

## Instanton sheaves on complex projective spaces

MARCOS JARDIM

IMECC - UNICAMP, Departamento de Matemática, Caixa Postal 6065

13083-970 Campinas-SP, Brazil

E-mail: jardim@ime.unicamp.br

Received April 13, 2005. Revised September 14, 2005

### ABSTRACT

We study a class of torsion-free sheaves on complex projective spaces which generalize the much studied mathematical instanton bundles. Instanton sheaves can be obtained as cohomologies of linear monads and are shown to be semistable if its rank is not too large, while semistable torsion-free sheaves satisfying certain cohomological conditions are instanton. We also study a few examples of moduli spaces of instanton sheaves.

### Introduction

The study of vector bundles and more general sheaves on complex projective spaces has been a topic of great interest to algebraic geometers for many years, see for instance the excellent book by Okonek, Schneider and Spindler [20] and Hartshorne's problem list [11]. In this paper we concentrate on a particular class of sheaves defined as follows, generalizing the concept of admissible sheaves on  $\mathbb{P}^3$  due to Manin [18], see also [10].

DEFINITION. An *instanton sheaf* on  $\mathbb{P}^n$  ( $n \geq 2$ ) is a torsion-free coherent sheaf  $E$  on  $\mathbb{P}^n$  with  $c_1(E) = 0$  satisfying the following cohomological conditions:

1. for  $n \geq 2$ ,  $H^0(E(-1)) = H^n(E(-n)) = 0$ ;
2. for  $n \geq 3$ ,  $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$ ;
3. for  $n \geq 4$ ,  $H^p(E(k)) = 0$ ,  $2 \leq p \leq n-2$  and  $\forall k$ ;

The integer  $c = -\chi(E(-1))$  is called the charge of  $E$ .

If  $E$  is a rank  $2m$  locally-free instanton sheaf on  $\mathbb{P}^{2m+1}$  of trivial splitting type (i.e. there exist a line  $\ell \subset \mathbb{P}^{2m+1}$  such that  $E|_\ell \simeq \mathcal{O}_\ell^{2m}$ ), then  $E$  is a *mathematical instanton bundle* as originally defined by Okonek and Spindler [21]. There is an extensive

---

*Keywords:* Monads, semistable sheaves.

*MSC2000:* 14J60; 14F05.

literature on such objects, see for instance [2, 24]. The nomenclature is motivated by gauge theory: mathematical instanton bundles on  $\mathbb{P}^{2n+1}$  correspond to quaternionic instantons on  $\mathbb{P}\mathbb{H}^n$  [22].

The goal of this paper is to extend the discussion in two directions: the inclusion of even-dimensional projective spaces and the analysis of more general sheaves, allowing non-locally-free sheaves of arbitrary rank. Such extension is motivated by the concept that in order to better understand moduli spaces of stable vector bundles over a projective variety one must also consider semistable torsion-free sheaves [11]. It turns out that many of the well-known results regarding mathematical instanton bundles on  $\mathbb{P}^{2n+1}$  generalize in sometimes surprising ways to more general instanton sheaves.

The paper is organized as follows. We start by studying linear monads and their cohomologies in Section 1, spelling out criteria to decide whether the cohomology of a given monad is torsion-free, reflexive or locally-free. We then show that every instanton sheaf is the cohomology of a linear monad, and that rank  $r$  instanton sheaves on  $\mathbb{P}^n$  exist if and only if  $r \geq n - 1$ . Further properties of instanton sheaves are also studied in Section 2.

The bulk of the paper lies in Section 3, where we analyze the semistability (in the sense of Mumford-Takemoto) of instanton sheaves. It is shown, for instance, that every rank  $r \leq 2n - 1$  locally-free instanton sheaf on  $\mathbb{P}^n$  is semistable, while every rank  $r \leq n + 1$  reflexive instanton sheaf on  $\mathbb{P}^n$  is semistable. We also determine when a semistable torsion-free sheaf on  $\mathbb{P}^n$  is an instanton sheaf, showing for instance that every semistable torsion-free sheaf on  $\mathbb{P}^2$  is an instanton sheaf.

In Section 4 it is shown that every rank  $n - 1$  instanton sheaf on  $\mathbb{P}^n$  is simple, generalizing a result of Ancona and Ottaviani for mathematical instanton bundles [2, Proposition 2.11]. We then conclude in Section 5 with a few results concerning the moduli spaces of instanton sheaves.

It is also worth noting that Buchdahl has studied monads over arbitrary blow-ups of  $\mathbb{P}^2$  [3] while Costa and Miró-Roig have initiated the study of locally-free instanton sheaves over smooth quadric hypersurfaces within  $\mathbb{P}^n$  [5], obtaining some results similar to ours. Many of the results here obtained are also valid for instanton sheaves suitably defined over projective varieties with cyclic Picard group, see [16].

**Notation.** We work over an algebraically closed field  $\mathbb{F}$  of characteristic zero. It might be interesting from the algebraic point of view to study how the results here obtained generalize to finite fields. Throughout this paper,  $U$ ,  $V$  and  $W$  are finite dimensional vector spaces over the fixed field  $\mathbb{F}$ , and we use  $[x_0 : \cdots : x_n]$  to denote homogeneous coordinates on  $\mathbb{P}^n$ . If  $E$  is a sheaf on  $\mathbb{P}^n$ , then  $E(k) = E \otimes \mathcal{O}_{\mathbb{P}^n}(k)$ , as usual; by  $H^p(E)$  we actually mean  $H^p(\mathbb{P}^n, E)$  and  $h^p(E)$  denotes the dimension of  $H^p(\mathbb{P}^n, E)$ .

**Acknowledgment.** The author is partially supported by the FAEPEX grants number 1433/04 and 1652/04, and the CNPq grant number 300991/2004-5. We thank Giorgio Ottaviani and Rosa Maria Miró-Roig for their valuable comments on the first version of this paper. This material was the topic of a two-week course during the 2005 Summer Program at IMPA; we thank Eduardo Esteves for the invitation and IMPA for the financial support.

## 1. Monads

Let  $X$  be a smooth projective variety. A *monad* on  $X$  is a complex  $\mathcal{V}_\bullet$  of the following form:

$$\mathcal{V}_\bullet : 0 \rightarrow V_{-1} \xrightarrow{\alpha} V_0 \xrightarrow{\beta} V_1 \rightarrow 0 \quad (1)$$

which is exact on the first and last terms. Here,  $V_k$  are locally free sheaves on  $X$ . The sheaf  $E = \ker \beta / \text{Im } \alpha$  is called the cohomology of the monad  $\mathcal{V}_\bullet$ .

Monads were first introduced by Horrocks, who has shown that every rank 2 locally free sheaf on  $\mathbb{P}^3$  can be obtained as the cohomology of a monad where  $V_k$  are sums of line bundles [15].

In this paper, we will focus on the so-called *linear monads* on  $\mathbb{P}^n$ , which are of the form:

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0, \quad (2)$$

where  $\alpha \in \text{Hom}(V, W) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$  is injective (as a sheaf map) and  $\beta \in \text{Hom}(W, U) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$  is surjective. The *degeneration locus*  $\Sigma$  of the monad (2) consists of the set of points  $x \in \mathbb{P}^n$  such that the localized map  $\alpha_x \in \text{Hom}(V, W)$  is not injective; in other words  $\Sigma = \text{supp } \{\text{coker } \alpha^*\}$ .

Linear monads have appeared in a wide variety of contexts within algebraic geometry, like the construction of locally free sheaves on complex projective spaces and the study of curves in  $\mathbb{P}^3$  and surfaces in  $\mathbb{P}^4$ , see for instance [9] and the references therein. Our main motivation to study them comes from gauge theory; as it is well-known, linear monads on  $\mathbb{P}^2$  and  $\mathbb{P}^3$  are closely related to instantons on  $\mathbb{R}^4$  [7].

The existence of linear monads on  $\mathbb{P}^n$  has been completely classified by Fløystad in [9]; let  $v = \dim V$ ,  $w = \dim W$  and  $u = \dim U$ .

### Theorem 1 [9]

Let  $n \geq 1$ . There exists a linear monad on  $\mathbb{P}^n$  as above if and only if at least one of the following conditions hold:

- (i)  $w \geq 2u + n - 1$  and  $w \geq v + u$ ;
- (ii)  $w \geq v + u + n$ .

If the conditions hold, there exists a linear monad whose degeneration locus is a codimension  $w - v - u + 1$  subvariety.

In particular, if  $v, w, u$  satisfy condition (2) above, then the degeneration locus is empty.

*Remark.* A similar classification result for linear monads over  $n$ -dimensional quadric hypersurfaces within  $\mathbb{P}^{n+1}$  has been proved by Costa and Miró-Roig in [5], by adapting Fløystad's technique.

**DEFINITION.** A coherent sheaf on  $\mathbb{P}^n$  is said to be a linear sheaf if it can be represented as the cohomology of a linear monad.

The goal of this section is to study linear sheaves, with their characterization in

mind. First, notice that if  $E$  is the cohomology of (2) then

$$\text{rank}(E) = w - v - u, \quad c_1(E) = v - u$$

and

$$c(E) = \left( \frac{1}{1-H} \right)^v \left( \frac{1}{1+H} \right)^u.$$

### Proposition 2

If  $E$  is a linear sheaf on  $\mathbb{P}^n$ , then:

- (i) for  $n \geq 2$ ,  $H^0(E(k)) = H^0(E^*(k)) = 0$ ,  $\forall k \leq -1$ ;
- (ii) for  $n \geq 3$ ,  $H^1(E(k)) = 0$ ,  $\forall k \leq -2$ ;
- (iii) for  $n \geq 4$ ,  $H^p(E(k)) = 0$ ,  $2 \leq p \leq n-2$  and  $\forall k$ ;
- (iv) for  $n \geq 3$ ,  $H^{n-1}(E(k)) = 0$ ,  $\forall k \geq -n+1$ ;
- (v) for  $n \geq 2$ ,  $H^n(E(k)) = 0$  for  $k \geq -n$ ;
- (vi) for  $n \geq 2$ ,  $\mathcal{E}xt^1(E, \mathcal{O}_{\mathbb{P}^n}) = \text{coker} \alpha^*$  and  $\mathcal{E}xt^p(E(k), \mathcal{O}_{\mathbb{P}^n}) = 0$  for  $p \geq 2$  and all  $k$ .

In particular, note that linear sheaves have natural cohomology in the range  $-n \leq k \leq -1$ , i.e. for the values of  $k$  in this range at most one of the cohomology groups  $H^q(E(k))$  is nontrivial. It also follows that  $c = -\chi(E(-1)) = h^1(E(-1))$ .

Every rank  $2n$  locally-free sheaf on  $\mathbb{P}^{2n+1}$  with total Chern class  $c(E) = (1+H^2)^{-c}$  and natural cohomology in the rank  $-2n-1 \leq k \leq 0$  is linear [21], and therefore satisfy the stronger conclusion of the proposition above. However, not all sheaves on  $\mathbb{P}^n$  having natural cohomology in the range  $-n \leq k \leq -1$  are linear, the simplest example being  $\Omega_{\mathbb{P}^n}^1(n)$ .

*Proof.* Assume that  $E$  is the cohomology of the monad (2). The kernel sheaf  $K = \ker \beta$  is locally-free, and one has the sequences  $\forall k$ :

$$0 \rightarrow K(k) \rightarrow W \otimes \mathcal{O}_{\mathbb{P}^n}(k) \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(k+1) \rightarrow 0 \quad \text{and} \quad (3)$$

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(k-1) \xrightarrow{\alpha} K(k) \rightarrow E(k) \rightarrow 0. \quad (4)$$

From the first sequence, we see that:  $H^p(K(k)) = 0$  for  $p = 0, 1$  and  $p+k \leq -1$ ; for  $2 \leq p \leq n-1$ ,  $\forall k$ ; and for  $p = n$  and  $k \leq -n$ . The second sequence tell us that  $H^p(K(k)) \xrightarrow{\sim} H^p(E(k))$  for  $p = 0$  and  $k \leq 0$ , for  $1 \leq p \leq n-2$  and all  $k$ , and for  $p \geq n-1$  and  $k \geq -n$ . Putting these together, we obtain the conditions (i) through (v) in the statement of the proposition.

Dualizing sequence (3), we obtain that  $H^0(K^*(k)) = 0$  for  $k \leq -1$ . Now dualizing sequence (4), we get, since  $K$  is locally-free:

$$0 \rightarrow E^*(-k) \rightarrow K^*(-k) \xrightarrow{\alpha^*} V \otimes \mathcal{O}_{\mathbb{P}^n}(-k+1) \rightarrow \mathcal{E}xt^1(E(k), \mathcal{O}_{\mathbb{P}^n}) \rightarrow 0. \quad (5)$$

Condition (vi) and the second part of condition (i) follow easily.  $\square$

Conversely, linear sheaves can be characterized by their cohomology:

**Theorem 3**

If  $E$  is a torsion-free sheaf on  $\mathbb{P}^n$  satisfying:

- (i) for  $n \geq 2$ ,  $H^0(E(-1)) = H^n(E(-n)) = 0$ ;
- (ii) for  $n \geq 3$ ,  $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$ ;
- (iii) for  $n \geq 4$ ,  $H^p(E(k)) = 0$ ,  $2 \leq p \leq n-2$  and  $\forall k$ ;

then  $E$  is linear, and can be represented as the cohomology of the monad:

$$\begin{aligned} 0 \rightarrow H^1(E \otimes \Omega_{\mathbb{P}^n}^2(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \\ \rightarrow H^1(E \otimes \Omega_{\mathbb{P}^n}^1) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow H^1(E(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0. \end{aligned} \quad (6)$$

*Proof.* Given a hyperplane  $\varphi \subset \mathbb{P}^n$ , consider the restriction sequence:

$$0 \rightarrow E(k-1) \rightarrow E(k) \rightarrow E(k)|_{\varphi} \rightarrow 0.$$

Clearly,  $H^0(E(-1)) = 0$  implies that  $H^0(E(k)) = 0$  for  $k \leq -1$ , while  $H^n(E(-n)) = 0$  forces  $H^n(E(k)) = 0$  for  $k \geq -n$ .

Since  $H^0(E(-1)) = H^1(E(-2)) = 0$ , it follows that  $H^0(E(-1)|_{\varphi}) = 0$ , hence  $H^0(E(k)|_{\varphi}) = 0$  for  $k \leq -1$ . So we have the sequence:

$$0 \rightarrow H^1(E(k-1)) \rightarrow H^1(E(k)), \quad \text{for } k \leq -2$$

thus by induction  $H^1(E(k)) = 0$  for  $k \leq -2$ .

Since  $H^n(E(-n)) = H^{n-1}(E(1-n)) = 0$ , it follows that  $H^{n-1}(E(1-n)|_{\varphi}) = 0$ , hence by further restriction  $H^{n-1}(E(k)|_{\varphi}) = 0$  for  $k \geq 1-n$ . So we have the sequence:

$$H^{n-1}(E(k-1)) \rightarrow H^{n-1}(E(k)) \rightarrow 0, \quad \text{for } k \geq 1-n$$

thus by induction  $H^{n-1}(E(k)) = 0$  for  $k \geq 1-n$ .

The key ingredient of the proof is the Beilinson spectral sequence [20]: for any coherent sheaf  $E$  on  $\mathbb{P}^n$  there exists a spectral sequence  $\{E_r^{p,q}\}$  whose  $E_1$ -term is given by  $(q = 0, \dots, n$  and  $p = 0, -1, \dots, -n)$ :

$$E_1^{p,q} = H^q(E \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p)$$

which converges to

$$E^i = \begin{cases} E, & \text{if } p+q=0 \\ 0 & \text{otherwise.} \end{cases}$$

Applying the Beilinson spectral sequence to  $E(-1)$ , we must show that

$$H^q(E(-1) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0 \quad \text{for } q \neq 1 \quad \text{and for } q = 1, p \leq -3. \quad (7)$$

It then follows that the Beilinson spectral sequence degenerates at the  $E_2$ -term and the monad

$$\begin{aligned} 0 \rightarrow H^1(E(-1) \otimes \Omega_{\mathbb{P}^n}^2(2)) \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \\ \rightarrow H^1(E(-1) \otimes \Omega_{\mathbb{P}^n}^1(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow H^1(E(-1)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow 0 \end{aligned} \quad (8)$$

has  $E(-1)$  as its cohomology. Tensoring (8) by  $\mathcal{O}_{\mathbb{P}^n}(1)$ , we conclude that  $E$  is the cohomology of (6), as desired.

The claim (7) follows from repeated use of the exact sequence

$$\begin{aligned} H^q(E(k))^{\oplus m} &\rightarrow H^q(E(k+1) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p-1)) \\ &\rightarrow H^{q+1}(E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \rightarrow H^{q+1}(E(k))^{\oplus m} \end{aligned} \quad (9)$$

associated with Euler sequence for  $p$ -forms on  $\mathbb{P}^n$  twisted by  $E(k)$ :

$$0 \rightarrow E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p) \rightarrow E(k)^{\oplus m} \rightarrow E(k) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p) \rightarrow 0, \quad (10)$$

where  $q = 0, \dots, n$ ,  $p = -1, \dots, -n$  and  $m = \binom{n+1}{-p}$ .

For instance, it is easy to see that:

$$\begin{aligned} H^0(E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) &= 0 \text{ for all } p \text{ and } k \leq -1; \\ H^q(E(-1) \otimes \Omega_{\mathbb{P}^n}^n(n)) &= H^q(E(-2)) = 0 \text{ for all } q; \\ H^q(E(-1)) &= 0 \text{ for all } q \neq 1; \\ H^n(E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) &= 0 \text{ for all } p \text{ and } k \geq -n. \end{aligned}$$

Setting  $q = n - 1$ , we also obtain:

$$H^{n-1}(E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0 \text{ for } p \geq -n + 1 \text{ and } k \geq -n - 1,$$

and so on.  $\square$

Clearly, the cohomology of a linear monad is always coherent, but more can be said if the codimension of the degeneration locus of  $\alpha$  is known.

#### Proposition 4

Let  $E$  be a linear sheaf.

- (i)  $E$  is locally-free if and only if its degeneration locus is empty;
- (ii)  $E$  is reflexive if and only if its degeneration locus is a subvariety of codimension at least 3;
- (iii)  $E$  is torsion-free if and only if its degeneration locus is a subvariety of codimension at least 2.

*Proof.* Let  $\Sigma$  be the degeneration locus of the linear sheaf  $E$ . From Proposition 2, we know that  $\mathcal{E}xt^p(E, \mathcal{O}_{\mathbb{P}^n}) = 0$  for  $p \geq 2$  and

$$\Sigma = \text{supp } \mathcal{E}xt^1(E, \mathcal{O}_{\mathbb{P}^n}) = \{x \in \mathbb{P}^n \mid \alpha_x \text{ is not injective}\}.$$

The first statement is clear; so it is now enough to argue that  $E$  is torsion-free if and only if  $\Sigma$  has codimension at least 2 and that  $E$  is reflexive if and only if  $\Sigma$  has codimension at least 3.

Recall that the  $m^{\text{th}}$ -singularity set of a coherent sheaf  $\mathcal{F}$  on  $\mathbb{P}^n$  is given by:

$$S_m(\mathcal{F}) = \{x \in \mathbb{P}^n \mid dh(\mathcal{F}_x) \geq n - m\}$$

where  $dh(\mathcal{F}_x)$  stands for the homological dimension of  $\mathcal{F}_x$  as an  $\mathcal{O}_x$ -module:

$$dh(\mathcal{F}_x) = d \iff \begin{cases} \text{Ext}_{\mathcal{O}_x}^d(\mathcal{F}_x, \mathcal{O}_x) \neq 0 \\ \text{Ext}_{\mathcal{O}_x}^p(\mathcal{F}_x, \mathcal{O}_x) = 0 \quad \forall p > d. \end{cases}$$

In the case at hand, we have that  $dh(E_x) = 1$  if  $x \in \Sigma$ , and  $dh(E_x) = 0$  if  $x \notin \Sigma$ . Therefore  $S_0(E) = \cdots = S_{n-2}(E) = \emptyset$ , while  $S_{n-1}(E) = \Sigma$ . It follows that [23, Proposition 1.20]:

- (i) if  $\text{codim } \Sigma \geq 2$ , then  $\dim S_m(E) \leq m - 1$  for all  $m < n$ , hence  $E$  is a locally 1<sup>st</sup>-syzygy sheaf;
- (ii) if  $\text{codim } \Sigma \geq 3$ , then  $\dim S_m(E) \leq m - 2$  for all  $m < n$ , hence  $E$  is a locally 2<sup>nd</sup>-syzygy sheaf.

The desired statements follow from the observation that  $E$  is torsion-free if and only if it is a locally 1<sup>st</sup>-syzygy sheaf, while  $E$  is reflexive if and only if it is a locally 2<sup>nd</sup>-syzygy sheaf [20, p. 148-149].  $\square$

**A splitting criterion for locally-free linear sheaves.** Given a coherent sheaf  $E$  on  $\mathbb{P}^n$ , we define

$$H_*^p(E) = \bigoplus_{k \in \mathbb{Z}} H^p(E(k))$$

which has the structure of a graded module over  $S^n = \bigoplus_{k \in \mathbb{Z}} H^p(\mathcal{O}(k))$ . Kumar, Peterson and Rao prove the following result [17]:

### Theorem 5

Let  $E$  be a rank  $r$  locally-free sheaf on  $\mathbb{P}^n$ ,  $n \geq 4$ .

- (i) If  $n$  is even and  $r \leq n - 1$ , then  $E$  splits as a sum of line bundles if and only if  $H_*^p(E) = 0$  for  $2 \leq p \leq n - 2$ .
- (ii) If  $n$  is odd and  $r \leq n - 2$ , then  $E$  splits as a sum of line bundles if and only if  $H_*^p(E) = 0$  for  $2 \leq p \leq n - 2$ .

Thus we obtain as an easy consequence of (3) in Proposition 2 and the previous theorem:

### Corollary 6

Let  $E$  be a rank  $r$  locally-free linear sheaf on  $\mathbb{P}^n$ .

- (i) If  $n$  is even and  $r \leq n - 1$ , then  $E$  splits as a sum of line bundles.
- (ii) If  $n$  is even and  $r \leq n - 2$ , then  $E$  splits as a sum of line bundles.

This means that linear monads are not useful to produce locally free sheaves of low rank on  $\mathbb{P}^n$ , one of the problems suggested by Hartshorne in [11] and still a challenge in the subject.

Let us also point out that Kumar, Peterson and Rao's result is optimal, in the sense that there exist rank  $2n$  locally-free sheaves on  $\mathbb{P}^{2n+1}$  and on  $\mathbb{P}^{2n}$  ( $n \geq 2$ ) satisfying  $H_*^p(E) = 0$  for  $2 \leq p \leq 2n - 1$  which are stable, and hence do not split as a sum of line bundles. Moreover, as we will see in examples below, it does not generalize to reflexive or torsion-free sheaves either.

Thus linear monads will only produced interesting locally-free sheaves when  $r = w - v - u \geq n$  if  $n$  is even and when  $r = w - v - u \geq n - 1$  if  $n$  is odd. However, we can still expect to use monads to produce interesting *reflexive* sheaves of low rank, see Example 7 below.

## 2. Basic properties of instanton sheaves

It follows from Proposition 2 that the cohomology of a linear monad with  $v = u$  and  $w - 2v \geq 1$  is a torsion-free instanton sheaf of rank  $w - 2v$  and charge  $v$ . In particular, by Fløystad's theorem, there are rank  $r$  instanton sheaves on  $\mathbb{P}^n$  for each  $r \geq n - 1$ . Moreover, for every  $r \geq 2n$ , there are rank  $r$  *locally-free* instanton sheaves on  $\mathbb{P}^{2n+1}$  or  $\mathbb{P}^{2n}$ .

By virtue of Theorem 3, every rank  $r$  instanton sheaf of charge  $c$  is the cohomology of a monad of the form:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0 \quad (11)$$

for some injective map  $\alpha$  degenerating in codimension at least 2 and some surjective map  $\beta$ , such that  $\beta\alpha = 0$ . It follows easily from Fløystad's theorem that there are no instanton sheaves on  $\mathbb{P}^n$  of rank  $r \leq n - 2$ .

### Corollary 7

*If  $E$  is an instanton sheaf then:*

- (i)  $H^0(E^*(k)) = 0, \forall k \leq -1$ ;
- (ii)  $\mathcal{E}xt^p(E, \mathcal{O}_{\mathbb{P}^n}) = 0$  for  $p \geq 2$ ;
- (iii)  $\text{Ext}^p(E, E) = 0$  for  $p \geq 3$ .

*Proof.* The first two statements follow easily from Proposition 2 and the fact that every instanton sheaf is linear. For the last statement, let  $K = \ker \beta$ ; taking the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \rightarrow K \rightarrow E \rightarrow 0,$$

we obtain:

$$\text{Ext}^p(E, \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c}) \rightarrow \text{Ext}^p(E, K) \rightarrow \text{Ext}^p(E, E) \rightarrow \text{Ext}^{p+1}(E, \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c}). \quad (12)$$

Thus  $\text{Ext}^p(E, K) \simeq \text{Ext}^p(E, E)$  for all  $p \geq 2$ . Now from the sequence

$$0 \rightarrow K \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0$$

we obtain:

$$\mathrm{Ext}^p(E, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c}) \rightarrow \mathrm{Ext}^{p+1}(E, K) \rightarrow \mathrm{Ext}^{p+1}(E, \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}). \quad (13)$$

Hence  $\mathrm{Ext}^p(E, K) = 0$  for  $p \geq 3$ , and the result follows.  $\square$

Furthermore,  $\mathrm{Ext}^1(E, E)$  is completely determined by the map  $\alpha$ , while  $\mathrm{Ext}^2(E, E)$  is completely determined by the map  $\beta$ . Indeed, first note that the maps  $\alpha$  and  $\beta$  induce linear maps:

$$\begin{aligned} \mathrm{Ext}^1 \alpha : \mathrm{Ext}^1(E, \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c}) &\rightarrow \mathrm{Ext}^1(E, K) \\ \mathrm{Ext}^1 \beta : \mathrm{Ext}^1(E, \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}) &\rightarrow \mathrm{Ext}^1(E, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c}). \end{aligned}$$

Setting  $p = 1$  on (12) and (13) we get:

$$\mathrm{Ext}^1(E, E) = \mathrm{coker} \{ \mathrm{Ext}^1 \alpha \} \quad \text{and} \quad \mathrm{Ext}^2(E, E) = \mathrm{coker} \{ \mathrm{Ext}^1 \beta \}.$$

In particular, note that  $\mathrm{Ext}^2(E, E) = 0$  is an open condition on  $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c})$ .

Given two linear sheaves, one can produce a new instanton sheaf of higher rank using the following result:

### Proposition 8

*An extension  $E$  of linear sheaves  $F'$  and  $F''$*

$$0 \rightarrow F' \rightarrow E \rightarrow F'' \rightarrow 0$$

*is also a linear sheaf. Moreover, if  $c_1(F') = -c_1(F'')$ , then  $E$  is instanton.*

*Proof.* The desired statement follows easily from the associated sequences of cohomology:

$$H^q(F'(k)) \rightarrow H^q(E(k)) \rightarrow H^q(F''(k)), \quad \forall q = 0, \dots, n,$$

so that  $H^q(E(k))$  vanishes whenever  $H^q(F'(k))$  and  $H^q(F''(k))$  do. Note that  $E$  is classified by  $\mathrm{Ext}^1(F'', F')$ .  $\square$

### Proposition 9

*If  $E$  is a locally-free instanton sheaf on  $\mathbb{P}^n$ , then  $E^*$  is also instanton.*

*Proof.* The statement is an easy consequence of Serre duality. In fact, if  $E$  arises as the cohomology of the monad

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0,$$

then  $E^*$  is the cohomology of the dual monad

$$0 \rightarrow U^* \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\beta^*} W^* \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\alpha^*} V^* \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0. \quad \square$$

In general, if  $E$  is not locally-free, its dual might not be an instanton sheaf, see Example 4 below. However, the dual of every semistable sheaf on  $\mathbb{P}^2$  is instanton.

**Proposition 10**

Let  $\varphi$  be a hyperplane in  $\mathbb{P}^n$ . The restriction  $E|_\varphi$  of an instanton sheaf  $E$  on  $\mathbb{P}^n$  is also an instanton sheaf, and the restriction map  $\rho : H^1(E(-1)) \rightarrow H^1(E(-1)|_\varphi)$  is an isomorphism.

*Proof.* Follows easily from the definition and the exact sequence

$$0 \rightarrow E(k-1) \xrightarrow{\sigma} E(k) \rightarrow E|_\varphi(k) \rightarrow 0. \quad \square$$

**Instanton sheaves and mathematical instanton bundles.** Mathematical instanton bundles have been defined in [21] as a rank  $2m$  locally-free sheaf on  $\mathbb{P}^{2m+1}$  satisfying the following conditions:

- $c(E) = \left(\frac{1}{1-H^2}\right)^c = (1 + H^2 + H^4 + \dots)^c$ ;
- $E$  has natural cohomology in the range  $-2m - 1 \leq k \leq 0$ ;
- $E$  is simple;
- $E$  has trivial splitting type (i.e. there exist a line  $\ell \subset \mathbb{P}^{2m+1}$  such that  $E|_\ell \simeq \mathcal{O}_\ell^{2m}$ ).

It was later shown by Ancona and Ottaviani have shown that the simplicity assumption is redundant [2, Proposition 2.11]: every rank  $2m$  locally-free sheaf on  $\mathbb{P}^{2m+1}$  satisfying the first two conditions is simple (in fact, more is true, see Lemma 23 below). The last condition is also redundant for  $m = 1$  and for  $c = 1$ ; there are however rank  $2m$  locally-free sheaf on  $\mathbb{P}^{2m+1}$  satisfying the first two conditions which are not of trivial splitting type.

Every mathematical instanton bundle as above can be represented as the cohomology of a linear monad with  $v = u = c$  and  $w = 2m + 2c$  [21], hence it is a rank  $2m$  locally-free instanton sheaf on  $\mathbb{P}^{2m+1}$  of charge  $c$ .

Conversely, a rank  $2m$  locally-free instanton sheaf  $E$  on  $\mathbb{P}^{2m+1}$  is a mathematical instanton bundle as above if  $H^0(E) = 0$  and it is of trivial splitting type. We show that the vanishing of  $H^0(E)$  is automatic.

**Proposition 11**

If  $E$  is a rank  $n - 1$  instanton sheaf on  $\mathbb{P}^n$ , then  $H^0(E) = 0$ . If  $E$  is locally-free, then  $H^0(E^*) = 0$ .

*Proof.* Let  $E$  be a rank  $n - 1$  locally-free instanton sheaf on  $\mathbb{P}^n$ , and assume that  $H^0(E) \neq 0$ . So let  $Q = E/\mathcal{O}_{\mathbb{P}^n}$  and consider the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow E \rightarrow Q \rightarrow 0.$$

Note that  $c_1(Q) = 0$ ,  $H^0(Q(k)) = 0$  for  $k \leq -1$ ,  $H^n(Q(k)) = 0$  for  $k \geq -n$  and  $H^q(Q(k)) = H^q(E(k))$  for  $1 \leq q \leq n - 1$ . It follows that  $Q$  must be a rank  $n - 2$  instanton sheaf, which cannot exist by Fløystad's theorem.

If  $E$  is locally-free, then  $E^*$  is also instanton and the vanishing of  $H^0(E^*)$  follows by the same argument. □

In particular, there are no rank  $2m - 1$  locally-free instanton sheaves on  $\mathbb{P}^{2m}$ . Indeed, by Theorem 5, any such sheaf must split as a sum of line bundles, and this contradicts  $c_1(E) = 0$  and  $H^0(E) = 0$ .

We also point out that there are rank  $n - 1$  properly reflexive instanton sheaves  $E$  on  $\mathbb{P}^n$  for which  $H^0(E^*) \neq 0$ , see Example 6 below.

### 3. Semistability of instanton sheaves

Recall that a torsion-free sheaf  $E$  on  $\mathbb{P}^n$  is said to be *semistable* if for every coherent subsheaf  $0 \neq F \hookrightarrow E$  we have

$$\mu(F) = \frac{c_1(F)}{\mathrm{rk}(F)} \leq \frac{c_1(E)}{\mathrm{rk}(E)} = \mu(E).$$

Furthermore, if for every coherent subsheaf  $0 \neq F \hookrightarrow E$  with  $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$  we have

$$\frac{c_1(F)}{\mathrm{rk}(F)} < \frac{c_1(E)}{\mathrm{rk}(E)},$$

then  $E$  is said to be *stable*. A sheaf is said to be properly semistable if it is semistable but not stable. It is also important to remember that  $E$  is (semi)stable if and only if  $E^*$  and  $E(k)$  are.

For any given torsion-free sheaf  $E$  of rank  $r$ , there is an uniquely determined integer  $k_E$  such that

$$c_1(E(k_E)) = c_1(E) + rk_E \in \{0, -1, \dots, -r + 1\};$$

$E_\eta = E(k_E)$  is called the normalization of  $E$ . A sheaf  $E$  is said to be normalized if  $-r + 1 \leq c_1(E) \leq 0$ .

**Lemma 12** ([20, p. 167])

Let  $E$  be a normalized torsion-free sheaf on  $\mathbb{P}^n$ . If  $E$  is stable then  $H^0(E) = 0$  and

- $H^0(E^*) = 0$  if  $c_1(E) = 0$ ;
- $H^0(E^*(-1)) = 0$  if  $c_1(E) < 0$ .

If  $E$  is semistable then  $H^0(E(-1)) = H^0(E^*(-1)) = 0$ .

For sheaves of rank 2 or 3, the above necessary criteria turns out to be also sufficient, as we recall in the next two lemmas. We fix  $n \geq 2$ .

**Lemma 13**

Let  $E$  be a normalized rank 2 torsion-free sheaf  $E$  on  $\mathbb{P}^n$ . If  $c_1(E) = 0$ , then:

- $E$  is stable if and only if  $H^0(E) = H^0(E^*) = 0$ ;
- $E$  is semistable if and only if  $H^0(E(-1)) = H^0(E^*(-1)) = 0$ .

If  $c_1(E) = -1$ , then  $E$  is stable if and only if it is semistable if and only if  $H^0(E) = H^0(E^*(-1)) = 0$ .

*Proof.* For  $E$  being reflexive, this result is in [20, p. 166]. In general, simply note that  $E^*$  is reflexive and use the result just mentioned.  $\square$

As an easy consequence of Proposition 2 and Lemma 13 we have:

**Proposition 14**

*If  $E$  is a rank 2 instanton sheaf on  $\mathbb{P}^n$  ( $n = 2, 3$ ), then  $E$  is semistable. It is stable if  $H^0(E) = 0$ .*

This result does not generalize for higher rank, even if we restrict ourselves to locally-free instanton sheaves, as we show in the example below.

EXAMPLE 1 For each  $n \geq 2$ , there are rank  $2n$  locally-free instanton sheaves on  $\mathbb{P}^n$  which are not semistable. Indeed, by Fløystad's theorem, there is a linear monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a+1} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+2a+1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus a} \rightarrow 0 \quad (a \geq 1), \quad (14)$$

whose cohomology  $F$  is a locally-free sheaf of rank  $n$  on  $\mathbb{P}^n$  and  $c_1(F) = 1$ . The dual  $F^*$  is a locally-free sheaf of rank  $n$  on  $\mathbb{P}^n$  and  $c_1(F^*) = -1$ . Any extension of  $E$  of  $F^*$  by  $F$ :

$$0 \rightarrow F \rightarrow E \rightarrow F^* \rightarrow 0$$

is a rank  $2n$  locally-free instanton sheaf which is clearly not semistable. Furthermore, one can adjust the parameter  $a$  depending on  $n$  to ensure the existence of nontrivial extensions.

For instanton sheaves of higher rank, the best statement one can have is the following:

**Theorem 15**

*Let  $E$  be a rank  $r$  instanton sheaf on  $\mathbb{P}^n$ .*

- *If  $E$  is reflexive and  $r \leq n + 1$ , then  $E$  is semistable;*
- *if  $E$  is locally-free and  $r \leq 2n - 1$ , then  $E$  is semistable.*

EXAMPLE 2 Note that the upper bound in the rank given in the second part of Theorem 15 is sharp, as seen in Example 1. We now show that there are rank  $n + 2$  reflexive instanton sheaves which are not semistable.

Indeed, let  $X = \mathbb{P}^n$ ,  $n \geq 3$ . By Fløystad's theorem [9], there is a linear monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n-2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0$$

whose cohomology  $F$  is a rank 2 reflexive linear sheaf on  $\mathbb{P}^n$  and  $c_1(F) = n - 3$ .

Next, consider the rank  $n$  locally free linear sheaf  $G$  associated to the linear monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus 2n+2a-3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+a-3} \rightarrow 0 \quad (a \geq 1).$$

Note that  $c_1(G) = 3 - n$ .

As in the previous example, an extension of  $G$  by  $F$  is a rank  $n + 2$  reflexive instanton sheaf which is not semistable. The choice of a suitable value of the parameter  $a$  guarantees the existence of non-trivial extensions.

The proof of Theorem 15 is based on Hoppe's criterion [14]: if  $E$  is a rank  $r$  reflexive sheaf on  $\mathbb{P}^n$  with  $c_1(E) = 0$  satisfying

$$H^0(\Lambda^q E(-1)) = 0 \quad \text{for } 1 \leq q \leq r - 1$$

then  $E$  is semistable. Indeed, assume  $E$  is not semistable, and let  $F$  be a rank  $q$  destabilizing sheaf with  $c_1(F) = d > 0$ . Then  $\Lambda^q F = \mathcal{O}_{\mathbb{P}^n}(d)$ , and the induced map  $\Lambda^q F \rightarrow \Lambda^q E$  yields a section in  $H^0(\Lambda^q E(-d))$ , which forces  $h^0(\Lambda^q E(-1)) \neq 0$ . Similarly, it is also easy to see that if

$$H^0(\Lambda^q E) = 0 \quad \text{for } 1 \leq q \leq r - 1$$

then  $E$  is stable.

*Proof of Theorem 15.* Every rank  $r$  reflexive instanton sheaf on  $\mathbb{P}^n$  can be represented as the cohomology of the monad (11). Taking the sequence

$$0 \rightarrow K \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(r+2c)} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0,$$

we consider the associated long exact sequence of exterior powers, twisted by  $\mathcal{O}_{\mathbb{P}^n}(-1)$ :

$$0 \rightarrow \Lambda^q K(-1) \rightarrow \Lambda^q(\mathcal{O}_{\mathbb{P}^n}^{\oplus(r+2c)})(-1) \rightarrow \dots$$

Hence  $H^0(\Lambda^q K(-1)) = 0$  for  $1 \leq q \leq r + c - 1$ . Now take the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \rightarrow K \rightarrow E \rightarrow 0,$$

and consider the associated long exact sequence of symmetric powers, twisted by  $\mathcal{O}_{\mathbb{P}^n}(-1)$ :

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-q-1)^{\binom{c+q-1}{q}} \rightarrow K(-q)^{\binom{c+q-2}{q-1}} \rightarrow \dots \\ \rightarrow \Lambda^{q-1} K \otimes \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus c} \rightarrow \Lambda^q K(-1) \rightarrow \Lambda^q E(-1) \rightarrow 0. \end{aligned}$$

Cutting into short exact sequences and passing to cohomology, we have obtain that every reflexive instanton sheaf satisfies:

$$H^0(\Lambda^p E(-1)) = 0, \quad \text{for } 1 \leq p \leq n - 1. \quad (15)$$

It follows from (15) that every rank  $r \leq n$  reflexive instanton sheaf is semistable. If  $E$  is a rank  $n + 1$  reflexive instanton sheaf, then because  $c_1(E) = 0$ :

$$H^0(\Lambda^n E(-1)) = H^0(E^*(-1)) = 0,$$

thus  $E$  is also semistable.

Now if  $E$  is locally-free, the dual  $E^*$  is also an instanton sheaf on  $X$ , so

$$H^0(\Lambda^q(E^*)(-1)) = 0, \quad \text{for } 1 \leq q \leq n - 1. \quad (16)$$

But  $\Lambda^p(E^*) \simeq \Lambda^{r-p}(E^*)$ , since  $\det(E) = \mathcal{O}_{\mathbb{P}^n}$ ; it follows that:

$$\begin{aligned} H^0(\Lambda^p E(-1)) &= H^0(\Lambda^{r-p}(E^*)(-1)) = 0, \quad \text{for } 1 \leq r-p \leq n-1 \\ &\implies r-n+1 \leq p \leq r-1. \end{aligned} \quad (17)$$

Together, (16) and (17) imply that if  $E$  is a rank  $r \leq 2n-1$  locally-free instanton sheaf, then:

$$H^0(\Lambda^p E(-1)) = 0, \quad \text{for } 1 \leq p \leq 2n-2$$

hence  $E$  is semistable by Hoppe's criterion.  $\square$

**EXAMPLE 3** A similar result for the semistability of torsion-free instanton sheaves beyond rank 2 is unclear. However, it is easy to construct rank  $n+1$  torsion-free instanton sheaves which are not semistable. Indeed, the cohomology of the monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n-1} \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0$$

is of the form  $\mathcal{I}_M(n-2)$ , where  $\mathcal{I}_M$  is the ideal sheaf of a codimension 2 subvariety  $M \hookrightarrow \mathbb{P}^n$  [9].

On the other hand, there is a rank  $n$  locally-free linear sheaf  $F$  on  $\mathbb{P}^n$  with  $c_1(F) = 2-n$  given by the cohomology of the monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus 2c+2n-2} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c+n-2} \rightarrow 0.$$

Thus the sheaf  $E$  given by the extension:

$$0 \rightarrow \mathcal{I}_M(n-2) \rightarrow E \rightarrow F \rightarrow 0$$

is a rank  $n+1$  torsion-free instanton sheaf which is not semistable.

In other words, Proposition 14 is sharp on  $\mathbb{P}^2$ , and the reasonable conjecture seems to be that every rank  $r = n-1, n$  torsion-free instanton sheaves on  $\mathbb{P}^n$  are semistable.

On the other hand, we have:

### Proposition 16

*For  $r > (n-1)c$ , there are no stable rank  $r$  instanton sheaves on  $\mathbb{P}^n$  of charge  $c$ .*

In other words, every stable rank  $r$  instanton sheaf on  $\mathbb{P}^n$  must be of charge  $c \geq r/(n-1)$ , and there are properly semistable rank  $r$  instanton sheaves for  $n \leq r \leq 2n-1$ .

*Proof.* Any rank  $r$  instanton sheaf of charge  $c$  is the cohomology of a monad of the form:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0$$

and note that

$$H^0(E) \simeq H^0(\ker \beta) \simeq \ker\{H^0\beta : H^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c})\}$$

thus if  $r > (n-1)c$ , then  $h^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}) > h^0(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c})$  and  $H^0(E) \neq 0$ , so that  $E$  is not stable by Lemma 12.  $\square$

Let us now analyze the inverse question: are all semistable sheaves of degree zero on  $\mathbb{P}^n$  instanton? The answer is positive for  $n = 2$ , but there are cohomological restrictions for  $n \geq 3$ .

### Theorem 17

Let  $E$  be a torsion-free sheaf on  $\mathbb{P}^2$  with  $c_1(E) = 0$ . If  $E$  is semistable, then  $E$  is instanton.

*Proof.* The semistability of  $E$  and  $E^*$  immediately implies that  $H^0(E(k)) = H^0(E^*(k)) = 0$  for  $k \leq -1$ . If  $E$  is a locally-free sheaf, then via Serre duality  $H^2(E(k)) = 0$  for  $k \geq -2$ , thus  $E$  is instanton.

Now if  $E$  is properly torsion-free, we consider the sequence:

$$0 \rightarrow E \rightarrow E^{**} \rightarrow Q \rightarrow 0 \quad (18)$$

where  $Q = E^{**}/E$  is supported on a zero dimensional subscheme. Clearly,  $E^{**}$  is a semistable locally-free sheaf with  $c_1(E) = 0$ , so it is instanton by the previous paragraph. It follows from (18) that:

$$H^0(E(k)) \hookrightarrow H^0(E^{**}(k)) = 0 \quad \text{for } k \leq -1,$$

and

$$H^2(E(k)) \xrightarrow{\simeq} H^2(E^{**}(k)) = 0 \quad \text{for } k \geq -2,$$

so  $E$  is also instanton.  $\square$

For  $n \geq 3$ , we have:

### Proposition 18

If  $E$  is a semistable locally-free sheaf on  $\mathbb{P}^n$  with  $c_1(E) = 0$  such that  $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$  and, for  $n \geq 4$ ,  $H_*^p(E) = 0$  for  $2 \leq p \leq n-2$ , then  $E$  is instanton.

*Proof.* If  $E$  is semistable, then  $H^0(E(k)) = H^0(E^*(k)) = 0$  for  $k \leq -1$ , hence  $H^n(E(k)) = 0$  for  $k \geq -n$  by Serre duality.  $\square$

As simple consequence of Proposition 14 and Proposition 18 we have:

- A rank 2 torsion-free sheaf on  $\mathbb{P}^3$  with  $c_1(E) = 0$  is instanton if and only if it is semistable and  $H^1(E(-2)) = H^2(E(-2)) = 0$ .
- A rank 3 reflexive sheaf on  $\mathbb{P}^3$  with  $c_1(E) = 0$  is instanton if and only if it is semistable and  $H^1(E(-2)) = H^2(E(-2)) = 0$ .

- A rank 3 reflexive sheaf on  $\mathbb{P}^4$  with  $c_1(E) = 0$  is instanton if and only if it is semistable and  $H^1(E(-2)) = H_*^2(E) = H^3(E(-3)) = 0$ .
- A rank  $4 \leq r \leq 2n - 1$  locally-free sheaf on  $\mathbb{P}^n$  ( $n \geq 3$ ) with  $c_1(E) = 0$  is instanton if and only if it is semistable and  $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$  and, for  $n \geq 4$ ,  $H_*^p(E) = 0$  for  $2 \leq p \leq n - 2$ .

*Remark.* Since every Gieseker semistable torsion-free sheaf on  $\mathbb{P}^n$  is semistable [20, p. 174], one can use the results above to decide when a Gieseker semistable torsion-free sheaf on  $\mathbb{P}^n$  is instanton. It is easy to see, however, that not all instanton sheaves are Gieseker semistable; indeed if  $E$  is an instanton sheaf satisfying  $H^0(E) \neq 0$ , then  $E$  is not Gieseker semistable. Thus, there are Gieseker unstable instanton sheaves of every rank.

A little more can be said about rank 2 reflexive instanton sheaves on  $\mathbb{P}^3$  and rank 4 locally-free instanton sheaves on  $\mathbb{P}^5$ .

### Proposition 19

*Every rank 2 reflexive instanton sheaf on  $\mathbb{P}^3$  is locally-free and stable.*

*Proof.* Hartshorne has shown that if  $E$  is a rank 2 reflexive sheaf on  $\mathbb{P}^3$  with  $c_3(E) = 0$ , then  $E$  is locally-free [12], thus every rank 2 reflexive instanton sheaf on  $\mathbb{P}^3$  is locally-free. By Proposition 11, we have that  $H^0(E) = 0$ , hence  $E$  is stable by Proposition 14.  $\square$

This result is sharp, in the sense that there are properly semistable rank 2 torsion-free instanton sheaves on  $\mathbb{P}^3$  and properly semistable rank 3 properly reflexive instanton sheaves on  $\mathbb{P}^3$ :

EXAMPLE 4 Consider the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad (19)$$

$$\alpha = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = (-x_2 \quad x_1 \quad x_3 \quad x_4).$$

Since  $\alpha$  is injective provided  $x_1, x_2 \neq 0$ , its cohomology is a rank 2 properly torsion-free instanton sheaf of charge 1. Moreover,  $E$  is not stable because it is a non-locally-free nullcorrelation sheaf [8, Remark 1.2.1].

Finally, note that  $E^*$  is a properly semistable rank 2 properly reflexive sheaf on  $\mathbb{P}^3$  with  $c_1(E^*) = 0$ ; by Proposition 19,  $E^*$  cannot be instanton.

EXAMPLE 5 Set  $w = 5$  and  $v = u = 1$  and consider the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad (20)$$

$$\alpha = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ x_3 \end{pmatrix} \quad \text{and} \quad \beta = (-x_2 \ x_1 \ x_3 \ x_4 \ 0).$$

It is easy to see that  $\beta$  is surjective for all  $[x_1 : \cdots : x_4] \in \mathbb{P}^3$ , while  $\alpha$  is injective provided  $x_1, x_2, x_3 \neq 0$ . It follows that  $E$  is reflexive, but not locally-free; its singularity set is just the point  $[0 : 0 : 0 : 1] \in \mathbb{P}^3$ .

In summary,  $E$  is a rank 3 properly reflexive instanton sheaf of charge 1 on  $\mathbb{P}^3$ . Note that  $E$  is properly semistable, by Theorem 15 and Proposition 16.

**Proposition 20**

*Every rank 4 locally-free instanton sheaf  $E$  on  $\mathbb{P}^5$  is stable.*

*Proof.* Noting that  $H^0(E) = 0$  by Proposition 11, the claim follows from [2, Theorem 3.6]. □

Again, this result is sharp, in the sense that there exists a properly semistable rank 4 properly reflexive instanton sheaf on  $\mathbb{P}^5$ ; it is also not true that every rank  $2n$  reflexive instanton sheaf on  $\mathbb{P}^{2n+1}$  is locally-free or stable, as Hartshorne’s result could suggest.

EXAMPLE 6 Consider the cohomology  $E$  of the monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^5}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^5}^{\oplus 6} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^5}(1) \rightarrow 0$$

with the maps  $\alpha$  and  $\beta$  given by:

$$\beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix} \quad \alpha = \begin{pmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 \\ 0 \\ 0 \end{pmatrix}.$$

Its degeneration locus is the line  $\{x_1 = \cdots = x_4 = 0\}$ , so its cohomology is indeed properly reflexive.

Finally,  $E$  is properly semistable because it is a non-locally-free nullcorrelation sheaf [8, Remark 1.2.1], and  $H^0(E^*) \neq 0$ .

*Remark.* It seems reasonable to conjecture that every rank  $2n$  locally-free instanton sheaf on  $\mathbb{P}^{2n+1}$  is stable. In support of this conjecture, see Lemma 23 and Proposition 25 below. Results in this direction were also obtained in [2] for *symplectic* mathematical instanton bundles.

EXAMPLE 7 Indecomposable rank 2 locally-free sheaves on  $\mathbb{P}^n$ ,  $n \geq 4$ , have been extremely difficult to construct, and linear monads do not help with this problem.

However, stable rank 2 reflexive sheaves on  $\mathbb{P}^n$  are easy to construct. Indeed, Fløystad's theorem guarantees the existence of linear monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+a-3} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+2a-1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus a} \rightarrow 0 \quad (a \geq 1)$$

whose cohomology is a rank 2 reflexive linear sheaf with  $c_1(E) = n - 3$ . To see that it is also stable, note that  $E_\eta = E(k)$  for some  $k \leq -1$ , thus  $H^0(E_\eta) = 0$  and it follows from Lemma 13 that  $E$  must be stable.

It is interesting to contrast the existence of such stable rank 2 reflexive sheaves on  $\mathbb{P}^n$  with Hartshorne's conjecture: there are no indecomposable rank 2 locally-free sheaves on  $\mathbb{P}^n$  for  $n \geq 7$  [13].

Furthermore, this example implies that Kumar, Peterson and Rao's result (Theorem 5) is sharp, in the sense it cannot be extended to more general sheaves: there are rank 2 reflexive sheaves  $E$  on  $\mathbb{P}^n$  with  $H_*^p(E) = 0$  for  $2 \leq p \leq n - 2$  which do not split as a sum of rank 1 sheaves.

For instance, with  $n = 4$  and  $a = 1$ , we get the monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}(1) \rightarrow 0$$

with the maps  $\alpha$  and  $\beta$  given by:

$$\beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} \quad \alpha = \begin{pmatrix} -x_2 & -x_5 \\ x_1 & x_3 \\ -x_4 & -x_2 \\ x_3 & 0 \\ 0 & x_1 \end{pmatrix},$$

where  $[x_1 : \cdots : x_5]$  are homogeneous coordinates in  $\mathbb{P}^4$ . Note that the degeneration locus of this monad is given by the union of two lines:

$$\Sigma(E) = \{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_2 = x_3 = x_5 = 0\}.$$

One can thus hope to construct stable rank 2 locally-free sheaves on  $\mathbb{P}^n$  via some mechanism that turns reflexive into locally-free sheaves without introducing new global sections.

**Lifting of instantons.** It is known that for every locally-free instanton sheaf  $E$  of trivial splitting type on  $\mathbb{P}^2$ , there exists a locally-free instanton sheaf  $\tilde{E}$  of trivial splitting type on  $\mathbb{P}^3$  and a hyperplane  $\wp \subset \mathbb{P}^3$  such that  $\tilde{E}|_\wp \simeq E$  [7].  $\tilde{E}$  is called a *lifting* of  $E$ .

It would be interesting to see whether this generalizes to higher dimensional projective spaces and/or to more general sheaves; more precisely, we propose the following conjecture:

**Conjecture.** If  $E$  is a rank  $r \geq 2n$  locally-free instanton sheaf of trivial splitting type on  $\mathbb{P}^{2n}$  of charge  $c$ , then there is a rank  $r$  locally-free instanton sheaf  $\tilde{E}$  on  $\mathbb{P}^{2n+1}$  of charge  $c$  and a hyperplane  $\wp \subset \mathbb{P}^{2n+1}$  such that  $\tilde{E}|_\wp \simeq E$ .

Donaldson's argument in [7] for the  $n = 1$  case is "unashamedly computational", relying in the correspondence between instanton sheaves of trivial splitting type on  $\mathbb{P}^2$

and  $\mathbb{P}^3$  and solutions of the ADHM equations, and it is not clear how to generalize it for  $n \geq 2$ . As far as we know, there is no alternative, more conceptual proof of Donaldson's result.

It is not difficult to see that if a locally-free instanton sheaf on  $\mathbb{P}^{2n}$  can be lifted to  $\mathbb{P}^{2n+1}$ , then the lifted sheaf must be instanton. The hard part is determining whether a given instanton sheaf can be lifted; the condition that the sheaf on  $\mathbb{P}^{2n}$  is of trivial splitting type might be crucial here.

**Proposition 21**

*Let  $E$  be a locally-free instanton sheaf on  $\mathbb{P}^n$  ( $n \geq 4$ ) of charge  $c$ . If there is a locally-free sheaf  $\tilde{E}$  on  $\mathbb{P}^{n+1}$ , and a hyperplane  $\wp \subset \mathbb{P}^{n+1}$  such that  $\tilde{E}|_{\wp} \simeq E$ , then  $\tilde{E}$  is also an instanton sheaf of charge  $c$ .*

*Proof.* The desired result follows from the restriction sequence:

$$0 \rightarrow \tilde{E}(k-1) \rightarrow \tilde{E}(k) \rightarrow E(k) \rightarrow 0$$

together with repeated use of Serre's duality and Serre's vanishing theorem.

Since  $H^0(E(k)) = 0$  for all  $k \leq -1$ , we get that

$$H^0(\tilde{E}(k-1)) \simeq H^0(\tilde{E}(k))$$

for all  $k \leq -1$ . Hence also  $H^0(\tilde{E}(-1)) = 0$ .

Similarly, since  $H^n(E(k)) = 0$  for all  $k \geq -n$ , we get that

$$H^{n+1}(\tilde{E}(k-1)) \simeq H^{n+1}(\tilde{E}(k))$$

for all  $k \geq -n$ . Hence also  $H^{n+1}(\tilde{E}(-n-1)) = 0$ .

Since  $H^1(E(k)) = 0$  for all  $k \leq -2$ , we get that

$$H^1(\tilde{E}(k-1)) \simeq H^1(\tilde{E}(k)) \rightarrow 0$$

for all  $k \leq -2$ . But  $H^1(\tilde{E}(l)) = 0$  for  $l \ll 0$  by Serre's vanishing theorem, thus it follows that  $H^1(\tilde{E}(-2)) = 0$ .

Similarly, since  $H^{n-1}(E(k)) = 0$  for all  $k \geq 1-n$ , we get that

$$0 \rightarrow H^n(\tilde{E}(k-1)) \rightarrow H^n(\tilde{E}(k))$$

for all  $k \geq 1-n$ . But  $H^1(\tilde{E}(l)) = 0$  for  $l \gg 0$  by Serre's vanishing theorem, thus it follows that  $H^n(\tilde{E}(-n)) = 0$ . This completes the proof for the case  $n = 2$ .

Since  $n \geq 4$ , we have that  $H^p(E(k)) = 0$  for  $2 \leq p \leq n-2$  and all  $k$ ; thus

$$H^2(\tilde{E}(k-1)) \simeq H^2(\tilde{E}(k))$$

for all  $k \leq -2$ . Again Serre's vanishing theorem forces  $H^2(\tilde{E}(k)) = 0$  for all  $k \leq -2$ . Moreover, we have that

$$H^2(\tilde{E}(k-1)) \rightarrow H^2(\tilde{E}(k)) \rightarrow 0$$

for all  $k$ , hence  $H^2(\tilde{E}(k)) = 0$  for all  $k$ .

Finally, for  $n \geq 5$  we have that

$$H^p(\tilde{E}(k-1)) \xrightarrow{\sim} H^p(\tilde{E}(k))$$

for  $3 \leq p \leq n-2$  and all  $k$ . It follows that  $H^p(\tilde{E}(k)) = 0$  for  $3 \leq p \leq n-2$  and all  $k$ , which completes the proof of the first statement.

Setting  $k = -1$  on the restriction sequence, we get  $h^1(\tilde{E}(-1)) = h^1(E(-1))$ , showing that the charge is preserved.  $\square$

#### 4. Simplicity of linear sheaves

Recall that a torsion-free sheaf  $E$  on a projective variety  $X$  is said to be *simple* if  $\dim \text{Ext}^0(E, E) = 1$ . Every stable torsion-free sheaf on  $\mathbb{P}^n$  is simple.

##### Theorem 22

*If  $E$  is the cohomology of the linear monad*

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0,$$

*for which  $K = \ker \beta$  is simple, then  $E$  is simple.*

*Proof.* Applying  $\text{Ext}^*(\cdot, E)$  to the sequence

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow K \rightarrow E \rightarrow 0,$$

we get

$$0 \rightarrow \text{Ext}^0(E, E) \rightarrow \text{Ext}^0(K, E) \rightarrow \dots \quad (21)$$

Now applying  $\text{Ext}^*(K, \cdot)$  we get:

$$V \otimes \text{Ext}^0(K, \mathcal{O}_{\mathbb{P}^n}(-1)) \rightarrow \text{Ext}^0(K, K) \rightarrow \text{Ext}^0(K, E) \rightarrow V \otimes \text{Ext}^1(K, \mathcal{O}_{\mathbb{P}^n}(-1)).$$

But it follows from the dual of sequence (3) with  $k = 1$  that  $h^0(K^*(-1)) = h^1(K^*(-1)) = 0$ , thus

$$\dim \text{Ext}^0(K, E) = \dim \text{Ext}^0(K, K) = 1$$

because  $K$  is simple. It then follows from (21) that  $E$  is also simple.  $\square$

As a consequence of [2, Theorem 2.8(a)], we have in particular the following generalization of [2, Theorem 2.8(b)]:

##### Lemma 23

*Every rank  $n-1$  linear sheaf on  $\mathbb{P}^n$  is simple.*

The above result is sharp, in the sense that there are rank  $n$  instanton sheaves on  $\mathbb{P}^n$  which are not simple. For example, recall that a rank 2 locally-free sheaf is simple if and only if it is stable; since every rank 2 instanton sheaf on  $\mathbb{P}^2$  of charge 1 is properly semistable (by Proposition 16), it follows that these are not simple, as desired.

## 5. Moduli spaces of instanton sheaves

Let  $\mathcal{I}_{\mathbb{P}^n}(r, c)$  denote the moduli space of equivalence classes of rank  $r$  instanton sheaves of charge  $c$  on  $\mathbb{P}^n$ . Let  $\mathcal{I}_{\mathbb{P}^n}^{\text{lf}}(r, c)$  denote the open subset of  $\mathcal{I}_{\mathbb{P}^n}(r, c)$  consisting of locally-free sheaves. Note that  $\mathcal{I}_{\mathbb{P}^n}^{\text{lf}}(r, c)$  might be empty even though  $\mathcal{I}_{\mathbb{P}^n}(r, c)$  is not.

Very little is known in general about  $\mathcal{I}_{\mathbb{P}^n}(r, c)$ ; research so far has concentrated on  $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, c)$ . Here is a summary of some of the known facts:

- $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, c)$  is affine [6];
- $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, 1)$  is an open subset of  $\mathbb{P}^{n(2n+1)-1}$  [8];
- $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, 2)$  is an irreducible, smooth variety of dimension  $4n^2 + 12n - 3$  [2];
- $\mathcal{I}_{\mathbb{P}^3}^{\text{lf}}(2, c)$  is an irreducible, smooth variety of dimension  $8c - 3$  for  $1 \leq c \leq 5$ , see [4] and the references therein;
- $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, c)$  is singular for all  $n \geq 2$  and  $c \geq 3$  [19].

The smoothness of  $\mathcal{I}_{\mathbb{P}^3}^{\text{lf}}(2, c)$  for arbitrary charge  $c$  is still an open problem; it is known however that its closure in the moduli space of semistable locally-free sheaves with Chern character  $2 - cH^2$  is in general singular, see [1].

In this section, we will generalize the second statement, study the moduli spaces of instanton sheaves on  $\mathbb{P}^2$  and conclude with a general conjecture that generalizes the first statement.

**Instanton sheaves and nullcorrelation sheaves.** Recall that a nullcorrelation sheaf  $N$  on  $\mathbb{P}^n$  is a rank  $n - 1$  torsion-free sheaf defined by the short exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\sigma} \Omega_{\mathbb{P}^n}^1(1) \rightarrow N \rightarrow 0,$$

where  $\sigma \in H^0(\Omega_{\mathbb{P}^n}^1(2)) = \Lambda^2 H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ . If  $n$  is odd and  $\sigma$  is generic, then  $N$  is locally-free. If  $n$  is even, then  $N$  is never locally-free; however, the generic one is reflexive if  $n \geq 4$ .

It is easy to see that any nullcorrelation sheaf is instanton of charge 1. The converse is also true: every rank  $n - 1$  torsion-free instanton sheaf of charge 1 is a nullcorrelation sheaf. Indeed, let  $E$  be a rank  $n - 1$  instanton sheaf of charge 1, so that it is the cohomology of the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0.$$

Comparing this with the Euler sequence, it follows that  $\ker \beta$  coincides  $\Omega_{\mathbb{P}^n}^1(1)$ , up to an automorphism of  $\mathbb{P}^n$ . Thus  $E$  fits into the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} \Omega_{\mathbb{P}^n}^1(1) \rightarrow E \rightarrow 0,$$

hence  $E$  is nullcorrelation.

It follows from the above correspondence and the fact that every nullcorrelation sheaf is simple that any nullcorrelation sheaf is completely determined by a section  $\sigma \in H^0(\Omega_{\mathbb{P}^n}^1(2))$ . Hence, we have that the set of equivalence classes of nullcorrelation sheaves on  $\mathbb{P}^n$  is exactly  $\mathbb{P}(H^0(\Omega_{\mathbb{P}^n}^1(2)))$ . Thus we conclude:

**Theorem 24**

$$\mathcal{I}_{\mathbb{P}^n}(n-1, 1) \simeq \mathbb{P}^{\frac{n(n+1)}{2}-1}.$$

It is known that every nullcorrelation locally-free sheaf is stable, while nullcorrelation sheaves which are not locally-free are not stable [8, Remark 1.2.1]; they are however properly semistable. In particular, we have:

**Proposition 25**

*Every rank  $2n$  locally-free instanton sheaf of charge 1 on  $\mathbb{P}^{2n+1}$  is stable.*

**Moduli spaces of instanton sheaves on  $\mathbb{P}^2$ .** The simplest possible instanton sheaves are the rank 1 instanton sheaves on  $\mathbb{P}^2$ . It is not difficult to see that such sheaves are exactly the ideals of points in  $\mathbb{P}^2$ .

Indeed, let  $Z$  be a closed zero dimensional subscheme in  $\mathbb{P}^2$ , and let  $I_Z$  denote its ideal sheaf; it fits into the sequence:

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_Z \rightarrow 0. \quad (22)$$

After tensoring with  $\mathcal{O}_{\mathbb{P}^2}(k)$ , it follows that  $H^0(\mathbb{P}^2, I_Z(k)) = 0$  for  $k \leq -1$ , and  $H^2(\mathbb{P}^2, I_Z(k)) = 0$  for  $k \geq -2$ , so  $I_Z$  is indeed instanton. Moreover, the charge of  $I_Z$  is just the length of  $Z$ .

Conversely, let  $E$  be a rank 1 instanton sheaf of charge  $c$  on  $\mathbb{P}^2$ . Then  $E^{**}$  is a rank 1 locally-free sheaf with  $c_1(E^{**}) = c_1(E) = 0$ , so  $E^{**} = \mathcal{O}_{\mathbb{P}^2}$ . Thus  $E$  is the ideal sheaf associated with the zero-dimensional scheme  $\mathcal{O}_{\mathbb{P}^2}/E$ , whose length is equal to the charge of  $E$ .

In other words, there is a 1-1 correspondence between rank 1 instanton sheaves of charge  $c$  on  $\mathbb{P}^2$  and closed zero dimensional subschemes of length  $c$  in  $\mathbb{P}^2$ . This gives us the following identity:

**Theorem 26**

$$\mathcal{I}_{\mathbb{P}^2}(1, c) \simeq (\mathbb{P}^2)^{[c]}.$$

*Remark.* Let  $E$  be a locally-free instanton sheaf of charge  $c$ ; by tensoring sequence (22) with  $E(k)$ , it is easy to see that  $E_Z = E \otimes I_Z$  is a properly torsion-free instanton sheaf of charge  $c + r \cdot \text{length}(Z)$ . Moreover, if  $H^0(\mathbb{P}^2, E) = 0$ , then  $H^0(\mathbb{P}^2, E_Z) = 0$ . Is it true that every properly torsion-free instanton is a locally-free instanton sheaf tensored by an ideal of points?

As we have seen in Section 3, rank 2 and 3 instanton sheaves on  $\mathbb{P}^2$  are in 1-1 correspondence with semistable torsion-free sheaves with zero first Chern class. So we have:

**Corollary 27**

*For  $r = 2, 3$ ,  $\mathcal{I}_{\mathbb{P}^2}(r, c) \simeq \mathcal{M}_{\mathbb{P}^2}(r, 0, c)$ , the moduli space of rank  $r$  semistable torsion-free sheaves  $E$  on  $\mathbb{P}^2$  with  $c_1(E) = 0$  and  $c_2(E) = c$ . In particular,  $\mathcal{I}_{\mathbb{P}^2}(2, c)$  and  $\mathcal{I}_{\mathbb{P}^2}(3, c)$  are quasi-projective varieties.*

In general, we conjecture that  $\mathcal{I}_{\mathbb{P}^n}(r, c)$  is always a quasi-projective variety.

## References

1. V. Ancona and G. Ottaviani, On singularities of  $M_{\mathbb{P}^3}(c_1, c_2)$ , *Internat. J. Math.* **9** (1988), 407–419.
2. V. Ancona and G. Ottaviani, Stability of special instanton bundles on  $\mathbb{P}^{2n+1}$ , *Trans. Amer. Math. Soc.* **341** (1994), 677–693.
3. N. Buchdahl, Monads and bundles on rational surfaces, *Rocky Mountain J. Math.* **34** (2004), 513–540.
4. I. Coandă, A. Tikhomirov, and G. Trautmann, Irreducibility and smoothness of the moduli space of mathematical 5-instantons over  $\mathbb{P}^3$ , *Internat. J. Math.* **14** (2003), 1–45.
5. L. Costa and R.M. Miró-Roig, Monads and instanton bundles over smooth projective varieties, (preprint, 2004).
6. L. Costa and G. Ottaviani, Nondegenerate multidimensional matrices and instanton bundles, *Trans. Amer. Math. Soc.* **355** (2003), 49–55.
7. S.K. Donaldson, Instantons and geometric invariant theory, *Comm. Math. Phys.* **93** (1984), 453–460.
8. L. Ein, Some stable vector bundles on  $\mathbb{P}^4$  and  $\mathbb{P}^5$ , *J. Reine Angew. Math.* **337** (1982), 142–153.
9. G. Fløystad, Monads on projective spaces, *Comm. Algebra* **28** (2000), 5503–5516.
10. I. Frenkel and M. Jardim, Complex ADHM equations, sheaves on  $\mathbb{P}^3$  and quantum instantons, (preprint math.RT/0408027).
11. R. Hartshorne, Algebraic vector bundles on projective spaces: a problem list, *Topology* **18** (1979), 117–128.
12. R. Hartshorne, Stable reflexive sheaves, *Math. Ann.* **254** (1980), 121–176.
13. R. Hartshorne, Varieties of small codimension in projective space, *Bull. Amer. Math. Soc.* **80** (1974), 1017–1032.
14. H.J. Hoppe, Generischer spaltungstyp und zweite Chernklasse stabiler Vektorraumbündel vom rang 4 auf  $\mathbb{P}^4$ , *Math. Z.* **187** (1984), 345–360.
15. G. Horrocks, Vector bundles on the punctured spectrum of a local ring, *Proc. London Math. Soc.* **14** (1964), 689–713.
16. M. Jardim and R.M. Miró-Roig, On the semistability of instanton sheaves over certain projective varieties, (preprint, 2005).
17. N.M. Kumar, C. Peterson, and A.P. Rao, Monads on projective spaces, *Manuscripta Math.* **112** (2003), 183–189.
18. Y.I. Manin, *Gauge Field Theory and Complex Geometry*, Springer-Verlag, Berlin, 1997.
19. R.M. Miró-Roig and J.A. Orus-Lacort, On the smoothness of the moduli space of mathematical instanton bundles, *Compositio Math.* **105** (1997), 109–119.
20. C. Okonek, M. Schneider, and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Birkhäuser, Boston, 1980.
21. C. Okonek and H. Spindler, Mathematical instanton bundles on  $\mathbb{P}^{2n+1}$ , *J. Reine Angew. Math.* **364** (1986), 35–50.
22. S.M. Salamon, Quaternionic structures and twistor spaces, *Ellis Horwood Ser. Math. Appl.* (1984), 65–74.
23. Y.-T. Siu and G. Trautmann, *Gap-Sheaves and Extension of Coherent Analytic Subsheaves*, Springer-Verlag, Berlin-New York, 1971.
24. H. Spindler and G. Trautmann, Special instanton bundles on  $P_{2N+1}$ , their geometry and their moduli, *Math. Ann.* **286** (1990), 559–592.