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Special effect varieties in higher dimension

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In memory of my grandfather Annibale

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Abstract

Here we introduce the concept of special effect varieties in higher dimension and we generalize to $\mathbb{P}^n$, $n \geq 3$, the two conjectures given in [2] for the planar case. Finally, we propose some examples on the product of projective spaces and we show how these results fit with the ones of Catalisano, Geramita and Gimigliano.

1. Introduction

Let $\mathcal{L}_{n,d} := |\mathcal{O}_{\mathbb{P}^n}(d)|$ be the complete linear system of divisors of degree d in $\mathbb{P}^n$. Fix points $P_1, \dots, P_h$ on $\mathbb{P}^n$ in general position and positive integers $m_1, \dots, m_h$. We denote by $\mathcal{L}_{n,d}(-\sum_{i=1}^h m_i P_i)$ the subsystem of $\mathcal{L}$ given by all divisors having multiplicity at least m_i at P_i, $i = 1, \dots, h$. Since a point of multiplicity m imposes $\binom{m+n-1}{n}$ conditions we can define the **virtual dimension** ν of the system $\mathcal{L}_{n,d}(-\sum_{i=1}^h m_i P_i)$ as

$$\nu\left(\mathcal{L}_{n,d}(-\sum_{i=1}^h m_i P_i)\right) := \binom{d+n}{n} - 1 - \sum_{i=1}^h \binom{m_i+n-1}{n}.$$

The virtual dimension can be computed on the blow-up $\pi : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$ at the points $P_1, \dots, P_h$. In fact, let E_i, $i = 1, \dots, h$ be the exceptional divisors corresponding to

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the blow-up of the points $P_i, i = 1, \dots, h$ and denote by H the pull-back of a general hyperplane of $\mathbb{P}^n$ via π, in such a way we can write the strict transform of the system $\mathcal{L} := \mathcal{L}_{n,d}(\sum_{i=1}^h m_iP_i)$ as $\tilde{\mathcal{L}} = |dH - \sum_{i=1}^h m_iE_i|$. It is an easy application of the (generalized) Riemann-Roch theorem to observe that

$$\nu(\mathcal{L}) = \chi(\tilde{\mathcal{L}}) - 1 = h^0(\tilde{\mathbb{P}}^n, \tilde{\mathcal{L}}) - h^1(\tilde{\mathbb{P}}^n, \tilde{\mathcal{L}}) - 1. \tag{1.1}$$

We then define the **expected dimension** ϵ of $\mathcal{L}_{n,d}(-\sum_{i=1}^h m_iP_i)$ as

$$\epsilon\Big(\mathcal{L}_{n,d}(-\sum_{i=1}^h m_iP_i)\Big) := \max\left\{\nu\Big(\mathcal{L}_{n,d}(-\sum_{i=1}^h m_iP_i)\Big), -1\right\}.$$

Since the conditions imposed by the multiple points m_iP_i could be dependent, in general we have

$$\dim\Big(\mathcal{L}_{n,d}(-\sum_{i=1}^h m_iP_i)\Big) \geq \epsilon\Big(\mathcal{L}_{n,d}(-\sum_{i=1}^h m_iP_i)\Big).$$

We say that a system $\mathcal{L}_{n,d}(-\sum_{i=1}^h m_iP_i)$ is **special** if strict inequality holds, otherwise $\mathcal{L}_{n,d}(-\sum_{i=1}^h m_iP_i)$ is said to be **non-special**.

Starting with the case $X = \mathbb{P}^2$, we have some precise conjectures about the characterization of special linear systems and a rich series of results on the conjectures. The main Conjectures are the following.

Conjecture 1.1 ((SC) B. Segre, 1961). If a linear system of plane curves with general multiple base points $\mathcal{L}_{2,d}(-\sum_{i=1}^h m_iP_i)$ is special, then its general member is non-reduced, i.e. the linear system has, according to Bertini's theorem, some multiple fixed component.

Conjecture 1.2 ((HHC) Harbourne-Hirschowitz, 1989). A linear system of plane curves with general multiple base points $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^h m_iP_i)$ is special if and only if is $(-1)-$special, i.e. its strict transform on the blow-up along the points $P_1, \dots, P_h$ splits as $\tilde{\mathcal{L}} = \sum_{i=1}^k N_iC_i + \tilde{\mathcal{M}}$ where the C_i, $i = 1, \dots, k$, are $(-1)-$curves such that $C_i \cdot \tilde{\mathcal{L}} = -N_i < 0$, $\nu(\tilde{\mathcal{M}}) \geq 0$ and there is at least one index j such that $N_j > 2$.

In [10] C. Ciliberto and R. Miranda proved that the Harbourne–Hirschowitz and Segre Conjectures are equivalent.

In [2] the concepts of $\alpha-$special effect curve and h^1-special effect curve are introduced and two new conjectures are proposed (see Definitions 2.6 and 3.2 for "numerically" and "cohomologically" special).

Conjecture 1.3 ((NSEC) "Numerical Special Effect" Conjecture). A linear system of plane curves $\mathcal{L}_{2,d}(-\sum_{i=1}^h m_iP_i)$ with general multiple base points is special if and only if it is numerically special.

Conjecture 1.4 ((CSEC) "Cohomological Special Effect" Conjecture). A linear system of plane curves $\mathcal{L} := \mathcal{L}_{2,d}(-\sum_{i=1}^h m_iP_i)$ with general multiple base points is special if and only if it is cohomologically special.

The main result in [2] is the following.

Theorem 1.5

Conjectures (SC), (HHC), (NSEC) *and* (CSEC) *are equivalent.*

When we pass to $\mathbb{P}^n$, $n \geq 3$, very little is known about special linear systems. One of the most important result is the classification of the homogeneous special systems for double points:

Theorem 1.6 (Alexander–Hirschowitz, 1996).

The system $\mathcal{L}_{n,d}(2^h)$ *is non-special unless:*

n	*any*	2	3	4	4
d	2	4	4	4	3
h	$2,\ldots,n$	5	9	14	7

Continuing with $\mathbb{P}^n$, $n \geq 3$ we can notice that there is not a precise conjecture. Although the Segre Conjecture can be generalized in every ambient variety using the statement concerning $H^1 \neq 0$ (see, for example, [1], [3] or [10]) there is nothing that characterizes the special systems from a geometric point of view as, for example, in the case of $(-1)-$curves in $\mathbb{P}^2$.

A worthy goal would be "find a conjecture (C) in $\mathbb{P}^n$, [or in a generic variety X] such that, when we read (C) in $\mathbb{P}^2$, (C) is equivalent to the Segre (1.1) and Harbourne–Hirschowitz (1.2) Conjectures".

A first conjecture in this direction was given in [7] where the speciality of a system in $\mathbb{P}^n$ was related to the existence of rational curves in the base locus with particular properties on their normal bundle. Recently, Laface and Ugaglia found a counterexample to this conjecture (see [15]). They showed that the linear system $\mathcal{L} := \mathcal{L}_{3,9}(-6P_0 - \sum_{i=1}^{8} 4P_i)$ in $\mathbb{P}^3$ is special and the only curve contained in its base locus has genus 2.

As already observed, Theorem 1.5 assure us that both *Numerical Special Effect Conjecture* and *Cohomological Special Effect Conjecture* are potential candidates for the above-mentioned goal.

In Sections 2 and 3 we generalize the special effect curves in $\mathbb{P}^2$ to special effect varieties in $\mathbb{P}^n$. The main goal of these sections is to prove that the Conjectures hold for every special system listed in Theorem 1.6.

In Section 4 we present some interesting examples of special effect varieties. In particular we show that the special system in the Laface–Ugaglia example is both numerically and cohomologically special.

In Section 5 we give some interesting evidence about a possible generalization of the Numerical Conjecture to linear systems in the product of projective spaces. In particular we observe how our results fit with similar results given in [4], [5], [6].

Whenever not otherwise specified, we work over the field $\mathbb{C}$.

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2. α-Special effect varieties in $\mathbb{P}^n, n\geq 3$

Let $\mathcal{L} := \mathcal{L}_{n,d}(-\sum_{i=1}^h m_i P_i)$ be an effective linear system on $\mathbb{P}^n$. When we blow up $\mathbb{P}^n$ at the points P_i, $i = 1, \ldots, h$, we can write

$$\nu(\mathcal{L}) := \chi(\tilde{\mathcal{L}}) - 1 = h^0(\tilde{\mathbb{P}}^n, \tilde{\mathcal{L}}) - h^1(\tilde{\mathbb{P}}^n, \tilde{\mathcal{L}}) - 1. \tag{2.1}$$

Let $Y \subset \mathbb{P}^n$ be a variety with $\operatorname{codim}(Y, \mathbb{P}^n) \geq 1$ passing through some of the points $P_1, \ldots, P_h$. We define $\mathcal{L} - Y := \tilde{\mathcal{L}} \otimes \mathcal{I}_{\tilde{Y}}$ The main question we could pose is if we can use the χ of a certain invertible sheaf as in the case of multiple points to compute $\nu(\mathcal{L} - \alpha Y)$. For example let $\tilde{\mathbb{P}}^n$ be the blow-up of $\mathbb{P}^n$ at the points $P_1, \ldots, P_h$ and let $\mathcal{L}' := \tilde{\mathcal{L}}$ be the strict transform of $\mathcal{L}$. After that, we blow up $\tilde{\mathbb{P}}^n$ along $\tilde{Y}$ and compute $\chi(\tilde{\mathcal{L}}' - \alpha R)$, where R is the exceptional divisor $\mathbb{P}(\mathcal{N}_{\tilde{Y}|\tilde{\mathbb{P}}^n})$. We can ask if $\nu(\mathcal{L} - \alpha Y) = \chi(\tilde{\mathcal{L}}' - \alpha R) - 1$. Unfortunately this method does not work for every Y. This is due to the fact that after the two blow-ups some extra-generators can appear in $H^i(\tilde{\mathcal{L}}' - \alpha R)$ for $i \geq 2$. Then it can happen that $h^0(\tilde{\mathcal{L}}' - \alpha R) = 0$, but $\chi(\tilde{\mathcal{L}}' - \alpha R) > 0$, that is the system is empty although we expect it to be nonempty. Thus we define the virtual dimension of a system $\mathcal{L} - Y$ as

$$\nu(\mathcal{L} - Y) = h^0(\mathcal{L} \otimes \mathcal{I}_Y) - h^1(\mathcal{L} \otimes \mathcal{I}_Y) - 1.$$

By (2.1) we see that this definition fits with the standard one. Moreover, it fits with the results of Laface and Ugaglia in [16].

We observe that, in this way, the speciality of the system is given by the non-vanishing of $H^1(\mathcal{L} \otimes \mathcal{I}_Y)$, that is exactly what we expected by the generalization of the Segre Conjecture (see [7]).

DEFINITION 2.1 Let $\mathcal{L}$ and $P_1, \ldots P_h$ as above. An irreducible variety Y has the $\alpha-$**special effect property** for $\mathcal{L}$ on $\mathbb{P}^n$ if there exist positive integer $\alpha, c_{j_1}, \ldots c_{j_s}$, such that

(i) Y contains the point P_{j_i} with multiplicity at least c_{j_i} for $j = 1, \ldots, s$, where $P_{j_i} \in \{P_1, \ldots, P_h\}$;
(ii) $\nu(\mathcal{L} - \alpha Y) > \nu(\mathcal{L})$.

and, if $\operatorname{codim}(Y, \mathbb{P}^n) = 1$, we require $\alpha e \leq d$ and $1 \leq \alpha \leq \min\{\lceil \frac{m_{j_i}}{c_{j_i}} \rceil, i = 1, \ldots, s\}$, where $e := \deg(Y)$. Moreover we require that α is the maximum admissible value for the $\alpha-$special effect property and, if $\beta > \alpha$ then $\nu(\mathcal{L} - \beta Y) < \nu(\mathcal{L} - \alpha Y)$.

DEFINITION 2.2 Let $\mathcal{L}$ and $P_1, \ldots P_h$ be as above. An irreducible variety Y is an $\alpha-$**special effect variety** for $\mathcal{L}$ on $\mathbb{P}^n$ if Y has the $\alpha-$special effect property for $\mathcal{L}$ and moreover $\nu(\mathcal{L} - \alpha Y) \geq 0$.

DEFINITION 2.3 Let $\mathcal{L}$ be a system as above. Fix a sequence of (not necessarily distinct) irreducible varieties $Y_1, \ldots Y_t$, Suppose further that

(1) Y_j has the α_j-special effect property for $\mathcal{L} - \sum_{i=1}^{j-1} \alpha_i Y_i$, for $j = 1, \ldots, t$,
(2) $\nu(\mathcal{L} - \sum_{i=1}^{t} \alpha_i Y_i) \geq 0$.

Then we call both $X := \sum_{i=1}^{\alpha} Y_i$ and $\{Y_1, \dots, Y_r\}$ an $(\alpha_1, \dots, \alpha_r)-$**special effect configuration for** $\mathcal{L}$.

Remark 2.4 It is possible to use the χ of a certain invertible sheaf in several situations, for example in the case of homogeneous systems, i.e. when $m = m_1 = \cdots = m_h$ with Y smooth, irreducible, $c_1 = \cdots = c_h = 1$ and with $\alpha = m$, i.e. α exhausts the multiplicity at the points. In this situation we blow up $\mathbb{P}^n$ along Y obtaining an exceptional divisor R; then condition (ii) becomes

$$\chi(dH - \alpha R) > \chi\Big(dH - \sum_{i=1}^{h} mE_i\Big) \tag{2.2}$$

where the χ on the left side is taken on $X = Bl_Y(\mathbb{P}^n)$ while the χ on the right side is taken on $X' = Bl_{\{P_i\}}(\mathbb{P}^n)$, i.e. the blow-up of $\mathbb{P}^n$ at $P_1, \dots P_h$.

Remark 2.5 In general we refer to conditions (ii) of Definition 2.1, condition (2) of Definition 2.3 and formula (2.2) as the **special inequality**.

Let X be an $\alpha-$special effect variety or an $(\alpha_1, \dots, \alpha_r)-$special effect configuration for a system $\mathcal{L}$. Then X forces $\mathcal{L}$ to be special. In fact, one has

$$\dim(\mathcal{L}) \geq \dim(\mathcal{L} - X) \geq \nu(\mathcal{L} - X) > \nu(\mathcal{L})$$

and, together with condition $\nu(\mathcal{L} - X) \geq 0$, one has $\dim(\mathcal{L}) > \epsilon(\mathcal{L})$.

These facts permit us to define a particular kind of speciality.

Definition 2.6 A special system arising from the existence of an $\alpha-$special effect variety (or an $(\alpha_1, \dots, \alpha_r)-$special effect configuration) is called **Numerically Special**.

Finally, we can state the same conjecture as in the planar case:

Conjecture 2.7 ((NSEC) "Numerical Special Effect" Conjecture). A linear system $\mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i P_i)$ with general multiple base points is special if and only if it is numerically special.

We restrict now our attention to $2-$special effect varieties in $\mathbb{P}^n$, $n \geq 3$ for the homogeneous case $\mathcal{L}_{n,d}(2^h)$. In particular we consider as a special effect variety respectively a smooth divisor $Y = dH$, a linear space $Y = \mathbb{P}^s$, $1 \leq s \leq n$ and a rational normal curve $C_n \subset \mathbb{P}^n$, i.e. the image of $\mathbb{P}^1$ under the $n-$Veronese embedding.

2.1 Hypersurfaces in $\mathbb{P}^n$

If Y is a smooth hypersurface of degree e passing through $P_1, \dots, P_h$, then the conditions for Y to be a $2-$special effect variety for $\mathcal{L} := \mathcal{L}_{n,d}(2^h)$ become

$$\binom{e+n}{n} - 1 \geq h \tag{2.3}$$

$$\binom{d-2e+n}{n} > \binom{d+n}{n} - h(n+1). \tag{2.4}$$

$$\binom{d-2e+n}{n} \geq 1, \text{ i.e. } d \geq 2e \tag{2.5}$$

We have the following.

Proposition 2.8

Let Y be a smooth hypersurface passing through $P_1, \ldots, P_h$, $h \geq n$. Then Y is a $2-$special effect variety for $\mathcal{L}_{n,d}(2^h)$, $n \geq 2$ when
i) $Y = \mathbb{P}^{n-1}$, *for* $\mathcal{L}_{n,2}(2^n)$ $\forall n \geq 2$;
ii) $Y =$ *Conic* $\subset \mathbb{P}^2$, *for* $\mathcal{L}_{2,4}(2^5)$;
iii) $Y =$ *Quadric* $\subset \mathbb{P}^3$, *for* $\mathcal{L}_{3,4}(2^9)$;
iv) $Y =$ *Quadric* $\subset \mathbb{P}^4$, *for* $\mathcal{L}_{4,4}(2^{14})$.

Proof. From conditions (2.3) and (2.4) we obtain the following bounds on h:

$$\binom{e+n}{n} - 1 \geq h > \frac{1}{n+1}\left[\binom{d+n}{n} - \binom{d-2e+n}{n}\right].$$

Then our special effect variety exists if $d \geq 2e$ and if

$$\varphi(d,e,n) := \binom{d+n}{n} - \binom{d-2e+n}{n} - (n+1)\binom{e+n}{n} + n + 1 \tag{2.6}$$

is negative. By Pascal's triangle for binomials we have

$$\varphi(d,e,n) = \sum_{i=1}^{2e} \binom{d+n-i}{n-1} - (n+1)\binom{e+n}{n} + (n+1).$$

Thus the function $\varphi(d,e,n)$ is increasing monotone in d and we can fix our attention in the case $\varphi(2e,e,n)$ and, eventually increase the value of d up to the first d_0 such that $\varphi(d_0,e,n) \geq 0$. Then our equation becomes

$$\varphi(2e,e,n) := \binom{2e+n}{n} + n - (n+1)\binom{e+n}{n}. \tag{2.7}$$

Step 1a: $e = 1, d = 2$

$$\begin{aligned}\varphi(2,1,n) &= \binom{2+n}{n} + n - (n+1)\binom{1+n}{n} \\ &= \frac{1}{2}(n+1)(n+2) + n - (n+1)^2 = \frac{1}{2}n(1-n) < 0 \quad \forall n \geq 2.\end{aligned}$$

Step 1b: $e = 1, d \geq 3$

$$\begin{aligned}\varphi(d,1,n) &= \binom{d+n-1}{n-1} + \binom{d+n-2}{n-1} + n + 1 - (n+1)\binom{n+1}{n} \\ &\geq \frac{1}{6}n(n+1)(n+2) + \frac{1}{2}n(n+1) + n + 1 - (n+1)^2 \\ &= \frac{n}{6}(n^2-1) > 0 \quad \forall n \geq 2.\end{aligned}$$

From conditions (2.3) and $h \geq n$ (to avoid degenerate cases) one has $h = n$. Then the first case of the Proposition follows. We will obtain again this result in Proposition 2.13.

Step 2a: $e = 2, d = 4$

$$\begin{aligned}\varphi(4,2,n) &= \binom{4+n}{n} + n - (n+1)\binom{2+n}{n} \\ &= \frac{n}{24}(n^3 - 2n^2 - 13n + 14)\end{aligned}$$

and we found

$$\begin{aligned}&\varphi(4,2,2) < 0, \quad h = 5; \\ &\varphi(4,2,3) < 0, \quad h = 9; \\ &\varphi(4,2,4) < 0, \quad h = 14; \\ &\varphi(4,2,n) > 0, \quad \text{for } n \geq 5.\end{aligned}$$

Step 2b: $e = 2, d \geq 5$

$$\begin{aligned}\varphi(d,2,n) &= \sum_{i=1}^{4}\binom{d+n-i}{n-1} + n + 1 - (n+1)\binom{2+n}{n} \\ &\geq \frac{1}{120}n(n^4 + 15n^3 + 25n^2 - 15n - 26) > 0 \quad \forall n \geq 2.\end{aligned}$$

Then, also cases (ii), (iii) and (iv) of the Proposition are proved.

Step 3: at this point, we can reduce the study of $\varphi(d,e,n)$ with the condition $d \geq 2e \geq 6$. But, in this case, $\varphi(d,e,n) \geq 0$ $\forall n \geq 3$, as stated in the next lemma, and this concludes the proof. $\square$

Lemma 2.9 (Numerical Lemma 1)

Let $\varphi(d,e,n)$ be defined as in (2.6). If $d \geq 2e \geq 6$, then $\varphi(d,e,n) \geq 0$, $\forall n \geq 3$.

Proof. Since $\varphi(d,e,n)$ is non-decreasing in d, we fix our attention on the minimal value $d = 2e$. Thus we write

$$\begin{aligned}\varphi(2e,e,n) &= \frac{(n+1)\dots(n+2e)}{2e!} + n - \frac{(n+1)(n+1)\dots(n+e)}{e!} \\ &= n + \frac{(n+1)\dots(n+e)}{e!}[A(e)]\end{aligned}$$

where

$$A(e) := \frac{(n+e+1)\dots(n+2e)}{(e+1)\dots(2e)} - n - 1.$$

We show that $A \geq 0$, $\forall e \geq 3$ and $\forall n \geq 3$.

Claim: $A(e+1) > A(e)$

We have

$$A(e+1) := \frac{(n+e+2)\dots(n+2e+2)}{(e+2)\dots(2e+2)} - n - 1.$$

$$A(e) := \frac{(n+e+1)\dots(n+2e)}{(e+1)\dots(2e)} - n - 1.$$

Thus we compute

$$A(e+1) - A(e) = \frac{(n+e+2)\dots(n+2e)}{(e+1)\dots(2e+2)}[n^2(e+1) + n(e+1)] > 0$$

and the claim follows.

If we consider $e = 3$ we obtain

$$\begin{aligned} A(3) &= \frac{1}{120}[(n+4)(n+5)(n+6)] - n - 1 \\ &= \frac{1}{120}n(n^2 + 15n - 46) > 0 \quad \forall n \geq 3 \end{aligned}$$

and so, for the claim, we have

$$\varphi(d, e, n) \geq 0 \quad \forall d \geq 6, \forall e \geq 3, \forall n \geq 3. \ \square$$

Remark 2.10 It is easy to see that, under the hypothesis on Y as in Proposition 2.8, the case $\alpha = 1$ does not give any new special effect hypersurfaces other than the ones in Proposition 2.8. In fact the conditions for Y to be a $1-$special effect variety for a system $\mathcal{L}_{n,d}(2^h)$ are

$$\begin{aligned} &\binom{e+n}{n} - 1 \geq h \\ &\binom{d-e+n}{n} - h > \binom{d+n}{n} - h(n+1), \\ &\binom{d-e+n}{n} - h \geq 1. \end{aligned}$$

Then our special effect variety can exist if

$$\begin{aligned} \psi(d, 2, n) :&= \binom{d+n}{n} - \binom{d-e+n}{n} - n\binom{e+n}{n} + n \\ &= \sum_{i=1}^{e} \binom{d+n-i}{n-1} - n\binom{e+n}{n} + n < 0. \end{aligned} \tag{2.8}$$

Once again, we can consider the minimal value $d = 2e$:

$$\psi(2e, e, n) := \binom{2e+n}{n} + n - (n+1)\binom{e+n}{n} < 0. \tag{2.9}$$

Since $\psi(2e, e, n)$ is equal to $\varphi(2e, e, n)$ in (2.7) the case $\alpha = 1$ does not produce any new examples of special effect hypersurfaces.

Remark 2.11 The argument of the proof of Proposition 2.8 can be used succesfully when the system $\mathcal{L}$ is homogeneous. In general we use the equations given by the numerical speciality to construct a function φ such that our problem of the existence of an $\alpha-$special effect variety can become a pure combinatorial problem. The function φ can change depending on the data of the variety Y, the system $\mathcal{L}$ and the ambient variety X.

2.2 Linear spaces in $\mathbb{P}^n$

Let Y be a linear space $\mathbb{P}^s \subset \mathbb{P}^n$ with $1 \leq s \leq n-1$; by changing the coordinates,we can suppose that Y is defined by $x_0 = \cdots = x_{n-s-1} = 0$. It is not difficult to verify that the expected dimension of $|dH - mY|$, $m \geq 2$, is given by

$$\binom{d+n}{n} - 1 - \sum_{i=0}^{m-1} \binom{d+s-i}{d-i}\binom{n-s-1+i}{i}.$$

In particular, the expected dimension of $|dH - 2Y|$ is given by

$$\binom{d+n}{n} - 1 - \binom{d+s}{d} - \binom{d+s-1}{d-1}(n-s). \tag{2.10}$$

Consider now the system $\mathcal{L} := \mathcal{L}_{n,d}(2^h)$, with $s+1 \leq h$, and suppose that the first $s+1$ points span $Y = \mathbb{P}^s$. Since Y does not pass through all double points in $\mathcal{L}$ we need to study the system $|dH - 2Y - \sum_{i=s+2}^{h} 2P_i|$. Then, for $s+1 \leq h$, $Y = \mathbb{P}^s$ is a $2-$special effect variety for $\mathcal{L}_{n,d}(2^h)$ if

$$(s+1)(n+1) - \binom{d-s}{d} - \binom{d-s-1}{d-1}(n-s) > 0 \tag{2.11}$$

and

$$\binom{d+n}{n} - 1 - \binom{d+s}{d} - \binom{d+s-1}{d-1}(n-s) - (h-s-1)(n+1) \geq 0. \tag{2.12}$$

If we check the special inequality (2.11) for $d = 2$ we obtain

$$(n+1)(s+1) - \frac{(s+1)(s+2)}{2} - (s+1)(n-s) > 0.$$

Simplifying we obtain $2n + 2 - s - 2 - 2n + 2s > 0$ then $s > 0$.

Let us consider the case $d \geq 3$. We write the special inequality as

$$\begin{aligned}(n+1)(s+1) - \binom{d+s}{s} - \binom{d-1+s}{d-1}(n-s) \\ &= (n-s+s+1)(s+1) - \binom{d+s}{s} - \binom{d-1+s}{d-1}(n-s) \\ &= (n-s)(s+1) + (s+1)^2 - \binom{d+s}{s} - \binom{d-1+s}{d-1}(n-s).\end{aligned}$$

Since $d \geq 3$ we have

$$\binom{d+s}{s} \geq \binom{s+3}{s} > (s+1)^2$$

and

$$\binom{d-1+s}{d-1} \geq \binom{s+2}{2} > s+1.$$

Thus, for $d \geq 3$, the special inequality (2.11) is false for every s and n.

We pass now to study the equation (2.12) assuming $d = 2$. We obtain

$$\frac{1}{2}n^2 + \frac{3}{2}n + \frac{1}{2}s^2 + \frac{1}{2}s - h(n+1) \geq 0. \tag{2.13}$$

If we solve by respect to s we find

$$s \geq \left\lfloor \frac{\sqrt{1 - 12n - 4n^2 + 8hn + 8h}}{2} - \frac{1}{2} \right\rfloor.$$

Thus we have the following.

Proposition 2.12

Let $\mathcal{L} := \mathcal{L}_{n,d}(2^h)$ with $2 \leq h \leq n$. Then $Y = \mathbb{P}^s$ is a 2−special effect variety for $\mathcal{L}$ if $\rho(n,h) \leq s \leq h-1$, where

$$\rho(n,h) = \begin{cases} \lfloor \frac{\sqrt{1 - 12n - 4n^2 + 8hn + 8h}}{2} - \frac{1}{2} \rfloor & if \quad h > \frac{n^2+3n}{2(n+1)} \\ 1 & otherwise\,. \end{cases}$$

Corollary 2.13

$Y = \mathbb{P}^{h-1} \subset \mathbb{P}^n$ is a 2−special effect variety for $\mathcal{L}_{n,2}(2^h)$ for $2 \leq h \leq n$ and $\forall n \geq 2$.

Proof. It follows easily from the proof of Proposition 2.12 by observing that formula (2.13) is always verified for $s + 1 = h$, for $2 \leq h \leq n$ and $\forall n \geq 2$. $\square$

2.3 Rational normal curves of degree n in $\mathbb{P}^n$

Let $C_n \subset \mathbb{P}^n$ be the image of $\mathbb{P}^1$ under the n−Veronese embedding ($n > 1$). Once we have fixed the dimension n of $\mathbb{P}^n$, the virtual dimension of $|dH - mC_n|$ can be computed using the generalized Riemann–Roch theorem on the blow-up of $\mathbb{P}^n$ along C_n. More generally, there are some classical results about the postulation of a multiple curve. See, for example, the works of B. Segre ([18]) and A. Franchetta ([12]).

Since we are interested, for the moment, only in the case $|dH - 2C_n|$ we can use some interesting results given by A. Conca in [11]. Thus, for $d \geq 3$, one has

$$\nu(|dH - 2C_n|) = \binom{d+n}{n} - 1 - ((d-1)n^2 + 2).$$

Supposing $h = n+3$ so that C_n is fixed, the special inequality becomes

$$(n+1)(n+3) - (d-1)n^2 - 2 > 0.$$

If we expand the previous inequality we obtain

$$\begin{aligned}(n+1)(n+3) - (d-1)n^2 - 2 &= -(d-1)n^2 - 2 + n^2 + 4n + 3 \\ &= (2-d)n^2 + 4n + 1 > 0.\end{aligned}$$

If we solve this equation with respect to n we find

$$\frac{2-\sqrt{2+d}}{d-2} < n < \frac{2+\sqrt{2+d}}{d-2}$$

then we restrict our solutions to $2 \leq n < \frac{2+\sqrt{2+d}}{d-2}$. If we substitute the values of d we find that the only possibilities are

d	3	4
n	$2, 3, 4$	2

At this point, we need to check $\nu(|dH - 2C_n|) \geq 0$. Since

$$\nu(|3H - 2C_2|) = \nu(|3H - C_3|) = -1,$$

we exclude $d = 3$ with $n = 2, 3$. This concludes the proof of the following.

Proposition 2.14

Let $C_n \subset \mathbb{P}^n$ be the rational normal curve, i.e. the image of $\mathbb{P}^1$ under the $n-$Veronese embedding. Then C_n is a $2-$special effect variety for $\mathcal{L}_{2,d}(2^{n+3})$ only when (n, d) is $(2, 4)$ or $(4, 3)$.

Remark 2.15 It is easy to check that C_n is not a $2-$special effect variety for $\mathcal{L} := \mathcal{L}_{n,d}(2^h)$ if $h \neq n+3$.

Let us analyze first the case $h \leq n+2$. The conditions for the speciality of C_n are

$$\binom{d+n}{n} - 1 - ((d-1)n^2 + 2) \geq 0 \tag{2.14}$$

$$h(n+1) - ((d-1)n^2 + 2) > 0. \tag{2.15}$$

Since $h \leq n+2$ and $d \geq 3$, in (2.15) we obtain

$$0 < h(n+1) - ((d-1)n^2 + 2) \leq (n+2)(n+1) - (2n^2+2) = 3n - n^2.$$

Thus the only possible value is $n = 2$ and equation (2.15) becomes

$$2 - 4d + 3h > 0.$$

Since C_2 is the conic in $\mathbb{P}^2$, we have to consider $d \geq 4$. Thus the previous equation has no solutions for $h \leq 4$.

For $h \geq n+4$ the speciality inequality is the same as in the case of Proposition 2.14, hence the allowed values are

d	3	4
n	$2, 3, 4$	2

But for this values, with the hypothesis $h \geq n+4$, one has

$$\nu\left(|dH - 2C_n - \sum_{i=n+4}^{h} 2P_i|\right) < 0.$$

2.4 The $\alpha-$special effect varieties and the Alexander–Hirschowitz Theorem

The examples in the previous sections fit with the Alexander–Hirschowitz Theorem. In particular we can state the following.

Theorem 2.16

The Numerical Conjecture holds for each of the special systems listed in Theorem 1.6

Proof. It is enough to find a $\alpha-$special effect variety Y for each of the special systems in the list of Thereom 1.6.

The cases $\mathcal{L}_{n,2}(2^h)$, $2 \leq h \leq n$ follow from Proposition 2.13 considering Y as the linear space $\mathbb{P}^{s-1}$.

The conic in $\mathbb{P}^2$ is a $2-$special effect curve for $\mathcal{L}_{2,4}(2^5)$ as shown in Example 3.8 in [2]. It follows also from Proposition 2.8, case *ii*), and from Proposition 2.14, case $(n, d) = (2, 4)$.

The cases $\mathcal{L}_{3,4}(2^9)$ and $\mathcal{L}_{4,4}(2^{14})$ are studied in Proposition 2.8 and Y is the quadric hypersurface respectively in $\mathbb{P}^3$ and $\mathbb{P}^4$.

Finally, using again Proposition 2.14 case $(n, d) = (4, 3)$, we obtain that the rational normal curve $C_4 \subset \mathbb{P}^4$ is a $2-$special effect variety for $\mathcal{L}_{4,3}(2^7)$. $\square$

Remark 2.17 Recently, A. Laface and L. Ugaglia proposed a conjecture for special linear systems in $\mathbb{P}^3$ ([16]). Although an equivalence between this conjecture and the Numerical Special Effect Conjecture is still unproved, it is easy to see some interesting evidence. In fact, in the Laface–Ugaglia Conjecture the speciality of a system $\mathcal{L}$ in standard form (i.e. after performing a series of Cremona transformations) is related to the existence of a quadric surface or a line in the base locus $Bs(\mathcal{L})$ which makes the value of $\nu(\mathcal{L})$ lower. In other terms, both the quadric or the line seem to be $\alpha-$special effect varieties.

3. h^1-Special effect varieties in $\mathbb{P}^n$, $n \geq 3$

We turn now to analyzing h^1-special effect varieties in higher dimension. Let $\mathcal{L} := \mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i P_i)$ be a linear system of hypersurfaces with general multiple base points and let X be the blow-up of $\mathbb{P}^n$ at the points $\{P_i\}$. Let $\tilde{\mathcal{L}}$ be the strict transform of $\mathcal{L}$. In general, if confusion cannot arise, we will denote both $\mathcal{L}$ and $\tilde{\mathcal{L}}$ by $\mathcal{L}$. We recall that, if we denote by $\tilde{Y}$ the strict transform of a variety $Y \subset \mathbb{P}^n$, then we define $\mathcal{L} - Y := \mathcal{L} \otimes \mathcal{I}_{\tilde{Y}}$. The definition of the h^1-special effect variety is slightly modified with respect to the planar case.

Definition 3.1 Let $\mathcal{L}$ and Y be as above with Y irreducible. Moreover, when $\operatorname{codim}(Y, \mathbb{P}^n) = 1$, we require $\mathcal{O}_{\mathbb{P}^n}(Y) \ncong \mathcal{L}$. Then $Y \subset \mathbb{P}^n$ is an h^1-**special effect variety** for the system $\mathcal{L}$ if the following conditions are satisfied:

(a) $h^0(\mathcal{L}_{|Y}) = 0$;
(b) $h^0(\mathcal{L} - Y) \neq 0$;
(c) $h^1(\mathcal{L}_{|Y}) > h^2(\mathcal{L} - Y)$.

As in the planar case, the speciality of the system $\mathcal{L}$ follows from the previous conditions and from the standard exact sequence

$$0 \to \mathcal{L} - Y \to \mathcal{L} \to \mathcal{L}_{|Y} \to 0. \tag{3.1}$$

In fact we have the following long exact sequence in cohomology:

$$0 \to H^0(\mathcal{L} - Y) \to H^0(\mathcal{L}) \to H^0(\mathcal{L}_{|Y}) \to H^1(\mathcal{L} - Y) \to H^1(\mathcal{L}) \to H^1(\mathcal{L}_{|Y}) \to \cdots$$

Conditions (a) and (b) assure us that $H^0(\mathcal{L}) \neq 0$, while condition (c) implies $H^1(\mathcal{L}) \neq 0$. Thus the existence of such Y forces the system $\mathcal{L}$ to have $h^0(\mathcal{L}) \cdot h^1(\mathcal{L}) \neq 0$ so that, by (1.1), $\mathcal{L}$ is special.

Definition 3.2 A special system arising from the existence of an h^1-special effect curve is called **Cohomologically Special**.

And again we can state a conjecture:

Conjecture 3.3 ((CSEC). "Cohomological Special Effect" Conjecture). A linear system $\mathcal{L} := \mathcal{L}_{n,d}(-\sum_{i=1}^{h} m_i P_i)$ with general multiple base points is special if and only if it is cohomologically special.

The h^1-special effect varieties seem easier to treat than the $\alpha-$special effect varieties. In fact we do not need to define the virtual dimension, but we just work with elements in cohomology. However, in several situations, it is very difficult to compute some cohomology groups, in particular $h^2(\mathcal{L} - Y)$.

As in the case of $\alpha-$special effect varieties, we do not have problems when Y is a divisor since $h^2(\mathcal{L} - Y) = 0$ if $\mathcal{L} - Y$ is effective. Unluckily, in this case, it can be difficult to study the behaviour of $\mathcal{L}_{|Y}$.

Instead, when $\operatorname{codim}(Y, \mathbb{P}^n) \geq 2$, the groups $h^i(\mathcal{L} - Y)$, $i = 1, 2$ can be computed on the blow-up of $\mathbb{P}^n$ along Y, but we need a deep understanding of the geometry and cohomology of Y.

We study now the situation in which $\mathcal{L} := \mathcal{L}_{n,d}(2^h)$, i.e. $\mathcal{L}$ is a linear system with imposed double points. The following Theorem is similar to Theorem 2.16. Thus, also the Cohomological Special Effect Conjecture fits with the Alexander–Hirschowitz Theorem.

Theorem 3.4

The Cohomological Conjecture holds for each of the special systems listed in Theorem 1.6.

Proof. We start with $\mathcal{L} := \mathcal{L}_{n,2}(2^h)$ with $2 \leq h \leq n$. Let Y be the span of the h points $P_0, \dots P_{h-1}$ in the linear system $\mathcal{L}$, i.e. $Y = \mathbb{P}^{h-1}$. Since $\mathcal{L}_{|Y} = \mathcal{L}_{h-1,2}(2^h)$, this system is clearly empty (see, for example, [17]) and one has $h^0(\mathcal{L}_{|Y}) = 0$ and the condition (a) for Y to be an h^1-special effect variety is satisfied. Let Z be the zero-dimensional scheme $\cup_{i=0}^{h-1} 2P_i$ on Y; then from the exact sequence

$$\begin{array}{ccl} 0 \to & \mathcal{I}_Z(2) & \to \mathcal{O}_{\mathbb{P}^{h-1}}(2) \to \mathcal{O}_Z \to 0 \\ & \| & \\ & \mathcal{L}_{|Y} & \end{array}$$

we obtain $h^1(\mathcal{L}_{|Y}) = \frac{h(h-1)}{2}$.

By Theorem 1.6 one has $h^0(\mathcal{L}) \neq 0$. Since $h^0(\mathcal{L}_{|Y}) = 0$ we conclude $h^0(\mathcal{L} - Y) \neq 0$, then condition (b) is satisfied.

Thus Y will be an h^1-special effect variety if we prove $h^1(\mathcal{L}_{|Y}) > h^2(\mathcal{L} - Y)$. From the discussion, in [17], about the matrices representing quadratic forms we easily compute

$$h^1(\mathcal{L}) = \frac{h(h-1)}{2}.$$

From the sequence (3.1) we obtain

$$0 \to H^1(\mathcal{L} - Y) \to H^1(\mathcal{L}) \xrightarrow{\theta} H^1(\mathcal{L}_{|Y}) \to H^2(\mathcal{L} - Y) \to 0.$$

Since $h^1(\mathcal{L}) = h^1(\mathcal{L}_{|Y})$ it is enough to prove that θ is not the zero map.

We can suppose that the h points are the coordinate points $P_i = [0, \dots, 1, \dots, 0]$, $i = 0, \dots, h-1$ so that Y has equation $x_h = \cdots = x_n = 0$. Let I_Y be the ideal of Y in $\mathbb{P}^n$ and let $\mathbf{m}_i, \mathbf{m}_{Y,i}$ be the ideals of P_i's respectively in $\mathbb{P}^n$ and Y. Let I and I' be respectively the ideals $\cap_{i=0}^{h-1} \mathbf{m}_i^2$ and $\cap_{i=0}^{h-1} \mathbf{m}_{Y,i}^2$.

Since $\mathbf{m}_{Y,i} = (x_0, \dots, \hat{x}_i, \dots x_{h-1})$ for $i = 0, \dots, h-1$, the generators of I' are

$$\begin{cases} x_k x_l x_m & k, l, m = 0, \dots, h-1, k \neq l, k \neq m, l \neq m, \\ x_k^2 x_l^2 & k, l = 0, \dots, h-1, k \neq l. \end{cases}$$

Moreover $\mathbf{m}_i = (x_0, \dots, \hat{x}_i, \dots x_n)$ for $i = 0, \dots, h-1$; hence if $j = h, \dots, n$ then $x_j^2 \in \cap_{i=0}^{h-1} \mathbf{m}_i^2$. Thus, after a straightforward computation we obtain

$$I = I_Y^2 \cup (x_k x_l x_m : l, m = 0, \dots, h-1, m \neq l, k = h, \dots, n) \cup I'.$$

We denote by $\mathcal{I}_Y$, $\mathcal{I}$ and $\mathcal{I}'$ the ideal sheaves corresponding to the previous ideals.

Consider the following diagram:

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}^n}(2)) & \xrightarrow{\alpha} & H^0(\mathcal{O}_{\mathbb{P}^n}/\mathcal{I}(2)) = B \\ \Big\downarrow{\scriptstyle <x_h,\dots,x_n>} & & \Big\downarrow{\scriptstyle <x_h,\dots,x_n>} \\ H^0(\mathcal{O}_Y(2)) & \xrightarrow{\alpha_Y} & H^0(\mathcal{O}_Y/\mathcal{I}'(2)) = B_Y\,. \end{array} \tag{3.2}$$

We call σ the map $B \to B_Y$ in the previous diagram (the map on the right-side). This map is given by the equations of Y.

From the previous computations of the ideals I and I' we know that σ is surjective. Since $H^1(\mathcal{L}_{|Y}) \neq \emptyset$, there exists an $\eta \in B_Y$ such that $\eta \notin \mathrm{Im}\alpha_Y$. Let $[\eta]$ be the image of η in $H^1(\mathcal{L}_{|Y})$. By the surjectivity of σ, η comes from an element $\eta_0 \in B$. Since diagram (3.2) is commutative, η_0 does not lies in $\mathrm{Im}\alpha$, because otherwise $\eta \in \mathrm{Im}\alpha_Y$. Thus we conclude that θ sends $[\eta_0] \in H^1(\mathcal{L})$ to $[\eta] \in H^1(\mathcal{L}_{|Y})$, i.e. $\theta \not\equiv 0$. Hence $h^1(\mathcal{L}_{|Y}) > h^2(\mathcal{L} - Y)$.

We turn to analyzing the cases $\mathcal{L}_{2,4}(2^5)$, $\mathcal{L}_{3,4}(2^9)$ and $\mathcal{L}_{4,4}(2^{14})$. We can treat them in an unified way just writing $\mathcal{L} := \mathcal{L}_{n,4}(2^s)$, where $s = \binom{2+n}{n} - 1$ and $n = 2, 3, 4$. Let Y be the divisor corresponding to $\mathcal{L}_{n,2}(1^s)$, i.e the conic in $\mathbb{P}^2$ through 5 points and the quadric in $\mathbb{P}^3$ and $\mathbb{P}^4$ respectively through 9 and 14 points. Since $\mathcal{L} - Y = \mathcal{L}_{n,2}(1^s)$ we have

$$h^0(\mathcal{L} - Y) = 1,$$
$$h^i(\mathcal{L} - Y) = 0 \text{ for } i \geq 1.$$

By Theorem 1.6 we know that $h^0(\mathcal{L}) = 1$ and

$$h^1(\mathcal{L}) = \begin{cases} 1 & \text{for } n = 2, 4 \\ 2 & \text{for } n = 3\,. \end{cases}$$

Finally, by sequence (3.1) we conclude $h^0(\mathcal{L}_{|Y}) = 0$ and $h^1(\mathcal{L}_{|Y}) = h^1(\mathcal{L}) > 0$. Hence the given $Y = \mathcal{L}_{2,n}(1^s)$ is an h^1-special effect variety for the system $\mathcal{L} := \mathcal{L}_{n,4}(2^s)$, $n = 2, 3, 4$.

The last case to treat is $\mathcal{L}_{4,3}(2^7)$. Let Y be the rational normal curve of degree 4 passing through the seven double points described in [8]. Since $\mathcal{L} \cdot Y = -2$ we have

$$h^0(\mathcal{L}_{|Y}) = 0 \qquad h^1(\mathcal{L}_{|Y}) = 1.$$

Moreover, by Theorem 1.6, we know that

$$h^0(\mathcal{L}) = 1 \qquad h^1(\mathcal{L}) = 1.$$

Thus we obtain $h^0(\mathcal{L} - Y) = 1$. To conclude the proof we need to show $h^2(\mathcal{L} - Y) = 0$. By the sequence

$$0 \to H^1(\mathcal{L} - Y) \to H^1(\mathcal{L}) \xrightarrow{\theta} H^1(\mathcal{L}_{|Y}) \to H^2(\mathcal{L} - Y) \to 0,$$

it is enough to prove that θ is surjective. If we tensor by $\mathcal{O}_{\tilde{Y}}$ the sequence (3.1) we obtain

$$\begin{array}{ccccccccc} & 0 & \to & \mathcal{L} & \to & \mathcal{O}_{\tilde{\mathbb{P}}^4}(3) & \to & \mathcal{O}_{\sum 2E_i} & \to & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \mathrm{Tor}_1(\mathcal{O}_{\sum 2E_i}, \mathcal{O}_{\tilde{Y}}) & \to & \mathcal{L} \otimes \mathcal{O}_{\tilde{Y}} & \to & \mathcal{O}_{\tilde{Y}}(3) & \to & \mathcal{O}_{\sum 2Q_i} & \to & 0 \end{array}$$

where $Q_i = \tilde{Y} \cap E_i$, $i = 1, \dots, 7$. Since $\mathcal{L} \otimes \mathcal{O}_{\tilde{Y}}$ corresponds to an invertible sheaf on $\mathbb{P}^1$, it cannot have torsion, thus $\mathrm{Tor}_1(\mathcal{O}_{\sum 2E_i}, \mathcal{O}_{\tilde{Y}}) = 0$ and we write

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{L} & \to & \mathcal{O}_{\tilde{\mathbb{P}}^4}(3) & \to & \mathcal{O}_{\sum 2E_i} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{L} \otimes \mathcal{O}_{\tilde{Y}} & \to & \mathcal{O}_{\tilde{Y}}(3) & \to & \mathcal{O}_{\sum 2Q_i} & \to & 0. \\
 & & \| & & & & & & \\
 & & \mathcal{L}_{|Y} & & & & & &
\end{array} \tag{3.3}
$$

Since every diagram in (3.3) is commutative, when we pass to cohomology we obtain the following commutative diagram

$$
\begin{array}{ccc}
H^0(\mathcal{O}_{\sum 2E_1}) & \xrightarrow{\delta_1} & H^1(\mathcal{L}) \\
{\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\theta}\big\downarrow \\
H^0(\mathcal{O}_{\sum 2Q_1}) & \xrightarrow{\delta_{1,Y}} & H^1(\mathcal{L}_{|Y})
\end{array}
$$

where δ_1 and $\delta_{1,Y}$ are the connection homomorphisms. We can observe that δ_1 and $\delta_{1,Y}$ are surjective because $H^1(\mathcal{O}_{\tilde{\mathbb{P}}^4}(3))$ and $H^1(\mathcal{O}_{\tilde{Y}}(3))$ are zero. Moreover α is surjective too. As a matter of fact, let $f_i \in k[\dots, x_j, \dots]$ be the polynomial defining E_i. We can fix our attention on a single polynomial f_0. Let $I(\tilde{Y})$ be the ideal of $\tilde{Y}$, thus $k[\dots, x_j, \dots]/I(\tilde{Y}) = k[t]$. Since f_0 is not tangent to $\tilde{Y}$ we have

$$f_0 \mod I(\tilde{Y}) = t + o(t^2).$$

Thus the map

$$k[\dots, x_j, \dots]/(f_0^2) \to k[t]/(f_0^2)$$

is surjective. Finally, from the surjectivity of $\delta_{1,Y} \circ \alpha = \theta \circ \delta_1$ it follows that θ is surjective. □

Remark 3.5 From Theorems 2.16 and 3.4 we can notice that each $\alpha-$special effect variety for special systems in Theorem 1.6 is an h^1-special effect variety too for the same system. However in $\mathbb{P}^n$, $n \geq 3$, this is not true in general, as will be shown in Example 4.5.

4. More examples of special effect varieties in $\mathbb{P}^n$

We collect in this section some special systems arising from the existence of different kind of special effect varieties. In particular we show a variety for the Laface–Ugaglia example ([15]) which is both $\alpha-$special effect and h^1-special effect.

EXAMPLE 4.1 **(Homogeneous special systems in $\mathbb{P}^n$)**: Let Y be a linear space $\mathbb{P}^s \subset \mathbb{P}^n$. It is not difficult to construct a family of homogeneous special systems $\mathcal{L}_{n,d}(m^{s+1})$ with Y as a special effect variety. Again we underline that the study of special effect varieties can turn in a pure combinatorial problem.

As an example we just consider $Y = \mathbb{P}^1 \subset \mathbb{P}^3$, i.e. $s = 1$ and $n = 3$. In this case, we write the special inequality as

$$\begin{aligned}
2\binom{m+2}{3} - \sum_{i=0}^{m-1}\binom{d+1-i}{d-i}\binom{i+1}{i}&\\
&= \frac{m(m+1)(m+2)}{3} - \sum_{i=0}^{m-1}(d+1-i)(i+1)\\
&= \frac{m(m+1)(m+2)}{3} - \sum_{i=0}^{m-1}\Big[(d+1) + di - i^2)\Big]\\
&= \frac{m(m+1)(m+2)}{3} - m(d+1) - \frac{d(m-1)m}{2} + \frac{(m-1)m(2m-1)}{6}\\
&= \frac{m}{6}(4m^2 + 3m - 1 - 3d - 3md) > 0.
\end{aligned}$$

Thus we ask for $4m^2 + 3m - 1 - 3d - 3md > 0$ and we obtain that $\mathbb{P}^1$ is an $m-$special effect variety for $\mathcal{L}_{3,d}(m^2)$ if $m \leq d < \frac{4m-1}{3}$. In a similar way we can prove that $Y = \mathbb{P}^2$ is an $m-$special effect variety for $\mathcal{L}_{3,d}(m^3)$ if

$$m \leq d \leq \frac{m}{2} - 2 + \frac{\sqrt{84 + 108m + 33m^2}}{6}.$$

Example 4.2 **(Rational curves in $\mathbb{P}^3$)**: Let Y be a smooth rational curve in $\mathbb{P}^3$ and define X as the blow-up of $\mathbb{P}^3$ along Y. Thus we have the diagram

$$\begin{array}{ccc}
R & \xrightarrow{j} & X \\
\big\downarrow g & & \big\downarrow \pi \\
Y & \xrightarrow{i} & \mathbb{P}^3
\end{array} \tag{4.1}$$

where $R = \mathbb{P}(\mathcal{N}_{C|\mathbb{P}^3})$ is the exceptional divisor along Y. Let $\tilde{H}$ be the pull-back via π of a general hyperplane section of $\mathbb{P}^3$.

The virtual dimension of $|dH - 2Y|$ can be computed as $\chi(\mathcal{O}_X(d\tilde{H} - 2R))$ on X. Using the generalized Riemann–Roch Theorem ([13] pages 286–295) we obtain

$$\chi(\mathcal{O}_X(D)) = \frac{1}{12}D \cdot (D-K) \cdot (2D - K) + \frac{1}{12}D \cdot c_2 + 1$$

where $K := K_X$. Since $c_1(X) = \pi^*(c_1(\mathbb{P}^3)) - R$ (for the proof, see [14], page 608) we have $K = -4\tilde{H} + R$.

Suppose that Y has degree e. Since $c_2(\mathbb{P}^3) = 6H^2$, from [14] (Lemma at pages 609–610), we obtain $c_2 = (6+e)\tilde{H}^2 - 4\tilde{H} \cdot R$. Thus we can write $\chi(\mathcal{O}_X(d\tilde{H} - 2R))$ as

$$\begin{aligned}
\chi(\mathcal{O}_X(d\tilde{H} - 2R)) &= \frac{1}{12}(d\tilde{H} - 2R) \cdot ((d+4)\tilde{H} - 3R) \cdot ((2d+4)\tilde{H} - 5R)\\
&+ \frac{1}{12}(d\tilde{H} - 2R) \cdot \Big[(6+e)\tilde{H}^2 - 4\tilde{H} \cdot R\Big] + 1.
\end{aligned}$$

We recall that $\tilde{H}^3 = 1$, $\tilde{H}^2 \cdot R = 0$, $\tilde{H} \cdot R^2 = (\tilde{H} \cdot R) \cdot R = eF \cdot R = -e$ and $R^3 = 2 - 4e$ (the last one can be computed by using Proposition at page 606 in [14]). Using these results we obtain

$$\chi(\mathcal{O}_X(d\tilde{H} - 2R)) = \frac{1}{6}d^3 - 5 + d^2 - 3de + \frac{11}{6}d + 4e = \binom{d+3}{3} - 3de + 4e - 5.$$

A rational curve of degree e in $\mathbb{P}^3$ can be defined by four polynomials of degree e. The set of their coefficients defines a projective space of dimension $4(e+1) - 1 - \mathrm{Aut}(\mathbb{P}^1) = 4e + 4 - 1 - 3 = 4e$. Since a simple point imposes two conditions on the equations of the curve, we can use, in a first analysis, the bound $4e \geq 2h$ where h is the number of points.

Obviously we need to check if the h points are in general position. In fact this is not a consequence of the previous bound.

Thus we want Y passing through the h points of $\mathcal{L}_{3,d}(2^h)$. The conditions for the 2−speciality of Y are

$$2e \geq h, \tag{4.2}$$

$$\binom{d+3}{3} - 3de + 4e - 6 \geq 0, \tag{4.3}$$

$$4h > 3de - 4e + 5. \tag{4.4}$$

From (4.2) and (4.4) we obtain $8e \geq 4h > 3de - 4e + 5$, then $(12 - 3d)e > 5$. This forces $d \leq 3$ and we finally find

(a) the line is a 2−special effect curve for $\mathcal{L}_{3,2}(2^2)$,
(b) the conic is a 2−special effect curve for $\mathcal{L}_{3,2}(2^3)$.

Case (a) was already discovered in Proposition 2.13. Case (b) exhibits a new 2−special effect variety for the system $\mathcal{L}_{3,2}(2^3)$: the other one was the plane $\mathbb{P}^2$, by Proposition 2.13. Moreover, since we fix only three points, the conic can move and it fills exactly a $\mathbb{P}^2$. More generally it is possible to prove that the special system $\mathcal{L}_{n,2}(2^h)$, for a fixed h, has at least two special effect varieties: the linear space $\mathbb{P}^{h-1}$ and the rational normal curve $C_{h-1} \subset \mathbb{P}^{h-1}$.

EXAMPLE 4.3 **(Particular unions of lines in $\mathbb{P}^n$)**: Let $Y \subset \mathbb{P}^n$ be the union of the $\binom{n+1}{2}$ lines passing through the $n+1$ coordinate points $P_i = [0, 0, \ldots, 1, \ldots 0]$, for $i = 0, \ldots, n$. Then Y is a $(n-1, \ldots, n-1)$−special effect configuration for $\mathcal{L}_{n,n+1}(n^{n+1})$, $n \geq 3$ (the proof is left to the reader).

EXAMPLE 4.4 **(The Laface–Ugaglia Example)**: In [15] Laface and Ugaglia show a counterexample to a conjecture presented in [7] which requires, for a special system, the existence of a rational curve in the base locus. Laface and Ugaglia analyzed the linear system $\mathcal{L} := \mathcal{L}_{3,9}(-6P_0 - \sum_{i=1}^{8} 4P_i)$. It splits as $Q + \mathcal{L}'$, where $\mathcal{L}' = \mathcal{L}_{3,7}(-5P_0 - \sum_{i=1}^{8} 3P_i)$ and Q is the quadric in $\mathbb{P}^3$ passing through $P_0, \ldots, P_8$. Then $\mathcal{L}$ is special because $\nu(\mathcal{L}) = 3$ while $\nu(\mathcal{L}') = 4$. After that, they proved that the only curve contained in the base locus of $\mathcal{L}$ is a curve $C \subset Q$ of genus 2 given by the intersection of Q with the generic element in $\mathcal{L}'$.

From the previous considerations we see that Q is a $1-$special effect variety for $\mathcal{L}$. As a matter of fact we have

(i) $\nu(|Q|) = 0$,

(ii) $\nu(\mathcal{L} - Q) = 4 > 3 = \nu(\mathcal{L})$.

(iii) $\nu(\mathcal{L} - Q) = \nu(\mathcal{L}_{3,7}(-5P_0 - \sum_{i=1}^{8} 3P_i)) = 4$,

Consider now the restricted system $\mathcal{L}_{|Q} = |9L_1 + 9L_2 - 6P_0 - \sum_{i=1}^{8} 4P_i|$, where L_1, L_2 are the generators of $\mathrm{Pic}(Q)$. $\mathcal{L}_{|Q}$ is empty of virtual dimension -2 (see the Appendix in [15] for the proof). Hence $h^0(\mathcal{L}_{|Q}) = 0$ and working on the blow-up of Q at the P_i's we obtain

$$\begin{aligned} h^1(\mathcal{L}_{|Q}) = h^2(\mathcal{L}_{|Q}) \\ -\frac{(9\tilde{L}_1 + 9\tilde{L}_2 - 6E_0 - \sum_{i=1}^{8} 4E_i)(11\tilde{L}_1 + 11\tilde{L}_2 - 7E_0 - \sum_{i=1}^{8} 5E_i)}{2} - 1 \\ = h^2(\mathcal{L}_{|Q}) + 2 - 1 \geq 1. \end{aligned}$$

Finally $h^0(\mathcal{L} - Q) = h^0(\mathcal{L}_{3,7}(-5P_0 - \sum_{i=1}^{8} 3P_i)) = 4$ and $h^2(\mathcal{L} - Q) = 0$. Then Q verifies the conditions to be an h^1-special effect variety too.

Example 4.5 **(A $1-$special effect variety that is not an h^1-special effect variety)**: Consider the system $\mathcal{L} := \mathcal{L}_{3,d}(m^3)$ and let $Y \subset \mathbb{P}^3$ be the plane through the three points in $\mathcal{L}$. Writing down the conditions of speciality we see that Y is an $1-$special effect variety for $\mathcal{L}$ if

$$\begin{cases} \dfrac{-3 + \sqrt{1 + 12m^2 + 12m}}{2} > d \\ d^3 + 3d^2 + 2d - 6 \geq 3m^3 - 3m. \end{cases}$$

Consider now the system $\mathcal{L} := \mathcal{L}_{3,6}(4^3)$. For the previous computation, $Y = \mathbb{P}^2$ is a $1-$special effect variety for $\mathcal{L}$ and $\mathcal{L}$ is special. One has $\nu(\mathcal{L}) = 23$ and $\nu(\mathcal{L} - Y) = 25$ as we expect by the special effect of Y. Moreover we can observe that Y is not a $2-$special effect variety since $\nu(\mathcal{L} - 2Y) = 22$.

If we restrict the system to Y we obtain the planar system $\mathcal{L}_{|Y} = \mathcal{L}_{2,6}(4^3)$. This system is special, so both $h^0(\mathcal{L}_{|Y})$ and $h^1(\mathcal{L}_{|Y})$ are different from zero. Hence $Y = \mathbb{P}^2$ does not satisfy condition (a) to be an h^1-special effect variety for $\mathcal{L}$.

However there is an h^1-special effect variety for the system $\mathcal{L}$. As a matter of fact, if we compute the effective dimension of the system $\mathcal{L}$ by a computer algebra program (e.g Maple) we discover $\dim(\mathcal{L}) = 26$ then $\mathcal{L} - Y$ represents a subsystem of the system of divisors of degree 6 with three points of multiplicity 4 (i.e $\mathcal{L}$ does not split as $Y + \mathcal{L}_{3,5}(3^3)$). Hence the generic element $D \in \mathcal{L}$ cannot be written as the sum of Y and of elements in $\mathcal{L}_{3,5}(3^3)$. Suppose the points P_i's are $P_1 := [0, 1, 0, 0]$, $P_2 := [0, 0, 1, 0]$ and $P_3 := [0, 0, 0, 1]$ and the coordinates are $x_0, \ldots x_3$. Then Y is the plane defined by $x_0 = 0$. Moreover $\mathcal{L}$ is generated by the span of $< Y + \mathcal{L}_{3,5}(3^3), F >$ where F is $(x_1x_2x_3)^2$ (as we expect from Theorem 2.4 in [9]).

We can observe that F contains twice the lines $L_{ij} := \overline{P_iP_j}$, $i, j = 1, 2, 3$ and $i \neq j$. Moreover, every element in $Y + \mathcal{L}_{3,5}(3^3)$ contains the same lines with multiplicity at least 2. Thus $Y' = L_{12} + L_{13} + L_{23}$ is a $(2, 2, 2)-$special effect configuration for $\mathcal{L}$.

Finally it is easy to check that if we just consider one of the previous lines L_{ij} we obtain that L_{ij} is an h^1-special effect variety for both $\mathcal{L}$ and $\mathcal{L}_{|Y}$.

5. $\alpha-$Special effect varieties in the product of projective spaces

We show now several examples of $\alpha-$special effect varieties on $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ with $t \geq 2$ and $n_i \geq 1$ for $i = 1, \ldots, t$. We treat only the case case $m = \alpha = 2$ and we suppose that the special effect variety Y is a divisor on X. Surely this does not exhaust all possible special effect varieties (and special linear systems) on X, but we will observe at the end of the Section how our results fit with the ones of Catalisano, Geramita and Gimigliano on secant varieties of products of projective spaces ([4], [5], [6]).

Notation. Let r be a positive integer. For any integer z we define $(r)_{(z)}$ as follows

$$(r)_{(z)} := \begin{cases} \Pi_{i=1}^{z}(r+i) & \text{if } z > 0 \\ 1 & \text{if } z = 0 \\ 0 & \text{if } z < 0. \end{cases}$$

We have the following fact: let r, s and t be positive integers, one has the equality

$$(r+s)_{(t)} = (s)_{(t)} + r\left(\sum_{i=1}^{t}(s)_{(i-1)}(r+s+i)_{(t-i)}\right). \tag{5.1}$$

Since each term $(s)_{(i-1)}(r+s+i)_{(t-i)}$ in the summation is greater than $(s)_{(t-1)}$ we can write the following inequality

$$(r+s)_{(t)} \geq (s)_{(t)} + rt\left((s)_{(t-1)}\right) = (s)_{(t-1)}(s+t+rt). \tag{5.2}$$

Let Y be a divisor of multidegree $(e_1, \ldots .e_t)$ on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ and consider the system $\mathcal{L} := \mathcal{L}_{(d_1,\ldots,d_t)}(2^h)$ of divisors of multidegree $(d_1, \ldots, d_t)$ passing through h general double points. We require that Y passes through the h points of $\mathcal{L}$. Then Y is a $2-$special effect varieties for $\mathcal{L}$ if

$$\Pi_{i=1}^{t}\binom{e_i+n_i}{n_i} - 1 \geq h; \tag{5.3}$$

$$\Pi_{i=1}^{t}\binom{d_i-2e_i+n_i}{n_i} > \Pi_{i=1}^{t}\binom{d_i+n_i}{n_i} - h\left(\sum_{i=1}^{t} n_i + 1\right) \tag{5.4}$$

$$\Pi_{i=1}^{t}\binom{d_i-2e_i+n_i}{n_i} \geq 1, \text{ i.e. } d_i \geq 2e_i, \text{ for } i = 1, \ldots, t. \tag{5.5}$$

We apply the same argument of Proposition 2.8. Again, the previous conditions give us the bounds on the number of points h:

$$\Pi_{i=1}^{t}\binom{e_i+n_i}{n_i} - 1 \geq h > \frac{1}{\sum_{i=1}^{t} n_i + 1}\left[\Pi_{i=1}^{t}\binom{d_i+n_i}{n_i} - \Pi_{i=i}^{t}\binom{d_i-2e_i+n_i}{n_i}\right]$$

so that we can study when the function

$$(5.6)\quad \varphi(d_1,\ldots,d_t,e_1,\ldots,e_t,n_1,\ldots,n_t) := \Pi_{i=1}^t \binom{d_i+n_i}{n_i} - \Pi_{i=1}^t \binom{d_i-2e_i+n_i}{n_i} - \left(\Pi_{i=1}^t \binom{e_i+n_1}{n_i} - 1\right)\left(\sum_{i=1}^t n_i + 1\right)$$

is negative.

Lemma 5.1 (Numerical Lemma)

Let $\varphi(d_1,\ldots,d_t,e_1,\ldots,e_t,n_1,\ldots,n_t)$ be defined as in (5.6). Then the function

$$\eta(e_1,\ldots,e_t,n_1,\ldots,n_t) := \varphi(2e_1,\ldots,2e_t,e_1,\ldots,e_t,n_1,\ldots,n_t)$$

is non-decreasing in the n_i's for

a) $t=2$ with $e_i \geq 2$ and $n_i \geq 2$, $i=1,2$.
b) $t \geq 3$ with $e_i \geq 1$ and $n_i \geq 1$, $i=1,\ldots,t$;

Proof. By definition of φ, we have

$$\begin{aligned}\eta &:= \varphi(2e_1,\ldots,2e_t,e_1,\ldots,e_t,n_1,\ldots,n_t) \\ &= \Pi_{i=1}^t \binom{2e_i+n_i}{n_i} - \left(\Pi_{i=1}^t \binom{e_i+n_1}{n_i} - 1\right)\left(\sum_{i=1}^t n_i + 1\right) - 1.\end{aligned}$$

For the symmetry of η, it is enough to prove the lemma for one n_{i_o}, with $i_o \in \{1,\ldots,t\}$.

$\boxed{\text{case } (a)}$ We prove that $\varphi(2e_1,2e_2,e_1,e_2,n_1,n_2)$ is increasing in n_1. Thus, after a tedious computation, one has

$$\begin{aligned}\gamma_{n_1} &:= \varphi(2e_1,2e_2,e_1,e_2,n_1+1,n_2) - \varphi(2e_1,2e_2,e_1,e_2,n_1,n_2) \\ &= P \cdot \Big[(n_1+e_1)_{(e_1)}(n_2+e_2)_{(e_2)} \\ &\quad - (e_1)_{(e_1-1)}(e_2)_{(e_2)}\left((n_1+e_1+1)(n_1+n_2+2) - (n_1+1)(n_1+n_2+1)\right)\Big] + 1 \\ &> P \cdot \left((e_1)_{(e_1-1)}(e_2)_{(e_2)}\right)[e_1e_2n_1n_2 - 2e_2n_1 - 2e_2] + 1 \geq 0\end{aligned}$$

where

$$P := \frac{(n_1+1)_{(e_1-1))}(n_2)_{(e_2)}}{(2e_2)!(2e_1-1)!}$$

and, for the inequality we use (5.2) with $r=n_i$, $s=t=e_i$, $i=1,2$.

Thus φ is increasing in n_i and (a) follows.

$\boxed{\text{case } (b)}$ As in case (a) we look for a good way to collect terms in

$$\gamma_{n_{i_0}} := \eta(e_1,\ldots,e_t,\ldots,n_{i_0}+1,\ldots,n_t) - \eta(e_1,\ldots,e_t,\ldots,n_{i_0},\ldots,n_t).$$

Using again (5.2) one has

$$
\begin{aligned}
\gamma_{n_{i_0}} &> \tilde{P} \cdot \left[\frac{2e_{i_0}\Pi_{s=1}^{t}(n_s+e_s)_{(e_s)}}{\left(\Pi_{s=1}^{t}(e_s)_{(e_s)}\right)(n_{i_0}+e_{i_0}+1)} - \left(\sum_{i=1}^{t} n_i + 2\right)\right] + 1 \\
&= \tilde{P} \cdot [C(e_i)] + 1
\end{aligned} \tag{5.7}
$$

where

$$
P := \frac{\Pi_{s=1}^{t}(n_s)_{(e_s)}}{(n_{i_0}+1)(\Pi_{k=1}^{t}(e_k!))} \quad \text{and} \quad \tilde{P} := (n_{i_0}+e_{i_0}+1)\cdot P.
$$

Now we can apply the same argument of claim of Lemma 2.9 and we obtain that the term $C(e_i)$ is increasing in e_i and $\gamma_{n_{i_0}}$ too.

If we substitute $e_1 = \cdots = e_t = 1$ in $\gamma_{n_{i_0}}$ we obtain

$$
\gamma^*_{n_{i_0}} = \frac{P(e_i=1)}{2^{t-1}} \cdot \left[\sum_{m=1}^{t} \left(2^{t-m} \sum_{|I|=m} n_{|I|} \right) - 2^{t-1} n_{i_0} - 2^{t-1} \right] + 1 \tag{5.8}
$$

where $n_{|I|} = n_{i_1} \cdots n_{i_m}$ if $I = \{i_1, \ldots, i_m\}$.

If $t \geq 4$ we have at least two terms of the form $2^{t-2} n_j n_{i_0}$ and four terms of the form $2^{t-3} n_j n_k n_l$. Then we have

$$
2^{t-2} \sum n_j n_{i_0} \geq 2^{t-1} n_{i_0}
$$

and

$$
2^{t-3} \sum n_j n_k n_l \geq 2^{t-1}.
$$

Thus the expression between square brackets in (5.8) is always positive and then η is increasing on n_{i_0}.

When $t = 3$, we obtain, for example for $i_0 = 1$,

$$
\gamma^*_{n_1} = \frac{P}{4} \left[n_1n_2n_3 + 2n_1n_2 + 2n_1n_3 + 2n_2n_3 - 4n_1 - 4 \right] + 1
$$

and the expression between square brackets is positive except for $n_1 = n_2 = n_3 = 1$, but, for these values we have

$$
\gamma_{n_1} = \frac{P}{4} \left[n_1n_2n_3 + 2n_1n_2 + 2n_1n_3 + 2n_2n_3 - 4n_1 - 4 \right]_{|n_i=1} + 1 = \frac{4}{4}[-1] + 1 = 0
$$

then, also for the case $t = 3$, η is non-decreasing in n_i, $i = 1, 2, 3$. Hence (b) is proved. □

Proposition 5.2

Let $\varphi(d_1, \ldots, d_t, e_1, \ldots, e_t, n_1, \ldots, n_t)$ be defined as in (5.6). Then

(a) *If $t = 2$ then $\varphi \geq 0$ for $e_1, e_2 \geq 2$, for $n_1, n_2 \geq 2$ and $d_i \geq 2e_i$, $i = 1, 2$.*

(b) *If $t = 3$ then $\varphi(d_1, d_2, d_3, e_1, e_2, e_3, n_1, n_2, n_3) \geq 0$, except for $(d_1, d_2, d_3) = (2, 2, 2)$, $(e_1, e_2, e_3) = (1, 1, 1)$ and $(n_1, n_2, n_3) = (1, 1, \gamma)$ with $\gamma \leq 3$.*

(c) *If $t \geq 4$ then $\varphi(d_1, \ldots, d_t, e_1, \ldots, e_t, n_1, \ldots, n_t) \geq 0$, for $e_i, n_i \geq 1$ and $d_i \geq 2e_i$, $i = 1, \ldots t$.*

Proof. Since the function φ is non-decreasing in d_i we can start from the value $d_i = 2e_i$, $i = 1, \dots, t$. Then, using the previous lemma it is enough to substitute the minimal values of n_i, $i = 1, \dots, t$ in η and then study the positivity of this easier function.

As an example, we prove (c). In this case, after the substitution of $n_i = 1$ in η we obtain

$$\varphi = \Pi_{i=1}^t (2e_i + 1) - (t+1)\Pi_{i=1}^t (e_i + 1) + t.$$

The previous expression is increasing in e_i, $i = 1, \dots t$ and, if we finally substitute $e_1 = \cdots = e_t = 1$ we obtain

$$\varphi = 3^t - 2^t(t+1) + t$$

and it is positive for $t \geq 4$. Hence (c) is proved. □

We now search for 2−special effect divisors on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$. We analyze first the case $t = 2$.

Proposition 5.3

Let $\mathcal{L} := \mathcal{L}_{(d_1,d_2)}(2^h)$ be a linear system of bidegree (d_1, d_2) on $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ passing through h double points in general position, with $d_1 \cdot d_2 \neq 0$. Let $Y \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ be a divisor of bidegree (e_1, e_2), with $e_i \neq 0$ for at least one i. Moreover we require that Y passes simply through the h points in $\mathcal{L}$. Then Y is a 2−special effect variety for $\mathcal{L}_{(d_1,d_2)}(2^h)$ in the following cases

$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$	(d_1, d_2)	(e_1, e_2)	h
$\mathbb{P}^1 \times \mathbb{P}^1$	$(2, 2e_2)$	$(1, e_2)$	$2e_2 + 1$
$\mathbb{P}^1 \times \mathbb{P}^1$	$(2e_1, 2)$	$(e_1, 1)$	$2e_1 + 1$
$\mathbb{P}^1 \times \mathbb{P}^{n_2}$	$(2e_1, 2)$	$(e_1, 1)$	$m_1(e_1, n_2) \leq h \leq M_1(e_1, n_2)$
$\mathbb{P}^2 \times \mathbb{P}^{n_2}$	$(2, 2)$	$(1, 1)$	$m_2(n_2) \leq h \leq M_2(n_2)$
$\mathbb{P}^3 \times \mathbb{P}^3$	$(2, 2)$	$(1, 1)$	15
$\mathbb{P}^3 \times \mathbb{P}^4$	$(2, 2)$	$(1, 1)$	19

where

$$m_1(e_1, n_2) := \lfloor \tfrac{(2e_1+1)(n_2+1)}{2} \rfloor \quad m_2(n_2) := \lfloor \tfrac{3n_2^2 + 9n_2 + 5}{n_2 + 3} \rfloor$$

$$M_1(e_1, n_2) := e_1 n_2 + e_1 + n_2 \quad M_2(n_2) := 3n_2 + 2.$$

Proof. We start with the case $e_1 \cdot e_2 \neq 0$. From Proposition 5.2 we can restrict our analysis to $e_1 + e_2 \leq 3$ or when at least one between n_1 and n_2 is equal to one. We divide the proof in three steps, analyzing some different situations.

Step 1: $n_1 = n_2 = 1$.

$$\varphi(2e_1, 2e_2, e_1, e_2, 1, 1) = e_1 e_2 - e_1 - e_2\,. \tag{5.9}$$

Obviously, (5.9) is negative only for

$$\begin{cases} e_1 = 1 \\ e_2 \geq 1 \end{cases} \quad \text{or} \quad \begin{cases} e_1 \geq 1 \\ e_2 = 1\,. \end{cases}$$

For the symmetry of the function, we can fix our attention on the case $e_1 = 1$. The only possible value for (d_1, d_2) is $(2, 2e_2)$; as a matter of fact, if $(d_1, d_2) = (2+i, 2e_2+j)$ with $i, j \geq 1$, we have

$$\varphi(2+i, 2e_2+j, 1, e_2, 1, 1) = 2e_2 i + 2j - 1 > 0 \qquad \forall i, j \geq 1.$$

Thus we obtain the first two cases of the list.

Step 2: $n_1 = 1$, $n_2 \geq 2$.

Since the case $e_1 = e_2 = 1$ will be studied in the next step in a more general context, we start with the case $e_1 \geq 1$, $e_2 = 1$. One has

$$\varphi(2e_2, 2, e_1, 1, 1, n_2) = -\frac{1}{2}n_2^2 - \frac{1}{2}n_2 < 0 \qquad \forall n_2, e_1.$$

Moreover one has $\varphi(d_1, d_2, e_1, e_2, 1, n_2) > 0$ when $d_i > 2e_i$, $i = 1, 2$ or when $e_2 \geq 2$. Finally, the case $e_1, e_2 \geq 2$ is already studied in Proposition 5.2, case (a). Thus the only possibility for $n_1 = 1$ and $n_2 \geq 2$ is $e_1 \geq 1$ and $e_2 = 1$ and the number of points h is given by

$$(e_1+1)(n_2+1) - 1 \geq h \geq \frac{1}{n_2+2}\left[\frac{(2e_1+1)(n_2+1)(n_2+2)}{2} - 1\right]$$

and we obtain the third case of the list.

Step 3: $e_1 = e_2 = 1$.

In this case we write

$$\varphi(2, 2, 1, 1, n_1, n_2) = \frac{(n_1^2 n_2^2 - n_1 n_2^2 - n_1^2 n_2 - 2n_1^2 - 2n_2^2 - 3n_1 n_2 + 2n_1 + 2n_2)}{4}.$$

Since we can suppose $n_2 \geq n_1 \geq 2$ we have that $\varphi(2, 2, 1, 1, n_1, n_2)$ is negative for

n_1	n_2
2	*any*
3	$3, 4$.

It is easy to see that, if d_1 or d_2 are strictly greater than 2, we do not have special effect varieties; as a matter of fact

$$\begin{aligned}\varphi(d_1, d_2, 1, 1, n_1, n_2) &> \varphi(3, 2, 1, 1, n_1, n_2) \\ &= \frac{(n_1+1)(n_2+1)}{12}\left[n_1^2 n_2 + 2n_1^2 + 5n_1 n_2 - 2n_1 - 6n_2\right] + n_2\end{aligned}$$

and the last term is positive for the previous values of n_1 and n_2. Thus we can conclude that Y is a $2-$special effect variety in the following cases

$\mathbb{P}^2 \times \mathbb{P}^{n_2}$	$(2, 2)$	$(1, 1)$	$m_2(n_2) < h \leq M_2(n_2)$
$\mathbb{P}^3 \times \mathbb{P}^3$	$(2, 2)$	$(1, 1)$	15
$\mathbb{P}^3 \times \mathbb{P}^4$	$(2, 2)$	$(1, 1)$	19

where m_1, m_2, M_1 and M_2 are defined by (5), i.e.

$$m_2(n_2) := \frac{3n_2^2 + 9n_2 + 5}{n_2 + 3} \qquad M_2(n_2) := 3n_2 + 2.$$

Finally, the non-existence of $2-$special effect varieties of bidegree (e_1, e_2), with $e_1 + e_2 \geq 3$ in $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, $n_1, n_2 \geq 2$ is a consequence of Proposition 5.2–(a) and the following claim.

Claim: $\varphi \geq 0$, for $e_1 = 1$, $e_2 \geq 2$ (resp. for $e_1 \geq 2$, $e_2 = 1$) for $n_1, n_2 \geq 2$ and $d_i \geq 2e_i$, $i = 1, 2$.

In fact, one has

$$\begin{aligned}\varphi(2, 2e_2, 1, e_2, n_1, n_2) &= \frac{(n_1+1)(n_1+2)}{2}\frac{(n_2)_{(2e_2)}}{(2e_2)!} + n_1 + n_2 \\ &\quad - \frac{(n_1+1)(n_2)_{(2e_2)}(n_1+n_2+1)}{e_2!} \\ &= \frac{(n_1+1)(n_2)_{(e_2)}}{e_2!} C(e_2) + n_1 + n_2\end{aligned}$$

where

$$C(e_2) = \frac{(n_1+2)(n_2+e_2)_{(e_2)}}{2(e_2)_{(e_2)}} - (n_1 + n_2 + 1).$$

Then the proof uses the same argument of the claim in Lemma 2.9, verifying, at the end, that $C(2) \geq 0$ for $n_1, n_2 \geq 2$.

We analyze now the case $e_1 \cdot e_2 = 0$. By symmetry, it is enough to treat the case $e_2 = 0$ (we recall that $d_1 \cdot d_2 \neq 0$). In this situation we have

$$\begin{aligned}\varphi(d_1, d_2, e_1, 0, n_1, n_2) :=& \binom{d_1+n_1}{n_1}\binom{d_2+n_2}{n_2} \\ & - \binom{d_1-2e_1+n_1}{n_1}\binom{d_2+n_2}{n_2} \\ & - \left[\binom{e_1+n_1}{n_1} - 1\right](n_1+n_2+1).\end{aligned} \tag{5.10}$$

Since the previous function is non-decreasing in d_1 and d_2 we can start from the minimal degree $(d_1, d_2) = (2e_1, 1)$ and we obtain

$$\varphi(2e_1, 1, e_1, 0, n_1, n_2) := \binom{2e_1+n_1}{n_1}(n_2+1) - \left[\binom{e_1+n_1}{n_1}\right](n_1+n_2+1) + n_1.$$

This function is clearly increasing in n_2. For the behaviour of $\varphi(2e_1, 1, e_1, 0, n_1, n_2)$ by respect to n_1 we can write

$$\varphi(2e_1, 1, e_1, 0, n_1, n_2) = A(n_1, n_2, e_1) \cdot B(n_1, n_2, e_1) + n_1$$

where

$$A(n_1, n_2, e_1) = \frac{(n_1)_{(e_1)}}{(2e_1)!}$$

and

$$B(n_1, n_2, e_1) = (n_2 + 1)(n_1 + e_1)_{(e_1)} - (n_1 + n_2 + 1)(e_1)_{(e_1)}.$$

Both $A(n_1, n_2, e_1)$ and $B(n_1, n_2, e_1)$ are increasing in n_1. Moreover $A(n_1, n_2, e_1) \geq 0$ for $n_1, n_2, e_1 \geq 1$ and, by a simple computation, one has

$$\begin{aligned} B(1, n_2, e_1) &= (n_2 + 1)(n_1 + e_1)_{(e_1)} - (n_1 + n_2 + 1)(e_1)_{(e_1)} \\ &= (e_1 + 1)_{(e_1-1)}(n_2 e_1 - 1) \geq 0 \text{ for } n_2, e_1 \geq 1\,. \end{aligned}$$

Hence $\varphi(2e_1, 1, e_1, 0, n_1, n_2)$ is non-decreasing in n_1 too and we can study it starting from $n_1 = n_2 = 1$. One has

$$\varphi(2e_1, 1, e_1, 0, 1, 1) = e_1 > 0 \quad \forall e_1 \geq 1.$$

Thus Y is not a $2-$special effect variety for $\mathcal{L}_{(d_1,d_2)}(2^h)$ if Y has bidegree (e_1, e_2), with $e_1 \cdot e_2 = 0$ and $d_1 \cdot d_2 \neq 0$. □

Let Q be the quadric in $\mathbb{P}^3$ and consider L_1 and L_2 the generators of $\mathrm{Pic}(Q)$. Denote by $\mathcal{L}(a, b)$ the linear system $|aL_1 + bL_2|$.

Corollary 5.4

A curve of type $(n, 1)$ (resp. of type $(1, n)$) on a quadric $Q \subset \mathbb{P}^3$ is a $2-$special effect variety on Q for $\mathcal{L}(2n, 2)(2^{2n+1})$ (resp. for $\mathcal{L}(2, 2n)(2^{2n+1})$).

Proof. It follows directly from the first two cases of Proposition 5.3. □

We pass now to analyze the case in which $t \geq 3$; we restrict our studying to the case $e_i \neq 0$, $\forall i = 1, \ldots, t$ and $n_t \geq n_{t-1} \geq \cdots \geq n_1$.

Proposition 5.5

Let $t \geq 3$. Let $\mathcal{L} := \mathcal{L}_{(d_1,\ldots d_t)}(2^h)$ be a linear system of multidegree $(d_1, \ldots, d_t)$, with $d_i \neq 0$ for $i = 1, \ldots, t$, on $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ passing through h double points in general position and let Y be a divisor of multidegree $(e_1, \ldots, e_t)$ on X with $e_i \neq 0$ for $i = 1, \ldots, t$. Moreover we require that Y passes simply through the h points in $\mathcal{L}$. Then Y is a $2-$special effect variety on X for $\mathcal{L}$ only if $t = 3$ and for the following values:

$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$	(d_1, d_2, d_3)	(e_1, e_2, e_3)	h
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(2, 2, 2)$	$(1, 1, 1)$	7
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	$(2, 2, 2)$	$(1, 1, 1)$	11
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$	$(2, 2, 2)$	$(1, 1, 1)$	15

Proof. The result follows immediately from cases (b) and (c) of Proposition 5.2. □

5.1 $\alpha-$Special effect varieties and Segre–Veronese varieties

It is known from the literature that the speciality of a linear system with imposed double points can be phrased in term of defectivity of certain varieties. The reader can find more topics on these subjects, for example, in [7], [17] and [19].

Catalisano, Geramita and Gimigliano, in [4], [5] and [6], study the secant varieties of Segre–Veronese varieties, i.e. the image of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ under the composition of the Veronese embeddings $\nu_{a_1} \times \cdots \times \nu_{a_t}$ followed by the Segre embedding ρ_s:

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \xrightarrow{\nu_{a_1} \times \cdots \times \nu_{a_t}} \mathbb{P}^{\binom{a_1+n_1}{n_1}-1} \times \cdots \times \mathbb{P}^{\binom{a_t+n_t}{n_t}-1} \xrightarrow{\rho_s} \mathbb{P}^N.$$

Their results on defective Segre–Veronese varieties, or equivalently on special linear systems on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ can be compared with our results on $\alpha-$special effect varieties.

A first result we mention is the following.

Theorem 5.6 (Theorem 2.1 in [6])

Let $\mathcal{L} := \mathcal{L}_{a_1,a_2}(2^h)$ be the linear system in $\mathbb{P}^1 \times \mathbb{P}^1$ of divisors of bidegree (a_1, a_2) with h imposed double points. Then $\mathcal{L}$ is non-special unless

$$a_1 = 2d, a_2 = 2, d \geq 1, \text{ and } h = 2d + 1.$$

Using the first two cases of Proposition 5.3 or Corollary 5.4 we obtain immediately the following.

Theorem 5.7

The Numerical Conjecture holds for each of the special systems listed in Theorem 5.6.

The second result we mention is related to the study of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 5.8 (Theorem 2.5 in [6])

Let $a_1 \geq a_2 \geq a_3 \geq 1$, $\rho \in \mathbb{N}$. Let $\mathcal{L} := \mathcal{L}_{a_1,a_2,a_3}(2^h)$ be the linear system in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of divisors of multidegree (a_1, a_2, a_3) with h imposed double points. Then $\mathcal{L}$ is non-special unless

$$(a_1, a_2, a_3) = (2\alpha, 1, 1) \text{ and } \ h = 2\alpha + 1.$$

$$(a_1, a_2, a_3) = (2, 2, 2) \text{ and } h = 7;$$

Once again we can try to check if there are special effect varieties for the special systems corresponding to the defective varieties listed before. It is easy to observe that, by numerical reasons, the second case cannot be treated with a $2-$special effect variety. However, using special effect configurations we can state a result as Theorem 5.7.

Theorem 5.9

The Numerical Conjecture holds for each of the special systems listed in Theorem 5.8.

Proof. For the case $(a_1, a_2, a_3) = (2, 2, 2)$, $h = 7$ there is a $2-$special effect variety as showed in Proposition 5.5, for $t = 3$, $n_1 = n_2 = n_3 = 1$ and $d_i = a_i$, $i = 1, 2, 3$. Let $\mathcal{L}$ be the special linear system $\mathcal{L}_{(2\alpha,1,1)}(2^{2\alpha+1})$ Let Y_1 [resp. Y_2] be a divisor corresponding to the system $\mathcal{L}_{(\alpha,0,1)}(1^{2\alpha+1})$ [resp. $\mathcal{L}_{(\alpha,1,0)}(1^{2\alpha+1})$]. We easily compute

$$\nu(\mathcal{L}) = -1$$
$$\nu(Y_1) = \nu(Y_2) = 0$$
$$\nu(\mathcal{L} - Y_1) = \nu(\mathcal{L}_{(\alpha,1,0)}(1^{2\alpha+1})) = 0$$
$$\nu(\mathcal{L} - Y_2) = \nu(\mathcal{L}_{(\alpha,0,1)}(1^{2\alpha+1})) = 0.$$

Then $Y + Y'$ is a $(1,1)-$special effect configuration for $\mathcal{L}$. $\square$

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