambda\,.$$

The third term in this expression can be handled using interior regularity estimate for the system (of the type stated in [11, Section 3]) and Lemma 6. By symmetrization we can assume that the sum of the first and second terms in the expression has the form

$$\int_0^\infty\int_{\mathbb{R}^n}[(u^s\circ\rho)(u^s\circ\rho)]_{t\lambda}\Big[1+2\frac{\partial P_{\gamma\lambda}A}{\partial\lambda}+\Big(\frac{\partial P_{\gamma\lambda}A}{\partial\lambda}\Big)^2\Big]\lambda dz d\lambda. \tag{45}$$

Partial integration w.r.t. t in (45), making use of Lemma 6 combined with Cauchy Schwarz with ϵ we can conclude that we have proved the inequality in (30) and hence part (i) of the Theorem 5. The deduction of (ii), (iii) and (iv) follows from (i) using the arguments in [11, p. 1322–1323]. □

Remark 2 An obvious consequence of the inequality in (30) is that

$$\int_0^\infty \int_{\mathbb{R}^n} |u^r_{x_i x_j} \circ \rho|^2 \lambda dz d\lambda \leq c_\beta \|N_*(\vec{u}_{x_j} \circ \rho)\|_2^2.$$

In the proof of the duality result in Theorem 4 we will need a L^p version of the inequality stated in Remark 2. It is well known that the appropriate L^p-version should follow from the L^2 result and real-variable techniques. Still in order to carry such an argument a localized version of the inequality in Remark 2 is needed. To formulate this properly let Δ_r be a cube on $\mathbb{R}^{n-1} \times \mathbb{R}$ of dimension $r \times .... \times r \times r^2$ and let

$$\tilde{\Delta}_r = \{(A(x,t), x, t) : (x,t) \in \Delta_r\}$$

be the associated surface cube on the boundary of our domain Ω.

Lemma 9

Let $\tilde{a} > 0$, $a > 0$ and let $\xi > 0$ be small number. For all cubes $\Delta_r \subset \mathbb{R}^{n-1} \times \mathbb{R}$ there exists a constant $C = C(||A||_{\text{comm}}, \tilde{a}, a, \xi)$ and a constant $\delta = \delta(||A||_{\text{comm}})$ such that

$$\int_{\tilde{\Delta}_r} |S^{\tilde{a},r}(u^k)(X,t)|^2 d\sigma(X,t) \leq C \int_{\tilde{\Delta}_{(1+\xi)r}} |\tilde{N}_*^{a,(1+\xi)\delta r}(u^s)(X,t)|^2 d\sigma(X,t).$$

The symbols $\tilde{N}_*^{a,(1+\xi)\delta r}(\cdot)$ and $S^{\tilde{a},r}(\cdot)$ refer to the truncated non-tangential maximal operator and square function operator introduced at the end of Section 2.1. Lemma 9 can be proved by arguing as in the proof of part (i) of Theorem 5. The main difference is that we instead start out by manipulating

$$I = -\int_{\mathbb{R}^n} Q^{rs}(u^r \circ \rho)(u^s \circ \rho)\tilde{\theta}^2 dz$$

where θ is an appropriate test function and $\tilde{\theta} = \theta \circ \rho$. The exact details of the proof (and much more) can be found in [12].

Remark 3 Based on Theorem 5 and Lemma 9 one can prove that for each $p \in (1, \infty)$ the following estimates is true,

$$\left(\int_{\mathbb{R}^n} \left(\int_0^\infty (u^r_{x_i x_j} \circ \rho)^2 \lambda d\lambda \right)^{p/2} dx dt \right)^{1/p} \leq c_\beta \|\tilde{N}_*(\nabla \vec{u})\|_p.$$

This is proved using real-variable techniques. In particular let $\lambda > 0$ and let $\Delta = \Delta_r \subset \mathbb{R}^{n-1} \times \mathbb{R}$ be a Whitney cube, of scale r, in the Whitney decomposition of $E_\lambda := \{(x,t) : S^{\tilde{a}}(u^k)(A(x,t),x,t) > \lambda\}$ such that $S^{\tilde{a}}(u^k)(A(x^*,t^*),x^*,t^*) \leq \lambda$ for some point $(x^*,t^*) \in 5\Delta$. 5Δ is the cube having the same center as Δ but being scaled by a factor 5 and $S^{\tilde{a}}(\cdot)$ is the square function operator introduced at the end of Section 2.1. Let also, for $\eta >> 1$ and $\epsilon << 1$,

$$F_\lambda(\eta,\epsilon) := \Big\{(x,t) \in \Delta : S^{\tilde{a}}(u^k)(A(x,t),x,t) > \eta\lambda, \sum_s \tilde{N}_*^a(u^s)(A(x,t),x,t) \leq \epsilon\lambda\Big\}.$$

By a standard argument based on interior regularity estimates we note that, if η large enough and ϵ is small enough then $S^{\tilde{a},r}(u^k)(A(x,t),x,t) > \eta\lambda/2$ if $(x,t) \in F_\lambda(\eta,\epsilon)$. If we define

$$\Omega(\Delta,\eta,\epsilon) := \cup_{(x,t)\in F_\lambda(\eta,\epsilon)} \tilde{\Gamma}_a(A(x,t),x,t)$$

then $\Omega(\Delta,\eta,\epsilon)$ is the region defined as the union of parabolic cones with vertex on $F_\lambda(\eta,\epsilon)$. Hence

$$(\eta\lambda/2)^2|F_\lambda(\eta,\epsilon)| \leq \int\limits_{F_\lambda(\eta,\epsilon)} |S^{\tilde{a},r}(u^k)(A(x,t),x,t)|^2 dxdt.$$

Let $\pi = \partial\Omega \to \mathbb{R}^n$ be the projection $\pi\big(A(x,t),x,t\big) = (x,t)$ then trivially $\pi^{-1}(F_\lambda(\eta,\epsilon)) \subset \partial\Omega(\Delta,\eta,\epsilon)$ and using Proposition 3.4 in [14] we can conclude that $\Omega(\Delta,\eta,\epsilon) = \{(x_0,x,t) : x_0 > \hat{A}(x,t)\}$ where $\|\hat{A}\|_{\text{comm}} < \beta$ with β independent of Δ. Based on Lemma 9 we can assume that there exist constants C and δ, independent of η, ϵ and Δ such that

$$\begin{aligned}
&\int\limits_{F_\lambda(\eta,\epsilon)} |S^{\tilde{a},r}(u^k)(A(x,t),x,t)|^2 dxdt \\
\leq\ &\int\limits_{F_\lambda(\eta,\epsilon)} |S^{\tilde{a},r}(u^k)(\hat{A}(x,t),x,t)|^2 dxdt \\
\leq\ & C\sum_s \int\limits_{(1+\xi)\tilde{\Delta}} |\tilde{N}^{a,(1+\xi)\delta r}_{*,\Omega(\Delta,\eta,\epsilon)}(u^s)(\hat{A}(x,t),x,t))|^2 dxdt.
\end{aligned}$$

The subscript in the maximal function indicates that it is defined w.r.t. $\Omega(\Delta,\eta,\epsilon)$. Hence by construction we can conclude that

$$(\eta\lambda/2)^2|F_\lambda(\eta,\epsilon)| \leq C\epsilon^2\lambda^2|\Delta|$$

and in particular we have proved that $|F_\lambda(\eta,\epsilon)| \leq C\epsilon^2/\eta^2|\Delta|$. From this we can conclude by standard arguments that for each $p \in (1,\infty)$ there exists a universal constant C such that

$$\int\limits_{\mathbb{R}^n} |S^{\tilde{a}}(u^k)(A(x,t),x,t)|^p dxdt \leq \sum_s C\int\limits_{\mathbb{R}^n} |\tilde{N}_*^a(u^s)(A(x,t),x,t)|^p dxdt.$$

The inequality stated in the beginning of this remark is now a consequence of this estimate and interior regularity estimates.

Based on the same techniques as in the proof of Theorem 5 we will also prove the following lemma which will be useful in the proof of Theorem 4.

Lemma 10

Let $\|A\|_{\text{comm}} \leq \beta < \infty$ and let $\vec{u}$ a solution to the second order parabolic system in (1) assuming that the system fulfills (2) and (10). Let δ be an arbitrary positive number. Then for any $p \in [2, \infty)$ such that the r.h.s. is finite the following is true,

$$\sum_r \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^p dz \leq C\delta \Big(\sum_r \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} |N_*(u^r_{x_i} \circ \rho)|^p dz \Big) + C(\delta) \|N_*(u^s_{x_0} \circ \rho)\|_p^p,$$

$$\int_{\mathbb{R}^n} |\mathbb{D}_n(u^r \circ \rho)|^p dz \leq C \|N_*(\nabla \vec{u} \circ \rho)\|_p^p.$$

Proof. We start by controlling the expression containing $(u^r \circ \rho)_{x_i}$. Integrating twice in the λ direction we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^p dxdt &= \int_0^\infty \int_{\mathbb{R}^n} \frac{\partial^2}{\partial \lambda^2} |(u^r \circ \rho)_{x_i}|^p \lambda dz d\lambda \\
&= p(p-1) \int_0^\infty \int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^{p-2} |(u^r \circ \rho)_{x_i\lambda}|^2 \lambda dz d\lambda \\
&\quad + p \int_0^\infty \int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^{p-1} (u^r \circ \rho)_{x_i\lambda\lambda} \lambda dz d\lambda.
\end{aligned}$$

Integrating by parts w.r.t. x_i in the last term we can conclude that

$$\begin{aligned}
\frac{1}{p(p-1)} \int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^p dxdt &= \int_0^\infty \int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^{p-2} |(u^r \circ \rho)_{x_i\lambda}|^2 \lambda dz d\lambda \\
&\quad - \int_0^\infty \int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^{p-2} (u^r \circ \rho)_{x_i x_i} (u^r \circ \rho)_{\lambda\lambda} \lambda dz d\lambda \\
&:= A + B. \qquad (46)
\end{aligned}$$

By a simple majorization using the non-tangential maximal operator and Cauchy-Schwarz we have for arbitrary positive numbers $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$,

$$\begin{aligned}
|A| &\leq C \Big(\int_{\mathbb{R}^n} |N_*((u^r \circ \rho)_{x_i})|^p dz \Big)^{1-2/p} \Big(\int_{\mathbb{R}^n} \Big(\int_0^\infty |(u^r \circ \rho)_{x_i\lambda}|^2 \lambda d\lambda \Big)^{p/2} dz \Big)^{2/p} \\
&\leq C\epsilon_1 \Big(\int_{\mathbb{R}^n} |N_*((u^r \circ \rho)_{x_i})|^p dz \Big) + C\epsilon_1^{-1} \Big(\int_{\mathbb{R}^n} \Big(\int_0^\infty |(u^r \circ \rho)_{x_i\lambda}|^2 \lambda d\lambda \Big)^{p/2} dz \Big) \\
&:= C\epsilon_1 A_1 + C\epsilon_1^{-1} A_2, \qquad (47)
\end{aligned}$$

$$
\begin{aligned}
|B| &\leq C\epsilon_2 \int_0^\infty \int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^{p-2} |(u^r \circ \rho)_{x_i x_i}|^2 \lambda dz d\lambda \\
&\quad + C\epsilon_2^{-1} \int_0^\infty \int_{\mathbb{R}^n} |(u^r \circ \rho)_{x_i}|^{p-2} |(u^r \circ \rho)_{\lambda\lambda}|^2 \lambda dz d\lambda \\
&\leq C\epsilon_2\epsilon_3 \Big(\int_{\mathbb{R}^n} |N_*((u^r \circ \rho)_{x_i})|^p dz \Big) + C\epsilon_2\epsilon_3^{-1} \Bigg(\int_{\mathbb{R}^n} \Big(\int_0^\infty |(u^r \circ \rho)_{x_i x_i}|^2 \lambda d\lambda \Big)^{p/2} dz \Bigg) \\
&\quad + C\epsilon_2^{-1}\epsilon_4 \Big(\int_{\mathbb{R}^n} |N_*((u^r \circ \rho)_{x_i})|^p dz \Big) + C\epsilon_2^{-1}\epsilon_4^{-1} \Bigg(\int_{\mathbb{R}^n} \Big(\int_0^\infty |(u^r \circ \rho)_{\lambda\lambda}|^2 \lambda d\lambda \Big)^{p/2} dz \Bigg) \\
&:= C\epsilon_2\epsilon_3 B_1 + C\epsilon_2\epsilon_3^{-1} B_2 + C\epsilon_2^{-1}\epsilon_4 B_3 + C\epsilon_2^{-1}\epsilon_4^{-1} B_4. \qquad (48)
\end{aligned}
$$

Hence we have to prove appropriate bounds on A_2, B_2 as well as B_4. As before

$$
\begin{aligned}
(u^r \circ \rho)_{x_i x_i} &= (u^r_{x_i x_i} \circ \rho) + 2(u^r_{x_i x_0} \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial x_i} + (u^r_{x_0} \circ \rho) \frac{\partial^2 P_{\gamma\lambda} A}{\partial x_i^2} \\
&\quad + (u^r_{x_0 x_0} \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial x_i} \frac{\partial P_{\gamma\lambda} A}{\partial x_i}, \\
(u^r \circ \rho)_{x_i \lambda} &= (u^r_{x_i x_0} \circ \rho) \Big(1 + \frac{\partial P_{\gamma\lambda} A}{\partial \lambda}\Big) + (u^r_{x_0 x_0} \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial x_i} \Big(1 + \frac{\partial P_{\gamma\lambda} A}{\partial \lambda}\Big) \\
&\quad + (u^r_{x_0} \circ \rho) \frac{\partial^2 P_{\gamma\lambda} A}{\partial x_i \partial \lambda}, \\
(u^r \circ \rho)_{\lambda\lambda} &= (u^r_{x_0 x_0} \circ \rho) \Big(1 + \frac{\partial P_{\gamma\lambda} A}{\partial \lambda}\Big)^2 + (u^r_{x_0} \circ \rho) \frac{\partial^2 P_{\gamma\lambda} A}{\partial \lambda^2}.
\end{aligned}
$$

Combining Remark 2 and 3 and using Lemma 6 we see that,

$$
A_2 + B_4 \quad \leq \quad c\|N_*(u^s_{x_0} \circ \rho)\|_p^p. \qquad (49)
$$

Similarly B_2 is bounded by

$$
c\Bigg[\|N_*(u^s_{x_i} \circ \rho)\|_p^p + \|N_*(u^s_{x_0} \circ \rho)\|_p^p + \int_{\mathbb{R}^n} \Bigg(\int_0^\infty |(u^r_{x_0} \circ \rho)|^2 \Big(\frac{\partial^2 P_{\gamma\lambda} A}{\partial x_i^2} \Big)^2 \lambda d\lambda \Bigg)^{p/2} dz \Bigg]. \qquad (50)
$$

By coping the argument in [7, p. 764–765] we can prove that the last term in (50) is bounded by $\|N_*(u^r_{x_0} \circ \rho)\|_p^p$. Hence

$$
B_2 \leq c\Big[\|N_*(u^s_{x_i} \circ \rho)\|_p^p + \|N_*(u^s_{x_0} \circ \rho)\|_p^p \Big]. \qquad (51)
$$

Returning to (46-48) and making appropriate choices of $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ we can conclude from (49) and (51) that

$$
|A| + |B| \leq C\delta \|N_*(u^r_{x_i} \circ \rho)\|_p^p + C(\delta) \|N_*(u^s_{x_0} \circ \rho)\|_p^p
$$

with δ arbitrary. This finishes the first part of the lemma.

To prove the second part let in the following $g \in C_0^\infty(\mathbb{R}^n)$, $||g||_q = 1$ where q is the index dual to p and let P_λ be the non-negative and even parabolic approximation of identity defined in (23). Again by integration by parts,

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \mathbb{D}_n(u^r \circ \rho) g dz &= -\int\limits_{\mathbb{R}^n}\int\limits_0^\infty \frac{\partial}{\partial\lambda}(\mathbb{D}_n(u^r \circ \rho) P_\lambda g) d\lambda dz \\ &= -\int\limits_{\mathbb{R}^n}\int\limits_0^\infty \mathbb{D}_n(u^r \circ \rho)\frac{\partial}{\partial\lambda} P_\lambda g d\lambda dz \\ &\quad -\int\limits_{\mathbb{R}^n}\int\limits_0^\infty \mathbb{D}_n(u^r \circ \rho)_\lambda P_\lambda g d\lambda dz := C + D. \end{aligned} \tag{52}$$

By duality we want to prove that both C and D in (52) are bounded as in the statement of the lemma. As $\mathbb{D}_n = \mathbb{D}^{-1} \circ \frac{\partial}{\partial t}$ we have

$$\begin{aligned} C &= -\int\limits_{\mathbb{R}^n}\int\limits_0^\infty (u^r \circ \rho)_t \mathbb{D}^{-1}\frac{\partial}{\partial\lambda} P_\lambda g d\lambda dz \\ &= -\int\limits_{\mathbb{R}^n}\int\limits_0^\infty (u^r_{x_0} \circ \rho)\mathbb{D}^{-1}\frac{\partial}{\partial\lambda} P_\lambda g \frac{\partial P_{\gamma\lambda} A}{\partial t} d\lambda dz \\ &\quad -\int\limits_{\mathbb{R}^n}\int\limits_0^\infty (u^r_t \circ \rho)\mathbb{D}^{-1}\frac{\partial}{\partial\lambda} P_\lambda g d\lambda dz := C_1 + C_2. \end{aligned}$$

Using Remark 1 and the equation we get

$$\begin{aligned} |C_2| &\le \left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty |\tilde{Q}_\lambda g|^2 \lambda^{-1} d\lambda\right)^{q/2} dz\right)^{1/q} \left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty (u^r_t \circ \rho)^2 \lambda d\lambda\right)^{p/2} dz\right)^{1/p} \\ &\le C\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty (u^s_{x_i x_j} \circ \rho)^2 \lambda d\lambda\right)^{p/2} dz\right)^{1/p} \end{aligned} \tag{53}$$

where we again have used summation convention. By Remark 3 we can therefore from (53) conclude that $|C_2| \le C\|N_*(\nabla\vec{u} \circ \rho)\|_p$. Similarly we have

$$|C_1| \;\le\; C\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty (u^r_{x_0} \circ \rho)^2 \left|\frac{\partial P_{\gamma\lambda} A}{\partial t}\right|^2 \lambda d\lambda\right)^{p/2} dz\right)^{1/p}. \tag{54}$$

By again copying the argument in [7, p. 764–765] we can conclude from (54) that $|C_1| \le C\|N_*(\nabla\vec{u} \circ \rho)\|_p$. To handle the term D we integrate once more w.r.t. λ,

$$\begin{aligned} D &= \int\limits_{\mathbb{R}^n}\int\limits_0^\infty \mathbb{D}_n(u^r \circ \rho)_{\lambda\lambda} P_\lambda g \lambda d\lambda dz \\ &\quad + \int\limits_{\mathbb{R}^n}\int\limits_0^\infty \mathbb{D}_n(u^r \circ \rho)_\lambda \frac{\partial}{\partial\lambda} P_\lambda g \lambda d\lambda dz := D_1 + D_2. \end{aligned} \tag{55}$$

To handle the term D_1 in (55) recall that $\mathbb{D}_n = \mathbb{D}^{-1} \circ \frac{\partial}{\partial t}$ and if we use that $\mathbb{D}^{-1}$ is a self-adjoint operator we get

$$\begin{aligned} D_2 &= \int_{\mathbb{R}^n}\int_0^\infty (u^r \circ \rho)_{t\lambda} \tilde{Q}_\lambda g \lambda d\lambda dz = \int_{\mathbb{R}^n}\int_0^\infty (u^r_{x_0 t} \circ \rho)\Big(1 + \frac{\partial P_{\gamma\lambda} A}{\partial \lambda}\Big) \tilde{Q}_\lambda g \lambda d\lambda dz \\ &\quad + \int_{\mathbb{R}^n}\int_0^\infty (u^r_{x_0 x_0} \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial t}\Big(1 + \frac{\partial P_{\gamma\lambda} A}{\partial \lambda}\Big) \tilde{Q}_\lambda g \lambda d\lambda dz \\ &\quad + \int_{\mathbb{R}^n}\int_0^\infty (u^r_{x_0} \circ \rho) \frac{\partial^2 P_{\gamma\lambda} A}{\partial \lambda \partial t} \tilde{Q}_\lambda g \lambda d\lambda dz \\ &:= D_{21} + D_{22} + D_{23}. \end{aligned} \tag{56}$$

In these expressions $\tilde{Q}_\lambda g = \mathbb{D}^{-1} \frac{\partial}{\partial \lambda} P_\lambda g$. Now in fact

$$|D_{21}| + |D_{22}| + |D_{23}| \le C \|N_*(\nabla \vec{u} \circ \rho)\|_p.$$

For D_{21} this follows from Remark 1, Remark 3 and interior regularity estimates. For D_{22} the conclusion is a consequence of the same two remarks combined with part (ii) of Lemma 6. Finally D_{23} is handled in the same way as we handled the term C_1 above. Left is to handle the term D_1 of (55). But spelling out

$$D_1 = \int_{\mathbb{R}^n}\int_0^\infty (u^r \circ \rho)_{t\lambda\lambda} \tilde{Q}_\lambda g \lambda d\lambda dz$$

in a way similar to the way we manipulated D_2 in (56) we can by analyzing all the terms conclude that all the terms can be handled using the arguments used to analyze the pieces D_{21}, D_{22} and D_{23}. This completes the proof of the Lemma. □

3. Proof of Theorem 4

By standard arguments (like the continuity method, see [10]) in order to prove Theorem 4 all we have to prove is the validity of the inequality

$$\|\vec{f}\|_q \le C_\beta \|S_b \vec{f}\|_{L^q_{1,1/2}(\partial\Omega)} \tag{57}$$

for q as in the statement of the theorem and where $S_b \vec{f}$ is defined in (29). Let

$$\begin{aligned} \Omega = \Omega_1 &= \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : \ x_0 > A(x,t)\}, \\ \Omega_2 &= \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : \ x_0 < A(x,t)\}. \end{aligned}$$

Let $S\vec{f}$ be the single layer potential defined in (27) and define $\vec{u}_1$ to be the restriction of $S\vec{f}$ to Ω_1. Similarly we define $\vec{u}_2$ to be the restriction of $S\vec{f}$ to Ω_2. Here If we assume that $\vec{f} \in L^q(\partial\Omega)$ then, as proved in [11, Section 3], a.e on $\partial\Omega$

$$\frac{\partial \vec{u}_1}{\partial \nu} = (I/2 + K^*_\nu)\vec{f}, \quad \frac{\partial \vec{u}_2}{\partial \nu} = (-I/2 + K^*_\nu)\vec{f}.$$

Here K_ν^* is an appropriate boundary operator having continuity in L^p as in Theorem 7 and the conormal is defined in (7). As

$$\|\vec{f}\|_q \leq \max\Big\{\|\frac{\partial \vec{u}_1}{\partial \nu}\|_q, \|\frac{\partial \vec{u}_2}{\partial \nu}\|_q\Big\} \tag{58}$$

we intend to prove that

$$\max\Big\{\|\frac{\partial \vec{u}_1}{\partial \nu}\|_q, \|\frac{\partial \vec{u}_2}{\partial \nu}\|_q\Big\} \leq C(\beta, n, q)\|S_b\vec{f}\|_{L^q_{1,1/2}(\partial\Omega)}. \tag{59}$$

Obviously (58) and (59) give us (57). In the following we just consider the case $\Omega = \Omega_1$ and we will for simplicity write $\vec{u} = \vec{u}_1$. By duality

$$\int\limits_{\partial\Omega} \left|\frac{\partial \vec{u}}{\partial \nu}\right|^q d\sigma_t dt = \sup_{\vec{v}\in L^p(\partial\Omega)} \Big|\int\limits_{\partial\Omega} \frac{\partial \vec{u}}{\partial \nu}\vec{v}\, d\sigma_t dt\Big| \tag{60}$$

and the idea is to manipulate the integral on the r.h.s. of (60). Note that $\frac{\partial \vec{u}}{\partial \nu}$ is vector of length m and that the multiplication on the r.h.s. of (60) should be understood as a scalar product. Using our assumption on the solvability of the Dirichlet problem any $\vec{v} \in L^p(\partial\Omega)$ is the trace in $L^p(\partial\Omega)$ of a solution to the adjoint system. Hence we can assume that there exists an extension to Ω of $\vec{v}$, also denoted by $\vec{v}$, such that $\|\tilde{N}_*(\vec{v})\|_p \leq C\|\vec{v}\|_p$ and such that

$$-\frac{\partial v^r}{\partial t} = \frac{\partial}{\partial x_i} A_{ij}^{rs} \frac{\partial v^s}{\partial x_j} \quad 0 \leq i, j \leq n-1 \quad 1 \leq r, s \leq m \quad \text{in} \quad \Omega. \tag{61}$$

We start out with some manipulations based on the divergence formula. Using that v solves the adjoint equation in (61) we have

$$\int\limits_{\Omega} \vec{v}\frac{\partial \vec{u}}{\partial t} dX dt = \int\limits_{\partial\Omega} \vec{v}\frac{\partial \vec{u}}{\partial \nu} d\sigma_t dt - \int\limits_{\Omega} \frac{\partial v^r}{\partial x_i} A_{ij}^{rs} \frac{\partial u^s}{\partial x_j} dX dt \tag{62}$$

$$\int\limits_{\Omega} \vec{u}\frac{\partial \vec{v}}{\partial t} dX dt = -\int\limits_{\partial\Omega} \vec{u}\frac{\partial \vec{v}}{\partial \nu} d\sigma_t dt + \int\limits_{\Omega} \frac{\partial u^r}{\partial x_i} A_{ij}^{rs} \frac{\partial v^s}{\partial x_j} dX dt. \tag{63}$$

Combining (62) and (63) using the symmetry condition in (2) we get

$$\int\limits_{\partial\Omega} \vec{v}\frac{\partial \vec{u}}{\partial \nu} d\sigma_t dt = \int\limits_{\partial\Omega} \vec{u}\frac{\partial \vec{v}}{\partial \nu} d\sigma_t dt + \int\limits_{\Omega} \Big(\vec{v}\frac{\partial \vec{u}}{\partial t} + \vec{u}\frac{\partial \vec{v}}{\partial t}\Big) dX dt := A + B. \tag{64}$$

Before continuing our manipulations we introduce some notation reflecting that our domain is a graph. In fact for $r = 1, ..., m$ we define

$$w^r(x_0, x, t) = \int\limits_{x_0}^{\infty} v^r(y, x, t)\, dy.$$

The vector $\vec{w} = (w^1, ..., w^m)$ solves the same adjoint equation as $\vec{v}$. When manipulating A and B we intend to explore that $\vec{w}$ is one degree smoother than $\vec{v}$ in the x_0-direction and in particular we intend to make use of Lemma 10. We start by considering the expression in A.

$$\begin{aligned}\Big(\frac{\partial \vec{v}}{\partial \nu}\Big)^r &:= A_{ij}^{rs} N^i \frac{\partial v^s}{\partial x_j} = A_{ij}^{rs} N^i \frac{\partial^2 w^s}{\partial x_j \partial x_0}\\ &= A_{ij}^{rs}\Big(N^i \frac{\partial}{\partial x_0} - N^0 \frac{\partial}{\partial x_i}\Big)\frac{\partial w^s}{\partial x_j} + A_{ij}^{rs} N^0 \frac{\partial^2 w^s}{\partial x_i \partial x_j}\\ &= A_{ij}^{rs}\Big(N^i \frac{\partial}{\partial x_0} - N^0 \frac{\partial}{\partial x_i}\Big)\frac{\partial w^s}{\partial x_j} - N^0 \frac{\partial w^r}{\partial t}. \qquad (65)\end{aligned}$$

We denote by $\frac{\partial}{\partial T_i}$, $i = 1, ..., n-1$, the tangential derivative $(N^i \frac{\partial}{\partial x_0} - N^0 \frac{\partial}{\partial x_i})$. Hence by (65)

$$\begin{aligned}A &= \int_{\partial\Omega} \vec{u} \frac{\partial \vec{v}}{\partial \nu} d\sigma_t dt = \int_{\partial\Omega} u^r A_{ij}^{rs} \frac{\partial}{\partial T_i} \frac{\partial w^s}{\partial x_j} d\sigma_t dt\\ &\quad - \int_{\partial\Omega} N^0 u^r \frac{\partial w^r}{\partial t} d\sigma_t dt := A_1 + A_2. \qquad (66)\end{aligned}$$

Trivially, by partial integration along the boundary,

$$|A_1| \ \le \ ||u^r||_{L^q_{1,1/2}(\partial\Omega)} \Big|\Big| \frac{\partial w^s}{\partial x_j} \Big|\Big|_{L^p(\partial\Omega)}. \qquad (67)$$

Manipulating A_2 in a by now standard way we have,

$$\begin{aligned}A_2 &= \int_{\mathbb{R}^n} (u^r \circ \rho)(w_t^r \circ \rho) dz = -\int_{\mathbb{R}^n} \int_0^\infty \frac{\partial}{\partial \lambda}[(u^r \circ \rho)(w_t^r \circ \rho)] d\lambda dz\\ &= -\int_{\mathbb{R}^n} \int_0^\infty \frac{\partial}{\partial \lambda}[(u^r \circ \rho)(w^r \circ \rho)_t] d\lambda dz + \int_{\mathbb{R}^n} \int_0^\infty \frac{\partial}{\partial \lambda}\Big[(u^r \circ \rho)(w_{x_0}^r \circ \rho)\frac{\partial P_{\gamma\lambda} A}{\partial t}\Big] d\lambda dz\\ &= \int_{\mathbb{R}^n} (u^r \circ \rho)(w^r \circ \rho)_t dz + \int_{\mathbb{R}^n} \int_0^\infty \frac{\partial}{\partial \lambda}\Big[(u^r \circ \rho)(w_{x_0}^r \circ \rho)\frac{\partial P_{\gamma\lambda} A}{\partial t}\Big] d\lambda dz := A_{21} + A_{22}.\end{aligned}$$

If H is the Hilbert transform in the t-variable we get using the definition of the half-time derivative in (16) that

$$|A_{21}| = \Big|\int_{\mathbb{R}^n} H D_{1/2}^t (u^r \circ \rho) D_{1/2}^t (w^r \circ \rho) dz\Big| \le ||D_{1/2}^t (u^r \circ \rho)||_q ||D_{1/2}^t (w^r \circ \rho)||_p. \ (68)$$

Continuing we have,

$$\begin{aligned}A_{22} &= \int_{\mathbb{R}^n} \int_0^\infty (u^r \circ \rho)_\lambda (v^r \circ \rho) \frac{\partial P_{\gamma\lambda} A}{\partial t} d\lambda dz + \int_{\mathbb{R}^n} \int_0^\infty (u^r \circ \rho)(v^r \circ \rho)_\lambda \frac{\partial P_{\gamma\lambda} A}{\partial t} d\lambda dz\\ &\quad + \int_{\mathbb{R}^n} \int_0^\infty (u^r \circ \rho)(v^r \circ \rho) \frac{\partial^2 P_{\gamma\lambda} A}{\partial t \partial \lambda} d\lambda dz. \qquad (69)\end{aligned}$$

We shortly stop here in order to manipulating the expression B of (64). Let in the following for simplicity $D = (1 + \frac{\partial P_{\gamma\lambda} A}{\partial \lambda})$. Then

$$B = \int_{\mathbb{R}^n} \int_0^\infty (v^r \circ \rho)(u_t^r \circ \rho) D d\lambda dz + \int_{\mathbb{R}^n} \int_0^\infty (u^r \circ \rho)(v_t^r \circ \rho) D d\lambda dz.$$

By familiar manipulations

$$\begin{aligned} B = & \int_{\mathbb{R}^n} \int_0^\infty (v^r \circ \rho)(u^r \circ \rho)_t D d\lambda dz - \int_{\mathbb{R}^n} \int_0^\infty (v^r \circ \rho)(u_{x_0}^r \circ \rho) D \frac{\partial P_{\gamma\lambda} A}{\partial t} d\lambda dz \\ & + \int_{\mathbb{R}^n} \int_0^\infty (u^r \circ \rho)(v^r \circ \rho)_t D d\lambda dz - \int_{\mathbb{R}^n} \int_0^\infty (u^r \circ \rho)(v_{x_0}^r \circ \rho) D \frac{\partial P_{\gamma\lambda} A}{\partial t} d\lambda dz \\ = & - \int_{\mathbb{R}^n} \int_0^\infty (v^r \circ \rho)(u^r \circ \rho) \frac{\partial^2 P_{\gamma\lambda} A}{\partial \lambda \partial t} d\lambda dz - \int_{\mathbb{R}^n} \int_0^\infty (v^r \circ \rho)(u^r \circ \rho)_\lambda \frac{\partial P_{\gamma\lambda} A}{\partial t} d\lambda dz \\ & - \int_{\mathbb{R}^n} \int_0^\infty (u^r \circ \rho)(v^r \circ \rho)_\lambda \frac{\partial P_{\gamma\lambda} A}{\partial t} d\lambda dz. \end{aligned} \tag{70}$$

Here we have slightly changed the order of the expressions and conducted one integration by parts. Adding A_{22} and B we see from (69) and (70) that we have perfect cancellation. Summarizing (64), (66-70) we can conclude that

$$\left| \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial \nu} \vec{v} d\sigma_t dt \right| \leq ||u^r||_{L^q_{1,1/2}(\partial\Omega)} \Big|\Big| \frac{\partial w^s}{\partial x_j} \Big|\Big|_{L^p(\partial\Omega)} + ||D^t_{1/2}(u^r \circ \rho)||_q ||D^t_{1/2}(w^r \circ \rho)||_p. \tag{71}$$

Our intention is now to apply Lemma 10 in order to complete the proof of (59) and hence of Theorem 4. Recall that by the definitions in (14), (16) and (21) $\mathbb{D}_n$, $D^t_{1/2}$ and $\mathbb{D}$ are defined using the multipliers $|\tau|^{1/2}$, $\tau||(\xi,\tau)||^{-1}$ and $||(\xi,\tau)||$. Also $||(\xi,\tau)||$ is defined through the relation in (12), i.e.,

$$\frac{\tau^2}{||(\xi,\tau)||^4} + \sum_{i=0}^{n-1} \frac{\xi_i^2}{||(\xi,\tau)||^2} = 1.$$

Define parabolic Riesz transforms R_j using the multipliers $\xi_j||(\xi,\tau)||^{-1}$ for $j \in \{1, ..., n-1\}$ and $\tau||(\xi,\tau)||^{-2}$ for $j = n$. Then

$$\mathbb{D} = \sum_{j=1}^{n-1} R_j \frac{\partial}{\partial x_j} + R_n \mathbb{D}_n$$

and by continuity of Riesz potentials

$$||\mathbb{D} f||_p \leq c_n \left[\sum_j \Big|\Big| \frac{\partial f}{\partial x_j} \Big|\Big|_p + ||\mathbb{D}_n f||_p \right] \tag{72}$$

for all smooth functions $f : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$. As $|\tau|^{1/2}||(\xi,\tau)||^{-1}$ is a L^p-multiplier for $p \in (1,\infty)$ we can conclude that the inequality in (72) holds with $\mathbb{D}_n$ replaced by $D^t_{1/2}$. Using our assumption on the solvability of the Dirichlet problem in (61) with the appropriate bound on the non-tangential maximal function we can conclude using Lemma 10 that

$$\begin{aligned}
\sum_r \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} |(w^r \circ \rho)_{x_i}|^p dz &\le C\delta \Bigg(\sum_r \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} |N_*(w^r_{x_i} \circ \rho)|^p dz \Bigg) + C(\delta) \|N_*(w^s_{x_0} \circ \rho)\|_p^p \\
&\le C\delta \Bigg(\sum_r \sum_{i=1}^{n-1} \int_{\mathbb{R}^n} |w^r_{x_i} \circ \rho|^p dz \Bigg) + C(\delta) \|w^s_{x_0} \circ \rho\|_p^p \\
\int_{\mathbb{R}^n} |\mathbb{D}_n(w^r \circ \rho)|^p dz &\le C\|N_*(\nabla \vec{w} \circ \rho)\|_p^p \le C\|\nabla \vec{w} \circ \rho\|_p^p
\end{aligned}$$

for an arbitrary positive δ. Combining these two inequalities, (72) and the discussion below that display we can conclude that

$$\Big|\Big| \frac{\partial w^s}{\partial x_j} \Big|\Big|_{L^p(\partial\Omega)} + ||D^t_{1/2}(w^r \circ \rho)||_p \le C\|w^s_{x_0} \circ \rho\|_p^p = C\|v^s \circ \rho\|_p^p. \tag{73}$$

In (73) the last equality follows by construction. Combining (71) and (73) we can conclude that,

$$\Bigg| \int_{\partial\Omega} \frac{\partial \vec{u}}{\partial \nu} \vec{v} d\sigma_t dt \Bigg| \le ||\vec{u}||_{L^q_{1,1/2}(\partial\Omega)} ||\vec{v}||_{L^p(\partial\Omega)}.$$

The proof of Theorem 4 is complete. $\square$

References

1. B.E.J. Dahlberg, C.E. Kenig, J. Pipher, and G.C. Verchota, Area integral estimates for higher order elliptic equations and systems, *Ann. Inst. Fourier (Grenoble)* **47** (1997), 1425–1461.
2. E.B. Fabes and M. Jodeit, L^p boundary value problems for parabolic equations, *Bull. Amer. Math. Soc.* **74** (1968), 1098–1102.
3. E.B. Fabes and N.M. Rivière, Symbolic calculus of kernels with mixed homogeneity, *Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)*, pp. 106–127, Amer. Math. Soc., Providence, R.I.
4. E.B. Fabes and N.M. Rivière, Singular intervals with mixed homogeneity, *Studia Math.* **27** (1966), 19–38.
5. S. Hofmann, Parabolic singular integrals of Calderón-type, rough operators and caloric layer potentials, *Duke Math. J.* **90** (1997), 209–259.
6. S. Hofmann and J.L. Lewis, L^2 solvability and representation by caloric layer potentials in time-varying domains, *Ann. of Math.* **144** (1996), 349–420.
7. S. Hofmann and J.L. Lewis, The L^p-regularity problem for the heat equation in non-cylindrical domains, *Illinois J. Math.* **43** (1999), 752–769.
8. S. Hofmann and J.L. Lewis, The L^p Neumann problem for the heat equation in non-cylindrical domains, *J. Funct. Anal.* **220** (2005), 1–54.
9. S. Hofmann and K. Nyström, Dirichlet problems for a non-stationary linearized system of Navier-Stokes equations in non-cylindrical domains, *Methods Appl. Anal.* **9** (2002), 13–98.

10. C.E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, CBMS Regional Conference Series in Mathematics, 83, American Mathematical Society, Providence, RI, 1994.
11. K. Nyström, Boundary value problems for parabolic Lamé systems in time-varying domains, *Indiana Univ. Math. J.* **48** (1999), 1285–1355.
12. K. Nyström, On area integral estimates for solutions to parabolic systems in time-varying and non-smooth cylinders, Preprint.
13. J. Pipher and G.C. Verchota, Dilation invariant estimates and the boundary Gårding inequality for higher order elliptic operators, *Ann. of Math. (2)* **142** (1995), 1–38.
14. J. Rivera-Noriega, Absolute continuity of parabolic measure and area integral estimates in non-cylindrical domains, *Indiana Univ. Math. J.* **52** (2003), 477–525.
15. G.C. Verchota, Layer potentials and regularity for Dirichlet problem for the Laplace's equation in Lipschitz domains, *J. Funct. Anal.* **59** (1984), 572–611.