Collectanea Mathematica (electronic version): http://www.imub.ub.es/collect

Collect. Math. **56**, 3 (2005), 275–297

Rough Marcinkiewicz integral operators on product spaces

H.M. Al-Qassem

Department of Mathematics, Yarmouk University, Irbid-Jordan

E-mail: husseink@yu.edu.jo

Received May 4, 2005

Abstract

In this paper, we study the Marcinkiewicz integral operators $\mathcal{M}_{\Omega,h}$ on the product space $\mathbb{R}^n\times\mathbb{R}^m$. We prove that $\mathcal{M}_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ $(1<p<\infty)$ provided that h is a bounded radial function and Ω is a function in certain block space $B_q^{(0,0)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ for some $q>1$. We also establish the optimality of our condition in the sense that the space $B_q^{(0,0)}$ $(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ cannot be replaced by $B_q^{(0,r)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ for any $-1<r<0$. Our results improve some known results.

1. Introduction

It is well-known that the theory of Marcinkiewicz integral operators is very useful in harmonic analysis. We refer the readers to [2], [5], [6], [7], [9], [10], [11], [12], [15], [21], [22], [24], [25] for the applications and the recent advances of this theory.

Our main focus in this paper will be on studying the L^p boundedness of Marcinkiewicz integral operators on the product space $\mathbb{R}^n\times\mathbb{R}^m$.

Suppose that $\mathbb{S}^{d-1}$ ($d=n$ or m) is the unit sphere of $\mathbb{R}^d$ $(d\geq 2)$ equipped with the normalized Lebesgue measure $d\sigma=d\sigma(x')$ which is normalized so that $\sigma(\mathbb{S}^{d-1})=1$. Let $h(t,s)$ be a locally integrable function on $\mathbb{R}_+\times\mathbb{R}_+$ and Ω be an integrable function on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$ and satisfies the following cancellation conditions:

$$\begin{cases} \int_{\mathbb{S}^{n-1}}\Omega(u,\cdot)\,d\sigma(u)=0, \\ \int_{\mathbb{S}^{m-1}}\Omega(\cdot,v)\,d\sigma(v)=0. \end{cases} \tag{1.1}$$

Keywords: Marcinkiewicz integrals; product domains; rough kernels; block spaces.
MSC2000: Primary 42B20; Secondary 42B15, 42B25.

The Marcinkiewicz integral operator on the product space $\mathbb{R}^n \times \mathbb{R}^m$ is defined by

$$\mathcal{M}_{\Omega,h}f(x,y) = \left(\int_{(0,\infty)\times(0,\infty)} |Y_{t,s}f(x,y)|^2 \frac{dtds}{(ts)^3} \right)^{1/2}, \tag{1.2}$$

where $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ and

$$Y_{t,s}f(x,y) = \int_{\{|u|\le t, |v|\le s\}} \frac{\Omega(u,v)h(|u|,|v|)}{|u|^{n-1}|v|^{m-1}} f(x-u, y-v)dudv.$$

For the sake of simplicity, we denote $\mathcal{M}_{\Omega,h} = \mathcal{M}_\Omega$ if $h(t,s) \equiv 1$.

The study of the L^p boundedness of $\mathcal{M}_{\Omega,h}$ has attracted the attention of many authors in recent years. In the one parameter setting (the non-product case), the operator $\mathcal{M}_\Omega$ was introduced by E. Stein in [21] as an extension of the notion of Marcinkiewicz function from one dimension to higher dimensions. Stein showed that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,2]$ if $\Omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{n-1})$ $(0 < \alpha \le 1)$. Subsequently, A. Benedek, A. Calderón, and R. Panzone proved that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$ if $\Omega \in C^1(\mathbb{S}^{n-1})$ (see [5]). Later on, the case of rough kernels (Ω satisfies only a size condition and a cancellation condition but no regularity is assumed) became the interest of many authors. For a sampling of past studies, see ([1], [2], [11], [12], [20], [24]). On the other hand, the investigations of the L^p boundedness of $\mathcal{M}_{\Omega,h}$ in the two parameter setting (the product case) began by Y. Ding and subsequently by many authors (see [7], [8], [9]). For example, by Fourier transform estimates, Ding [9] proved the $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ boundedness of $\mathcal{M}_{\Omega,h}$ when $h \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and the kernel function Ω belongs to $L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Subsequently, Chen-Ding-Fan proved that $\mathcal{M}_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ $(1 < p < \infty)$ provided that $h \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and Ω satisfies the stronger condition $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ $(q > 1)$ [7].

Results cited above naturally lead us to the following:

Problem. *Determine whether the L^p boundedness of the operator $\mathcal{M}_{\Omega,h}$ holds under a condition in the form of $\Omega \in B_q^{(0,\upsilon)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for $\upsilon > -1$, and, if so, what is the best possible value of υ.*

Here, $B_q^{(0,\upsilon)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ (for $\upsilon > -1$ and $q > 1$) represents a block space and its definition will be recalled in Section 2. We point out that $B_q^{(0,\upsilon)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ was introduced in [16] and can be traced back to [18] and [19]. Also, it is known that the following inclusions hold and are proper:

$$B_q^{(0,\upsilon_2)}\left(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}\right) \subset B_q^{(0,\upsilon_1)}\left(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}\right) \text{ for } -1 < \upsilon_1 < \upsilon_2 \tag{1.3}$$

$$\bigcup_{r>1} L^r(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subset B_q^{(0,\upsilon)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \tag{1.4}$$

for any $q > 1$ and $\upsilon > -1$.

The main focus of this paper is to obtain a solution to the above problem. Our main result in this paper can be stated as follows:

Theorem 1.1

Let Ω and $\mathcal{M}_{\Omega,h}$ be given as above. Then

(a) *If h is a bounded function and $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, $q > 1$, $\mathcal{M}_{\Omega,h}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$;*

(b) *There exists an Ω which belongs to $B_q^{(0,\upsilon)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for all $-1 < \upsilon < 0$ and satisfies (1.1) such that $\mathcal{M}_\Omega$ is not bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$.*

Remarks.

(1) We point out that the relationship between the spaces $B_q^{(0,\alpha-1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and $L(\log^+ L)^\alpha(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ (for $\alpha > 0$) remains open. One observes that Theorem 1.1 represents an improvement over the main result in [7] because Ω is allowed to be in the space $B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and bearing in mind the relation (1.4) remarked above. Also, it is worth mentioning that Theorem 1.1 (b) shows that the condition $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is nearly optimal.

(2) The complexity of the proof of Theorem 1.1 (b) is much more than we had anticipated. This is due in part to difficulties arising in product domain settings and the complexity of the multiplier of the Marcinkiewicz integral operator.

Throughout the rest of the paper, we always use the letter C to denote a positive constant that may vary at each occurrence, but it is independent of the essential variables.

2. Certain block spaces on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$

The block spaces originated in the work of M.H. Taibleson and G. Weiss on the convergence of the Fourier series (see [23]) in connection with the developments of the real Hardy spaces. Below we shall recall the definition of block spaces on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$. For further background information about the theory of spaces generated by blocks and its applications to harmonic analysis one can consult the book [18].

The special class of block spaces $B_q^{(0,\upsilon)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ (for $\upsilon > -1$ and $q > 1$) was introduced by Jiang and Lu with respect to the study of singular integral operators on product domains [16].

Definition 2.1 A q-block on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ is an L^q ($1 < q \le \infty$) function $b(x, y)$ that satisfies

(i) supp(b)$\subset I$;

(ii) $\|b\|_{L^q} \le |I|^{-1/q'}$,

where $1/q + 1/q' = 1$, $|\cdot|$ denotes the product measure on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$, and I is an interval on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$, *i.e.*,

$$I = \left\{x' \in \mathbb{S}^{n-1} : |x' - x_0'| < \alpha\right\} \times \left\{y' \in \mathbb{S}^{m-1} : |y' - y_0'| < \beta\right\}$$

for some α, $\beta > 0$, $x_0' \in \mathbb{S}^{n-1}$ and $y_0' \in \mathbb{S}^{m-1}$.

Definition 2.2 The block space $B_q^{(0,v)} = B_q^{(0,v)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ is defined by

$$B_q^{(0,v)} = \left\{ \Omega \in L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}) : \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu,\ M_q^{(0,v)}\left(\{\lambda_\mu\}\right) < \infty \right\},$$

where each λ_μ is a complex number; each b_μ is a *q-block supported on an interval* I_μ on $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$, $v > -1$ and

$$M_q^{(0,v)}\left(\{\lambda_\mu\}\right) = \sum_{\mu=1}^{\infty} |\lambda_\mu| \left\{1 + \log^{(v+1)}\left(|I_\mu|^{-1}\right)\right\}. \tag{2.1}$$

Let $\|\Omega\|_{B_q^{(0,v)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} = N_q^{(0,v)}(\Omega) = \inf\left\{M_q^{(0,v)}\left(\{\lambda_\mu\}\right)\right\}$, where the infimum is taken over all q-block decompositions of Ω.

We remark that the definition of $B_q^{(0,v)}([a,b]\times[c,d])$ for $a,b,c,d \in R$ will be the same as that of $B_q^{(0,v)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ except for minor modifications.

By following similar arguments as proving Lemma 5.1 in [3], we get the following.

Lemma 2.3

For any $v > -1, a,b,c,d \in \mathbb{R}$,

(i) $N_q^{(0,v)}$ *is a norm on* $B_q^{(0,v)}([a,b]\times[c,d])$ *and* $(B_q^{(0,v)}([a,b]\times[c,d]), N_q^{(0,v)})$ *is a Banach space;*

(ii) *If* $f \in B_q^{(0,v)}([a,b]\times[c,d])$ *and* g *is a measurable on* $[a,b]$ *with* $|g| \le |f|$, *then* $g \in B_q^{(0,v)}([a,b]\times[c,d])$ *with*

$$N_q^{(0,v)}(g) \le N_q^{(0,v)}(f);$$

(iii) *Let* I_1 *and* I_2 *be two disjoint intervals in* $[a,b]\times[c,d]$ *with* $|I_1|, |I_2| < 1$ *and* $\alpha_1, \alpha_2 \in \mathbb{R}_+$. *Then*

$$N_q^{(0,v)}(\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2}) \ge N_q^{(0,v)}(\alpha_1\chi_{I_1}) + N_q^{(0,v)}(\alpha_2\chi_{I_2});$$

(iv) *Let* I *be a interval in* $[a,b]\times[c,d]$ *with* $|I| < 1$. *Then*

$$N_q^{(0,v)}(\chi_I) \ge |I|\left(1 + \log^{v+1}(|I|^{-1}\right).$$

3. Proof of Theorem 1.1 (b)

It is clear that the operator $\mathcal{M}_\Omega$ is bounded on $L^2(\mathbb{R}^n\times\mathbb{R}^m)$ if and only if the multiplier

$$m(\xi,\eta) = \left(\int_{(0,\infty)\times(0,\infty)} \left| \int_{\{|u|\le t, |v|\le s\}} e^{-2\pi i (t\xi'\cdot u + s\eta\cdot' v)} \frac{\Omega(u,v)}{|u|^{n-1}|v|^{m-1}} du dv \right|^2 \frac{dtds}{(ts)^3}\right)^{1/2}$$

is an L^∞ function. Now,

$$\begin{aligned}(m(\xi,\eta))^2 \;&= \lim_{M\to\infty,\varepsilon_2\to 0}\,\lim_{N\to\infty,\varepsilon_1\to 0}\int_{(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})^2}\Omega\,(u,v)\,\overline{\Omega\,(x,y)}\\ &\quad\times\left(\int_{[0,1]^2}\left(\int_{\varepsilon_1}^{N}e^{-2\pi it\xi'\cdot(r_1u-r_2x)}\frac{dt}{t}\right)dr_1dr_2\right)\\ &\quad\times\left(\int_{[0,1]^2}\left(\int_{\varepsilon_2}^{M}e^{-2\pi is\eta'\cdot(w_1v-w_2y)}\frac{ds}{s}\right)dw_1dw_2\right)d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y).\end{aligned}$$

Note that

$$\begin{aligned}&\int_{\varepsilon_1}^{N}\left(e^{-2\pi it\xi'\cdot(r_1u-r_2x)}-\cos(2\pi t)\right)\frac{dt}{t}\to\log\left|\xi'\cdot(r_1u-r_2x)\right|^{-1}\\ &\qquad -i\frac{\pi}{2}\mathrm{sgn}(\xi'\cdot(r_1u-r_2x))\end{aligned}$$

as $N\to\infty$ and $\varepsilon_1\to 0$, and also this integral is bounded, uniformly in both ε_1 and N, by $C\left(1+\left|\log\left|\xi'\cdot(r_1u-r_2x)\right|\right|\right).$

For simplicity, let

$$S(\xi',r_1,r_2,u,x)=\log\left|\xi'\cdot(r_1u-r_2x)\right|^{-1}-i\frac{\pi}{2}\mathrm{sgn}(\xi'\cdot(r_1u-r_2x)).$$

Thus, using (1.1) and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}(m(\xi,\eta))^2 \;&= \int_{(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})^2}\Omega\,(u,v)\,\overline{\Omega\,(x,y)}\left(\int_{[0,1]^2}S(\xi',r_1,r_2,u,x)dr_1dr_2\right)\\ &\quad\times\left(\int_{[0,1]^2}S(\eta',w_1,w_2,v,y)dw_1dw_2\right)d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y).\end{aligned}$$

Now, if Ω is a real-valued function,

$$\begin{aligned}(m(\xi,\eta))^2 \;&= \int_{(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})^2}\int_{[0,1]^2}\int_{[0,1]^2}\Omega\,(u,v)\,\Omega\,(x,y)\\ &\quad\times\Big[\log\left|\xi'\cdot(r_1u-r_2x)\right|^{-1}\log\left|\eta'\cdot(w_1v-w_2y)\right|^{-1}\\ &\quad-\frac{\pi^2}{4}\mathrm{sgn}(\xi'\cdot(r_1u-r_2x))\\ &\quad\times\mathrm{sgn}(\eta'\cdot(w_1v-w_2y))\Big]dr_1dr_2dw_1dw_2d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y).\end{aligned}$$

Since if $\Omega\in L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$, the integral

$$\begin{aligned}&\int_{(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})^2}\Omega\,(u,v)\,\Omega\,(x,y)\int_{[0,1]^2}\int_{[0,1]^2}\left(\frac{\pi^2}{4}\mathrm{sgn}(\xi'\cdot(r_1u-r_2x))\right)\\ &\qquad\times\left(\mathrm{sgn}(\eta\xi'\cdot(w_1v-w_2y))\right)dr_1dr_2dw_1dw_2d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y)\end{aligned}$$

is bounded, uniformly in ξ' and η', we only to deal with $m_0(\xi,\eta)$, where

$$\begin{aligned}m_0(\xi,\eta \;&= \int_{(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})^2}\Omega\,(u,v)\,\Omega\,(x,y)\int_{[0,1]^2}\int_{[0,1]^2}\log\left|\xi'\cdot(r_1u-r_2x)\right|^{-1}\\ &\quad\times\log\left|\eta'\cdot(w_1v-w_2y)\right|^{-1}dr_1dr_2dw_1dw_2d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y).\end{aligned}$$

Now, for non-zero real numbers a and b, by straightforward computations, we have

$$\begin{aligned}&\int_{[0,1]^2} \log|r_1 a - r_2 b|^{-1}\, dr_1 dr_2\\ &= \frac{a}{2b}\log|a|^{-1} + \frac{b}{2a}\log|b|^{-1} - \frac{(a-b)^2}{2ab}\log|a-b|^{-1} - 1/2.\end{aligned}$$

Therefore, by the cancellation conditions on Ω, we immediately get

$$m_0(\xi,\eta) = \int_{(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})^2} \Omega(u,v)\,\Omega(x,y)\,F(u,x,\xi')F(v,y,\eta')d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y),$$

where

$$F(u,x,\xi') = \left(\left(1 - \frac{\xi'\cdot u}{\xi'\cdot x}\right)\log|\xi'\cdot(u-x)|^{-1} + \left(\frac{\xi'\cdot u}{\xi'\cdot x}\right)\log|\xi'\cdot u|^{-1}\right).$$

Now, we are ready to prove part (b) of Theorem 1.1. For the sake of simplicity we shall present the construction of our Ω only in the case $n = m = 2$ and $q = \infty$. Other cases can be obtained by making minor modifications. Also, we shall work on $[-1,1]^2$ instead of $\mathbb{S}^1\times\mathbb{S}^1$. We notice that the proof Theorem 1.1 (b) will be completed if we can construct an Ω on $[-1,1]^2$ with the following properties:

$$\int_{-1}^{1}\Omega(u,\cdot)\,du = \int_{-1}^{1}\Omega(\cdot,v)\,dv = 0; \tag{3.1}$$

$$\Omega \in B_\infty^{(0,\upsilon)}([-1,1]^2) \text{ for each } \upsilon, -1<\upsilon<0; \tag{3.2}$$

$$\Omega \notin B_\infty^{(0,0)}([-1,1]^2); \tag{3.3}$$

$$\int_{[0,1]^2}\int_{[0,1]^2} H(u,v,x,y)dudvdxdy = \infty; \tag{3.4}$$

$$\int_{[-1,1]^2\setminus[0,1]^2}\int_{[0,1]^2} |H(u,v,x,y)|\,dudvdxdy < \infty; \tag{3.5}$$

$$\int_{[0,1]^2}\int_{[-1,1]^2\setminus[0,1]^2} |H(u,v,x,y)|\,dudvdxdy < \infty; \tag{3.6}$$

$$\int_{[-1,1]^2\setminus[0,1]^2}\int_{[-1,1]^2\setminus[0,1]^2} |H(u,v,x,y)|\,dudvdxdy < \infty, \tag{3.7}$$

where

$$\begin{aligned}H(u,v,x,y) &= \Omega(u,v)\,\Omega(x,y)G(u,x)G(v,y),\\ G(u,x) &= \left(1-\frac{u}{x}\right)\log|u-x|^{-1} + \frac{u}{x}\log|u|^{-1}.\end{aligned}$$

To this end, for integers $k, j \geq 3$, let $I_j = [\frac{1}{j+1}, \frac{1}{j})$ and

$$\begin{aligned}\alpha_k &= \sum_{j=3}^{\infty}\frac{k}{(j+1)\,[\log(k+j)]^3},\\ \beta_{k,j} &= \frac{jk}{[\log(k+j)]^3}.\end{aligned}$$

Now, by definition of α_k, we have

$$\begin{aligned}\alpha_k &= \sum_{j=3}^{k}\frac{k}{(j+1)\left[\log(k+j)\right]^3}+\sum_{j=k+1}^{\infty}\frac{k}{(j+1)\left[\log(k+j)\right]^3}\\ &\leq \frac{k}{(\log k)^3}\left(\sum_{j=3}^{k}\frac{1}{(j+1)}\right)+k\left(\sum_{j=k+1}^{\infty}\frac{1}{(j+1)\,(\log j)^3}\right).\end{aligned}$$

Thus there exists a positive constant C independent of k such that

$$\alpha_k \leq C\frac{k}{(\log k)^2} \tag{3.8}$$

which easily implies

$$\sum_{k=3}^{\infty}\frac{\alpha_k}{k(k+1)}<\infty.$$

Define Ω on $[-1,1]^2$ by

$$\Omega\,(u,v)=\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}C_{k,j}\chi_{I_k\times I_j}(u,v),$$

where, for integers $k,j\geq 3$,

$$C_{2,2}=\left(\sum_{k=3}^{\infty}\frac{\alpha_k}{k(k+1)}\right),C_{2,k}=C_{k,2}=-\alpha_k\left|I_k\right|,C_{k,j}=\beta_{k,j}\left|I_k\times I_j\right|,$$

$$\begin{aligned}b_{2,2}(u,v) &= \chi_{[-1,0]^2}(u,v),b_{2,k}(u,v)=\left|I_k\right|^{-1}\chi_{[-1,0]\times I_k}(u,v),\\ b_{k,2}(u,v) &= \left|I_k\right|^{-1}\chi_{I_k\times[-1,0]}(u,v),\ b_{k,j}(u,v)=\left|I_k\times I_j\right|^{-1}\chi_{I_k\times I_j}(u,v)\end{aligned}$$

and χ_A represents the characteristic function of a set A.

By straightforward calculations, it is easy to see that (3.1)–(3.2) hold. To prove (3.3) we invoke Lemma 2.3. In fact, we notice that each $b_{k,j}$ is an ∞-block supported on the interval $I_k\times I_j$. So to prove (3.3), we only need to show that $N_\infty^{(0,0)}(\Omega)=\infty$. To this end, by Lemma 2.3 we have for each M,N,

$$\begin{aligned}&N_\infty^{(0,0)}\left(\Omega+\sum_{k=3}^{\infty}C_{2,k}b_{2,k}+\sum_{k=3}^{\infty}C_{k,2}b_{k,2}-C_{2,2}b_{2,2}\right)\\ &\geq\sum_{j=3}^{M}\sum_{k=3}^{N}C_{k,j}\left|I_k\times I_j\right|^{-1}N_\infty^{(0,0)}(\chi_{I_k\times I_j})\\ &\geq\sum_{j=3}^{M}\sum_{k=3}^{N}C_{k,j}\left(1+\log(\left|I_k\times I_j\right|^{-1})\right)\\ &\geq C\sum_{j=3}^{M}\sum_{k=3}^{N}\frac{(\log k+\log j)}{(k+1)(j+1)\left[\log(k+j)\right]^3}\end{aligned}$$

for some positive constant C independent of M and N. Letting $M, N \to \infty$, and since

$$\begin{aligned}\sum_{j=3}^{\infty}\sum_{k=3}^{\infty}\frac{(\log k+\log j)}{(k+1)(j+1)\left[\log(k+j)\right]^3} &\geq C\sum_{k=3}^{\infty}\frac{\log k}{k+1}\sum_{j\geq k}\frac{1}{(j+1)(\log j)^3}\\ &\geq C\sum_{k=3}^{\infty}\frac{1}{(k+1)\log k}=\infty\end{aligned}$$

we get

$$N_{\infty}^{(0,0)}\left(\Omega+\sum_{k=3}^{\infty}C_{2,k}b_{2,k}+\sum_{k=3}^{\infty}C_{k,2}b_{k,2}-C_{2,2}b_{2,2}\right)=\infty.$$

Since

$$N_{\infty}^{(0,0)}\left(\sum_{k=3}^{\infty}C_{2,k}b_{2,k}\right), N_{\infty}^{(0,0)}\left(\sum_{k=3}^{\infty}C_{k,2}b_{k,2}\right), N_{\infty}^{(0,0)}\left(C_{2,2}b_{2,2}\right)$$

are finite numbers, we obtain $N_{\infty}^{(0,0)}(\Omega)=\infty$. This ends the proof of (3.3).

At this point, let us introduce some notations and make some comments that will be used frequently in the proofs of (3.4)–(3.7).

Let

$$\begin{aligned}
E(k,l) &= \left(\int_{I_k\times I_l} G(x,u)dxdu\right);\\
E^{(1)}(k,l) &= \int_{I_k\times I_l}\left(1-\frac{u}{x}\right)\log|u-x|^{-1}\,dxdu;\\
E^{(2)}(k,l) &= \int_{I_k\times I_l}\frac{u}{x}\left(\log|u-x|-\log|u|\right)dxdu;\\
E^{(3)}(k,l) &= \int_{I_k\times I_l}\log|u-x|^{-1}\,dxdu;\\
E^{(4)}(k,l) &= \int_{I_k\times I_l}\frac{u}{x}\log|u|^{-1}\,dxdu;\\
\mathcal{I}(k) &= \left(\int_{I_k}\int_{-1}^{0} G(x,u)dxdu\right);\\
\mathcal{I}^*(k) &= \left(\int_{-1}^{0}\int_{I_k} G(x,u)dxdu\right);\\
\mathcal{I}_1(k) &= \int_{I_k}\int_{-1}^{0}\log|u-x|^{-1}\,dxdu;\\
\mathcal{I}_2(k) &= \int_{I_k}\int_{-1}^{0}\frac{u}{x}\left(\log|u-x|-\log|u|\right)dxdu;\\
\mathcal{I}_3(k) &= \int_{I_k}\int_{-1}^{0}\frac{u}{x}\left(\log|u-x|-\log|u|\right)dudx.
\end{aligned}$$

By straightforward calculations, it is easy to show that

$$|\mathcal{I}_1(j)| \leq C\frac{1}{j^2}; \tag{3.9}$$

$$E^{(3)}(j,s) \leq C\frac{\log j}{j^2 s^2} \quad \text{if } s > 2j; \tag{3.10}$$

$$E^{(3)}(j,s) \leq C\frac{\log s}{j^2 s^2} \quad \text{if } j > 2s; \tag{3.11}$$

$$E^{(3)}(j,s) \leq C\frac{\log s}{s^4} \quad \text{if } j/2 \leq s \leq 2j \tag{3.12}$$

for some positive constant C independent of j and s.[1]

Now, we need to point out to the following observations. First, if $(x,u) \in I_k \times I_l$ with $l \geq 2(k+1)$, we have $\frac{u}{x} \leq \frac{k+1}{l} \leq 1/2$ and hence $1 - \frac{u}{x} \geq 1/2 > 0$ which in turn yields

$$G(x,u) \geq \log |x-u|^{-1}. \tag{3.13}$$

Next, we notice that if $(x,u) \in I_k \times I_l$ with $k/2 - 1 < l < 2(k+1)$, we have $\frac{u}{x} \geq \frac{k}{l+1} > \frac{k}{3k} = \frac{1}{3}$. Thus $1 - \frac{u}{x} < \frac{2}{3}$ and hence by (3.12) we get

$$E^{(1)}(k,l) \leq \frac{2}{3}\int_{I_k \times I_l} \log |u-x|^{-1}\, dx du \leq C\frac{\log l}{l^4}, \tag{3.14}$$

for some positive constant independent of k, l. Thirdly, by the mean-value theorem we have

$$\left|\frac{u}{x}\left(\log|u-x| - \log|u|\right)\right| \leq \frac{|u|}{\min\{|u-x|, |u|\}}. \tag{3.15}$$

We notice that if $(x,u) \in I_k \times I_l$ with $3 \leq l \leq k/2 - 1$, we get

$$u - x \geq \frac{1}{l+1} - \frac{1}{k} = \frac{k-(l+1)}{(l+1)k} \geq \frac{1}{k} \geq x > 0.$$

Thus $u - x < u$ which yields

$$\left|\frac{u}{x}\left(\log|u-x| - \log|u|\right)\right| \leq \frac{u}{u-x} = \frac{1}{1-x/u}.$$

Since $x/u \leq 1/2$ we obtain

$$\left|\frac{u}{x}\left(\log|u-x| - \log|u|\right)\right| \leq 2.$$

Also, if $(x,u) \in I_k \times I_l$ with $l \geq 2(k+1)$, we have $x - u \geq u$ and thus

$$\left|\frac{u}{x}\left(\log|u-x| - \log|u|\right)\right| \leq 1.$$

So, in either case we have

$$E^{(2)}(k,l) \leq C\frac{1}{k^2 l^2} \tag{3.16}$$

---

1. The author is indebted to Prof. Yibiao Pan for a helpful discussion.

for some positive constant independent of k and l. Fourthly, if $(x,u)\in I_k\times I_l$ with $k/2-1<l<2(k+1)$, we have $\frac{u}{x}\le\frac{k+1}{l}\le\frac{2l+3}{l}\le 3$. Also, notice that if $u\in I_l$, we have $\log|u|^{-1}\le\log(l+1)$. Thus, we have

$$E^{(4)}(k,l)\le C\frac{\log l}{k^2l^2}. \tag{3.17}$$

Fifthly, by (3.15), it is easy to see that

$$|\mathcal{I}_2(k)|\le C\frac{1}{k^2}; \tag{3.18}$$

$$|\mathcal{I}_3(k)|\le C\frac{1}{k^2}. \tag{3.19}$$

Now, we turn to the proof of (3.4). By the above remarks, we notice that

$$\begin{aligned}&\int_{[0,1]^2}\int_{[0,1]^2}H(u,v,x,y)dudvdxdy\\ &=\sum_{j=3}^{\infty}\sum_{k=3}^{\infty}\sum_{s=3}^{\infty}\sum_{l=3}^{\infty}\beta_{k,j}\beta_{l,s}E(k,l)E(j,s)\\ &\ge S_1+S_2+S_3+S_4+S_5+S_6+S_7+S_8+S_9,\end{aligned}$$

where

$$\begin{aligned}
S_1&=\sum_{j=3}^{\infty}\sum_{s\ge2(j+1)}\sum_{k=3}^{\infty}\sum_{l\ge2(k+1)}\beta_{k,j}\beta_{l,s}E^{(3)}(k,l)E^{(3)}(j,s);\\
S_2&=\sum_{j=3}^{\infty}\sum_{s\ge2(j+1)}\sum_{k=3}^{\infty}\sum_{k/2-1<l<2(k+1)}\beta_{k,j}\beta_{l,s}E^{(1)}(k,l)E^{(3)}(j,s);\\
S_3&=\sum_{j=3}^{\infty}\sum_{j/2-1<s<2(j+1)}\sum_{k=3}^{\infty}\sum_{l\ge2(k+1)}\beta_{k,j}\beta_{l,s}E^{(3)}(k,l)E^{(1)}(j,s);\\
S_4&=\sum_{j=3}^{\infty}\sum_{s\ge2(j+1)}\sum_{k=3}^{\infty}\sum_{3\le l\le k/2-1}\beta_{k,j}\beta_{l,s}E^{(2)}(k,l)E^{(3)}(j,s);\\
S_5&=\sum_{j=3}^{\infty}\sum_{3\le s\le j/2-1}\sum_{k=3}^{\infty}\sum_{l\ge2(k+1)}\beta_{k,j}\beta_{l,s}E^{(3)}(k,l)E^{(2)}(j,s);\\
S_6&=\sum_{j=3}^{\infty}\sum_{j/2-1<s<2(j+1)}\sum_{k=3}^{\infty}\sum_{k/2-1<l<2(k+1)}\beta_{k,j}\beta_{l,s}E^{(1)}(k,l)E^{(1)}(j,s);\\
S_7&=\sum_{j=3}^{\infty}\sum_{j/2-1<s<2(j+1)}\sum_{k=3}^{\infty}\sum_{3\le l\le k/2-1}\beta_{k,j}\beta_{l,s}E^{(2)}(k,l)E^{(1)}(j,s);\\
S_8&=\sum_{j=3}^{\infty}\sum_{3\le s\le j/2-1}\sum_{k=3}^{\infty}\sum_{k/2-1<l<2(k+1)}\beta_{k,j}\beta_{l,s}E^{(1)}(k,l)E^{(2)}(j,s);\\
S_9&=\sum_{j=3}^{\infty}\sum_{3\le s\le j/2-1}\sum_{k=3}^{\infty}\sum_{3\le l\le k/2-1}\beta_{k,j}\beta_{l,s}E^{(2)}(k,l)E^{(2)}(j,s).
\end{aligned}$$

It is clear that to prove (3.4), it suffices to prove the following:
(i) $S_1 = \infty$; and (iii)$|S_i| < \infty$ for $i = 2, \dots, 9$.
To prove (i), we invoke (3.10) to get,

$$\begin{aligned}
S_1 &\geq C \sum_{s\geq 2(j+1)} \sum_{j=3} \sum_{k=3}^{\infty} \sum_{l\geq 2(k+1)} \left(\frac{jkls}{[\log(k+j)]^3 [\log(l+s)]^3} \right) \left(\frac{\log k \log j}{k^2 j^2 s^2 l^2} \right) \\
&\geq C \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \sum_{l\geq 2(k+1)} \left(\frac{\log k \log j}{kjl\, [\log(k+j)]^3 [\log(l+j)]^2} \right) \\
&\geq C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \left(\frac{\log k \log j}{kj\, [\log(k+j)]^4} \right) \\
&\geq C \sum_{k=3}^{\infty} \frac{\log k}{k} \sum_{j\geq k}^{\infty} \left(\frac{1}{j\, (\log j)^3} \right) \\
&\geq C \sum_{k=3}^{\infty} \frac{1}{k \log k} = \infty.
\end{aligned}$$

Now, we turn to prove that $|S_2| < \infty$. By (3.10), (3.12) and (3.14), we have

$$\begin{aligned}
|S_2| &\leq \sum_{j=3}^{\infty} \sum_{s\geq 2(j+1)} \sum_{k=3}^{\infty} \sum_{k/2-1<l<2(k+1)} \left(\frac{jkls}{[\log(k+j)]^3 [\log(l+s)]^3} \right) \left(\frac{\log l \log j}{j^2 s^2 l^4} \right) \\
&\leq C \sum_{j=3}^{\infty} \sum_{s\geq 2(j+1)} \sum_{k=3}^{\infty} \left(\frac{\log k \log j}{kjs\, [\log(k+j)]^3 [\log(k+s)]^3} \right) \\
&\leq C \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \left(\frac{\log k \log j}{kj\, [\log(k+j)]^5} \right) \\
&\leq C \left(\sum_{j=3}^{\infty} \frac{\log k}{k\, [\log(k+3)]^{5/2}} \right) \left(\sum_{k=3}^{\infty} \frac{\log j}{j\, [\log(3+j)]^{5/2}} \right) < \infty.
\end{aligned}$$

Similarly, $|S_3| < \infty$. Now, we verify $|S_4| < \infty$. By (3.10) and (3.16), we get

$$\begin{aligned}
S_4 &\leq C \sum_{j=3}^{\infty} \sum_{s\geq 2(j+1)} \sum_{k=3}^{\infty} \sum_{3\leq l\leq k/2-1} \left(\frac{\log j}{kjls\, [\log(k+j)]^3 [\log(l+s)]^3} \right) \\
&\leq C \sum_{j=3}^{\infty} \sum_{s\geq 2(j+1)} \sum_{k=3}^{\infty} \left(\frac{\log j}{kjs\, [\log(k+j)]^3 [\log(k+s)]^2} \right) \\
&\leq C \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \left(\frac{\log j}{kj\, [\log(k+j)]^4} \right) \\
&\leq C \left(\sum_{k=3}^{\infty} \frac{1}{k\, [\log(k+3)]^{3/2}} \right) \left(\sum_{j=3}^{\infty} \frac{\log j}{j\, [\log(3+j)]^{5/2}} \right) < \infty.
\end{aligned}$$

Similarly, $|S_5| < \infty$. To prove $|S_6| < \infty$, we use (3.12). In fact, by (3.12), we have

$$\begin{aligned}
S_6 &\le C \sum_{j=3}^{\infty} \sum_{j/2-1<s<2(j+1)} \sum_{k=3}^{\infty} \sum_{k/2-1<l<2(k+1)} \left(\frac{jk \log l \log s}{s^3 l^3 \left[\log(k+j)\right]^3 \left[\log(l+s)\right]^3} \right) \\
&\le C \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \left(\frac{\log k \log j}{kj \left[\log(k+j)\right]^6} \right) \\
&\le C \sum_{j=3}^{\infty} \left(\frac{\log j}{j \left[\log(3+j)\right]^3} \right) \sum_{k=3}^{\infty} \left(\frac{\log k}{k \left[\log(k+3)\right]^3} \right) < \infty.
\end{aligned}$$

Now, we prove $|S_7| < \infty$. By (3.12) and (3.16), we have

$$\begin{aligned}
|S_7| &\le C \sum_{j=3}^{\infty} \sum_{j/2-1<s<2(j+1)} \sum_{k=3}^{\infty} \sum_{3\le l\le k/2-1} \left(\frac{j \log s}{kls^3 \left[\log(k+j)\right]^3 \left[\log(l+s)\right]^3} \right) \\
&\le C \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \sum_{3\le l\le k/2-1} \left(\frac{\log j}{kjl \left[\log(k+j)\right]^3 \left[\log(l+j)\right]^3} \right) \\
&\le C \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \left(\frac{\log j}{kj \left[\log(k+j)\right]^5} \right) \\
&\le C \sum_{j=3}^{\infty} \left(\frac{\log j}{j \left[\log(3+j)\right]^{7/2}} \right) \sum_{k=3}^{\infty} \left(\frac{1}{k \left[\log(k+3)\right]^{3/2}} \right) < \infty.
\end{aligned}$$

Similarly, $|S_8| < \infty$. Now, we need to verify $|S_9| < \infty$. By (3.16), we have

$$\begin{aligned}
S_7 &\le C \sum_{j=3}^{\infty} \sum_{3\le s\le j/2-1} \sum_{k=3}^{\infty} \sum_{3\le l\le k/2-1} \left(\frac{jkls}{\left[\log(k+j)\right]^3 \left[\log(l+s)\right]^3} \right) \left(\frac{1}{k^2 j^2 l^2 s^2} \right) \\
&\le C \sum_{j=3}^{\infty} \sum_{3\le s\le j/2-1} \sum_{k=3}^{\infty} \sum_{3\le l\le k/2-1} \left(\frac{1}{kjls \left[\log(k+j)\right]^3 \left[\log(l+s)\right]^3} \right) \\
&\le C \sum_{j=3}^{\infty} \sum_{3\le s\le j/2-1} \sum_{k=3}^{\infty} \left(\frac{1}{kjs \left[\log(k+j)\right]^3 \left[\log(k+s)\right]^2} \right) \\
&\le C \sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \left(\frac{1}{kj \left[\log(k+j)\right]^5} \right) \\
&\le C \left(\sum_{j=3}^{\infty} \frac{1}{j \left[\log(3+j)\right]^{5/2}} \right) \left(\sum_{k=3}^{\infty} \frac{1}{k \left[\log(k+3)\right]^{5/2}} \right) < \infty.
\end{aligned}$$

This completes the proof of (3.4). Now, we verify (3.5). To this end, we divide $[-1,1]^2 \backslash [0,1]^2$ into three parts: $[-1,0] \times [0,1]$, $[0,1] \times [-1,0]$, and $[-1,0] \times [-1,0]$. First the integral over $[-1,0] \times [0,1]$ $\times [0,1]^2$ is dominated from above by

$$\begin{aligned}
X &= \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \sum_{s=3}^{\infty} \beta_{k,j} \alpha_s \left|\mathcal{I}(k)\right| \left|E(j,s)\right| && (3.20) \\
&\le X_1 + X_2 + X_3, && (3.21)
\end{aligned}$$

where

$$
\begin{aligned}
X_1 &= \sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s\ge 2(j+1)} \beta_{k,j}\alpha_s\,|\mathcal{I}(k)|\,|E(j,s)|\,;\\
X_2 &= \sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{j/2-1<s<2(j+1)} \beta_{k,j}\alpha_s\,|\mathcal{I}(k)|\,|E(j,s)|\,;\\
X_3 &= \sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{3\le s\le j/2-1} \beta_{k,j}\alpha_s\,|\mathcal{I}(k)|\,|E(j,s)|\,.
\end{aligned}
$$

Thus to prove $X<\infty$, it suffices to prove that each $X_i<\infty$ for $i=1,2,3$. To this end, we notice that

$$
\begin{aligned}
X_1 &\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s\ge 2(j+1)} \beta_{k,j}\alpha_s\,|\mathcal{I}_1(k)+\mathcal{I}_2(k)|\left|E^{(2)}(j,s)\right|\\
&\quad + C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s\ge 2(j+1)} \beta_{k,j}\alpha_s\,|\mathcal{I}_1(k)+\mathcal{I}_2(k)|\left|E^{(3)}(j,s)\right|\\
&= X_1^{(1)}+X_1^{(2)}.
\end{aligned}
$$

Now, by (3.8)–(3.9), (3.16) and (3.18),

$$
\begin{aligned}
X_1^{(1)} &\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s\ge 2(j+1)} \frac{jks}{[\log(k+j)]^3(\log s)^2}\frac{1}{k^2s^2j^2}\\
&\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\frac{1}{kj\,[\log(k+j)]^3(\log j)}\\
&\le C\left(\sum_{k=3}^{\infty}\frac{1}{k\,[\log(k+3)]^2}\right)\left(\sum_{j=3}^{\infty}\frac{1}{j\,[\log(3+j)]\,(\log j)}\right)<\infty.
\end{aligned}
$$

By (3.8)–(3.10) and (3.18),

$$
\begin{aligned}
X_1^{(2)} &\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s\ge 2(j+1)} \frac{jks}{[\log(k+j)]^3(\log s)^2}\frac{\log j}{k^2s^2j^2}\\
&\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s\ge 2(j+1)}\frac{\log j}{kjs\,[\log(k+j)]^3(\log s)^2}\\
&\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\frac{1}{kj\,[\log(k+j)]^3}\\
&\le C\left(\sum_{k=3}^{\infty}\frac{1}{k\,[\log(k+3)]^{3/2}}\right)\left(\sum_{j=3}^{\infty}\frac{1}{j\,[\log(3+j)]^{3/2}}\right)<\infty.
\end{aligned}
$$

This completes the proof of $X_1<\infty$. Now, we turn to the proof of $X_2<\infty$. To this end, we notice that

$$\begin{aligned}
X_2 \;&\le\; \sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{j/2-1<s<2(j+1)} \beta_{k,j}\alpha_s \left|\mathcal{I}_1(k)+\mathcal{I}_2(k)\right| \left|E^{(1)}(j,s)\right| \\
&\quad + \sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{j/2-1<s<2(j+1)} \beta_{k,j}\alpha_s \left|\mathcal{I}_1(k)+\mathcal{I}_2(k)\right| \left|E^{(4)}(j,s)\right| \\
&= X_2^{(1)} + X_2^{(2)}.
\end{aligned}$$

By (3.8)–(3.9), (3.14) and (3.18),

$$\begin{aligned}
X_2^{(1)} &\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{j/2-1<s<2(j+1)} \frac{jks}{[\log(k+j)]^3(\log s)^2}\,\frac{\log s}{k^2 s^4} \\
&\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty} \frac{1}{kj\,[\log(k+j)]^3(\log j)} \\
&\le C\left(\sum_{k=3}^{\infty}\frac{1}{k\,[\log(k+3)]^{3/2}}\right)\left(\sum_{j=3}^{\infty}\frac{1}{j\,[\log(3+j)]^{1/2}(\log j)}\right) < \infty.
\end{aligned}$$

On other hand, by (3.8)–(3.9) and (3.17)–(3.18),

$$\begin{aligned}
X_2^{(2)} &\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{j/2-1<s<2(j+1)} \frac{jks}{[\log(k+j)]^3(\log s)^2}\,\frac{\log s}{k^2 j^2 s^2} \\
&\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty} \frac{1}{kj\,[\log(k+j)]^3(\log j)} \\
&\le C\left(\sum_{k=3}^{\infty}\frac{1}{k\,[\log(k+3)]^{3/2}}\right)\left(\sum_{j=3}^{\infty}\frac{1}{j\,[\log(3+j)]^{1/2}(\log j)}\right) < \infty.
\end{aligned}$$

A proof of $X_3 < \infty$ can be obtained by following a similar argument as in our proof of the finiteness of X_2. Thus, we get $X < \infty$ which in turn proves the finiteness of the integral over $[-1,0]\times[0,1]\times[0,1]^2$. Similarly, the integral over $[0,1]\times[-1,0]\times[0,1]^2$ is finite. Also, the integral over $[-1,0]\times[-1,0]\times[0,1]^2$ is bounded above by

$$\begin{aligned}
&C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\beta_{k,j}\left|\mathcal{I}(k)\right|\left|\mathcal{I}(j)\right| \\
&\le C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\beta_{k,j}\left(\left|\mathcal{I}_1(k)\right|+\left|\mathcal{I}_2(k)\right|\right)\left(\left|\mathcal{I}_1(j)\right|+\left|\mathcal{I}_2(j)\right|\right) \\
&\le C\left(\sum_{k=3}^{\infty}\frac{1}{k\,[\log(k+3)]^{3/2}}\right)\left(\sum_{j=3}^{\infty}\frac{1}{j\,[\log(3+j)]^{3/2}}\right) < \infty
\end{aligned}$$

where the second inequality is obtained from (3.9) and (3.18). This ends the proof of (3.5).

Similarly we can prove (3.6). Finally, we verify (3.7). As above, we divide $[-1,1]^2\backslash[0,1]^2$ into three parts: $[-1,0]\times[0,1]$, $[0,1]\times[-1,0]$, and $[-1,0]\times[-1,0]$. We shall only present the proof of the finiteness of the integral over $[-1,0]\times[0,1]\times[-1,0]\times[0,1]$ and over $[-1,0]\times[0,1]\times[0,1]\times[-1,0]$. The proof of the other cases either will be similar or easier. To this end, we notice that the integral over $[-1,0]\times[0,1]\times[-1,0]\times[0,1]$ is bounded from above by

$$\begin{aligned} &C\sum_{k=3}^{\infty}\sum_{l=3}^{\infty}\frac{kl}{(\log k)^2(\log l)^2}E(k,l)\left(\int_{-1}^{0}\int_{-1}^{0}G(v,y)dvdy\right)\\ \le\ &C\sum_{k=3}^{\infty}\sum_{l=3}^{\infty}\frac{kl}{(\log k)^2(\log l)^2}E(k,l)=S^*. \end{aligned}$$

Now, as above we split S^* as

$$S^*=S_1^*+S_2^*+S_3^*,$$

where

$$\begin{aligned} S_1^* &= \sum_{k=3}^{\infty}\sum_{l>2(k+1)}\frac{kl}{(\log k)^2(\log l)^2}E(k,l);\\ S_2^* &= \sum_{l=3}^{\infty}\sum_{k/2-1<l<2(k+1)}\frac{kl}{(\log k)^2(\log l)^2}E(k,l);\\ S_3^* &= \sum_{k=3}^{\infty}\sum_{3\le l\le k/2-1}\frac{kl}{(\log k)^2(\log l)^2}E(k,l). \end{aligned}$$

Now,

$$\begin{aligned} S_1^* &\le \sum_{k=3}^{\infty}\sum_{l>2(k+1)}\frac{kl}{(\log k)^2(\log l)^2}\left|E^{(2)}(k,l)\right|\\ &\quad+\sum_{k=3}^{\infty}\sum_{l>2(k+1)}\frac{kl}{(\log k)^2(\log l)^2}\left|E^{(3)}(k,l)\right|\\ &= S_{1,1}^*+S_{1,2}^*. \end{aligned}$$

By (3.16), we have

$$\begin{aligned} S_{1,1}^* &\le C\sum_{k=3}^{\infty}\sum_{l>2(k+1)}\frac{kl}{(\log k)^2(\log l)^2}\frac{1}{k^2l^2}\\ &\le C\sum_{k=3}^{\infty}\frac{kl}{k(\log k)^3}<\infty. \end{aligned} \tag{3.22}$$

On the other hand, by (3.10)

$$\begin{aligned} S_{1,2}^* &\le C\sum_{k=3}^{\infty}\sum_{l>2(k+1)}\frac{kl}{(\log k)^2(\log l)^2}\frac{\log k}{k^2l^2}\\ &\le C\sum_{k=3}^{\infty}\frac{1}{k(\log k)^2}<\infty. \end{aligned} \tag{3.23}$$

Thus, by (3.22)–(3.23) we get $S_1^* < \infty$. Similarly, $S_3^* < \infty$. Now,

$$\begin{aligned} S_2^* &\leq \sum_{l=3}^{\infty} \sum_{k/2-1<l<2(k+1)} \frac{kl}{(\log k)^2(\log l)^2} \left|E^{(1)}(k,l)\right| \\ &\quad + \sum_{l=3}^{\infty} \sum_{k/2-1<l<2(k+1)} \frac{kl}{(\log k)^2(\log l)^2} \left|E^{(4)}(k,l)\right| \\ &= S_{2,1}^* + S_{2,2}^*. \end{aligned}$$

By (3.14), we have

$$\begin{aligned} S_{2,1}^* &\leq C \sum_{k=3}^{\infty} \sum_{k/2-1<l<2(k+1)} \frac{kl}{(\log k)^2(\log l)^2} \frac{\log l}{l^4} \\ &\leq C \sum_{k=3}^{\infty} \frac{1}{k(\log k)^3} < \infty. \end{aligned} \tag{3.24}$$

On the other hand, by (3.17)

$$\begin{aligned} S_{2,2}^* &\leq C \sum_{k=3}^{\infty} \sum_{k/2-1<l<2(k+1)} \frac{kl}{(\log k)^2(\log l)^2} \frac{\log k}{k^2 l^2} \\ &\leq C \sum_{k=3}^{\infty} \frac{1}{k(\log k)^2} < \infty. \end{aligned} \tag{3.25}$$

This finishes the proof of the finiteness of the integral over $[-1,0]\times[0,1]\times[-1,0]\times[0,1]$. Now, we turn to the proof of the finiteness of the integral over $[-1,0]\times[0,1]\times[0,1]\times[-1,0]$. We notice that the integral over $[-1,0]\times[0,1]\times[0,1]\times[-1,0]$ is bounded from above by

$$\begin{aligned} &C \sum_{k=3}^{\infty} \sum_{l=3}^{\infty} \frac{kl}{(\log k)^2(\log l)^2} \left|\mathcal{I}(k)\mathcal{I}(j)\right| \\ &\leq C \sum_{k=3}^{\infty} \sum_{l=3}^{\infty} \frac{kl}{kl(\log k)^2(\log l)^2} \left(|\mathcal{I}_1(k)| + |\mathcal{I}_2(k)|\right)\left(|\mathcal{I}_1(l)| + |\mathcal{I}_2(l)|\right) \\ &\leq C \left(\sum_{k=3}^{\infty} \frac{1}{k(\log k)^2}\right)\left(\sum_{j=3}^{\infty} \frac{1}{j(\log j)^2}\right) < \infty. \end{aligned}$$

This ends the proof of (3.7) which in turn completes the proof of Theorem 1.1 (b). □

4. Main lemma

For a suitable family of measures $\sigma = \{\sigma_{t,s} : t,s \in \mathbb{R}_+\}$ on $\mathbb{R}^n \times \mathbb{R}^m$, we define the square operator F_σ and the corresponding maximal operator σ^* by

$$F_\sigma(f)(x,y) = \left(\int_{(0,\infty)\times(0,\infty)} |\sigma_{t,s} * f(x,y)|^2 \frac{dtds}{ts}\right)^{1/2}$$

and

$$\sigma^*(f)(x,y) = \sup_{t,s\in\mathbb{R}_+} ||\sigma_{t,s}| * f(x,y)|.$$

Also, we write $t^{\pm\alpha} = \inf\{t^{\alpha}, t^{-\alpha}\}$ and $\|\sigma_{t,s}\|$ for the total variation of $\sigma_{t,s}$. The proof of Theorem 1.1 (a) will rely heavily on the following lemma:

Lemma 4.1

Let $a, b \geq 2$, $B > 1, C > 0$ and $q_0 \in (1,\infty)$. Suppose that the family of measures $\{\sigma_{t,s} : t, s \in \mathbb{R}_+\}$ satisfies the following:

(i) $\|\sigma_{t,s}\| \leq 1$ *for $t, s \in \mathbb{R}_+$;*

(ii) $\int_{a^{kB}}^{a^{(k+1)B}} \int_{b^{jB}}^{b^{(j+1)B}} |\hat{\sigma}_{t,s}(\xi,\eta)|^2 \frac{dtds}{ts} \leq CB^2 (a^{kB}|\xi|)^{\pm\frac{\alpha}{B}} (b^{jB}|\eta|)^{\pm\frac{\alpha}{B}}$ *for $t, s \in \mathbb{R}_+$ and $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m$;*

(iii) $\|\sigma^*(f)\|_q \leq C\|f\|_q$ *for all $q > q_0$ and $f \in L^q(\mathbb{R}^n \times \mathbb{R}^m)$.*

Then, for every p satisfying $|1/p - 1/2| < 1/(2q_0)$, there exists a positive constant C_p which is independent of B such that

$$\|F_\sigma(f)\|_p \leq C_p B \|f\|_p \tag{4.1}$$

for $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Proof. For $\lambda \geq 2$, let $\{\varphi_{j,\lambda}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the intervals $E_{j,\lambda} = [\lambda^{-(j+1)B}, \lambda^{-(j-1)B}]$. More precisely, we require the following:

$$\begin{aligned}
\varphi_{j,\lambda} \;\in\; C^{\infty},\ 0 \leq \varphi_{j,\lambda} \leq 1,\ \sum_j \varphi_{j,\lambda}(t) = 1;\\
\operatorname{supp}\varphi_{j,\lambda} \;\subseteq\; E_{j,\lambda};\\
\left|\frac{d^s\varphi_{j,\lambda}(t)}{dt^s}\right| \;\leq\; \frac{C}{t^s},
\end{aligned}$$

where C can be chosen to be independent of B. Let $I_{k,a} = \left[a^{kB}, a^{(k+1)B}\right)$, $\widehat{\Phi_k}(\xi) = \varphi_{k,a}(|\xi|)$ and $\widehat{\Psi_j}(\eta) = \varphi_{j,b}(|\eta|)$ for $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m$. Then for $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ we have

$$\begin{aligned}
\sigma_{t,s} * f(x,y) &= \sum_{k,l\in\mathbb{Z}} \sum_{j,r\in\mathbf{Z}} \left(\sigma_{t,s} * (\Phi_{k+l} \otimes \Psi_{j+r}) * f\right)(x,y)\chi_{I_{k,a}}(t)\chi_{I_{j,b}}(s)\\
&:= \sum_{k,j\in\mathbb{Z}} H_{k,j}(x,y,t,s),
\end{aligned}$$

say and define

$$F_{k,j}f(x,y) = \left(\int_{(0,\infty)\times(0,\infty)} |H_{k,j}(x,y,t,s)|^2 \frac{dtds}{ts}\right)^{1/2}.$$

Then

$$F_\sigma f(x,y) \leq \sum_{k,j\in\mathbb{Z}} F_{k,j} f(x,y).$$

Therefore, (4.1) is proved if we can show that

$$\|F_{k,j}(f)\|_p \le C_p B a^{-\theta_p|k|} b^{-\theta_p|j|} \|f\|_p \tag{4.2}$$

for every p satisfying $|1/p - 1/2| < 1/(2q_0)$, for all $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$ and for some positive constants C_p and θ_p.

The proof of (4.2) follows by interpolation between a sharp L^2 estimate and a cruder L^p estimate.

First, we compute the L^2-norm of $F_{k,j}$. By Plancherel's theorem, we have

$$\begin{aligned}
&\|F_{k,j}(f)\|_2^2 \\
&= \sum_{l,r\in\mathbb{Z}} \int_{\mathbb{R}^n\times\mathbb{R}^m} \int_{I_{l,a}\times I_{r,b}} \left|\left(\sigma_{t,s} * (\Phi_{k+l}\otimes\Psi_{j+r}) * f\right)(x,y)\right|^2 \frac{dtds}{ts} dxdy \\
&\le \sum_{l,r\in\mathbb{Z}} \int_{E_{l+k,a}\times E_{r+j,b}} \left(\int_{I_{l,a}\times I_{r,b}} |\hat{\sigma}_{t,s}(\xi,\eta)|^2 \frac{dtds}{ts}\right) \left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta \\
&\le CB^2 \sum_{l,r\in\mathbb{Z}} \int_{E_{l+k,a}\times E_{r+j,b}} \left|a^{lB}\xi\right|^{\pm\frac{\alpha}{B}} \left|b^{rB}\eta\right|^{\pm\frac{\alpha}{B}} \left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta \\
&\le CB^2 a^{-\alpha|k|} b^{-\alpha|j|} \sum_{l,r\in\mathbb{Z}} \int_{E_{l+k,a}\times E_{r+j,b}} \left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta \\
&\le CB^2 a^{-\alpha|k|} b^{-\alpha|j|} \|f\|_2^2 .
\end{aligned}$$

Therefore,

$$\|F_{k,j}(f)\|_2 \le CB a^{-\frac{\alpha}{2}|k|} b^{-\frac{\alpha}{2}|j|} \|f\|_2 . \tag{4.3}$$

On the other hand, we compute the L^p-norm of $F_{k,j}(f)$. We start first with the case $2 \le p < 2q_0(q_0-1)^{-1}$. Choose g in $L^{(p/2)'}(\mathbb{R}^n \times \mathbb{R}^m)$ with $\|g\|_{(p/2)'} \le 1$ such that

$$\begin{aligned}
&\|F_{k,j}(f)\|_p^2 \\
&= \sum_{l,r\in\mathbb{Z}} \int_{\mathbb{R}^n\times\mathbb{R}^m} \int_{I_{l,a}\times I_{r,b}} \left|\left(\sigma_{t,s} * (\Phi_{k+l}\otimes\Psi_{j+r}) * f\right)(x,y)\right|^2 \frac{dtds}{ts} |g(x,y)|\, dxdy \\
&\le C \sum_{l,r\in\mathbb{Z}} \int_{\mathbb{R}^n\times\mathbb{R}^m} \int_{I_{l,a}\times I_{r,b}} |\sigma_{t,s}| * \left|\left((\Phi_{k+l}\otimes\Psi_{j+r}) * f\right)(x,y)\right|^2 \frac{dtds}{ts} |g(x,y)|\, dxdy \\
&\le CB^2 \int_{\mathbb{R}^n\times\mathbb{R}^m} \sum_{l,r\in\mathbb{Z}} \left|\left((\Phi_{k+l}\otimes\Psi_{j+r}) * f\right)(x,y)\right|^2 \sigma^*(\tilde{g})(-x,-y) dx \\
&\le CB^2 \left\| \sum_{l,r\in\mathbb{Z}} \left|(\Phi_{k+l}\otimes\Psi_{j+r}) * f\right|^2 \right\|_{p/2} \|\sigma^*(\tilde{g})\|_{(p/2)'} ,
\end{aligned}$$

where $\tilde{g}(x,y) = g(-x,-y)$. By using (iii), the Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [22, p. 96], we have

$$\|F_{k,j}(f)\|_p \le CB \|f\|_p \text{ for } 2 \le p < 2q_0(q_0-1)^{-1}. \tag{4.4}$$

Interpolating the two estimates (4.3) and (4.4), we get (4.2) for $2 \le p < 2q_0(q_0-1)^{-1}$.

Now we prove (4.2) for $2q_0(q_0+1)^{-1} < p < 2$. By the above argument, we only need to show that

$$\|F_{k,j}(f)\|_p \le CB\,\|f\|_p \text{ for } 2q_0(q_0+1)^{-1} < p < 2. \tag{4.5}$$

To this end, we prove the following proposition.

Proposition 4.2

Suppose that (i) and (iii) in Lemma 4.1 are satisfied. Let $g_{k,j}(x,y,t,s)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^2_+$ and $g_{k,j,t,s}(x,y) = g_{k,j}(x,y,t,s)$ for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(t,s) \in \mathbb{R}^2_+$. If $2 < p < 2q_0(q_0-1)^{-1}$, then

$$\begin{aligned}
&\left\| S =: \left(\sum_{k,j\in\mathbb{Z}} \int_{I_{k,a}\times I_{j,b}} |\sigma_{t,s} * g_{k,j,t,s}|^2 \frac{dtdds}{ts} \right)^{1/2} \right\|_p \\
&\le C \left\| \left(\sum_{k,j\in\mathbb{Z}} \int_{I_{k,a}\times I_{j,b}} |g_{k,j,t,s}|^2 \frac{dtds}{ts} \right)^{1/2} \right\|_p .
\end{aligned}$$

Proof. Since $p > 2$, there exists a function $h \in L^{(p/2)'}(\mathbb{R}^n \times \mathbb{R}^m)$ such that

$$S = \left(\sum_{k,j\in\mathbb{Z}} \int_{I_{k,a}\times I_{j,b}} |\sigma_{t,s} * g_{k,j,t,s}(x,y)|^2 \frac{dtds}{ts} h(x,y)dxdy \right)^{1/2}.$$

By the same argument as above, we have

$$\begin{aligned}
S &\le \left(\sum_{k,j\in\mathbb{Z}} \int_{I_{k,a}\times I_{j,b}} |\sigma_{t,s}| * |g_{k,j,t,s}(x,y)|^2 \frac{dtds}{ts} h(x,y)dxdy \right)^{1/2} \\
&\le \left(\left\| \sum_{k,j\in\mathbb{Z}} \int_{I_{k,a}\times I_{j,b}} |g_{k,j,t,s}|^2 \frac{dtds}{ts} \right\|_{p/2} \left\| \sigma^*(\tilde{h}) \right\|_{(p/2)'} \right)^{1/2}
\end{aligned}$$

which ends the proof of the Proposition. □

Now we are ready to prove (4.5) for the case $2q_0(q_0+1)^{-1} < p < 2$. By a duality argument, there exist functions $g_{k,j,t,s}(x,y) = g_{k,j}(x,y,t,s)$ defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^2_+$ such that $\left\| \left\| \|g_{k,j,t,s}\|_{L^2\left(I_{k,a}\times I_{k,b}, \frac{dtds}{ts}\right)} \right\|_{\ell^2} \right\|_{L^{p'}} \le 1$ and

$$\begin{aligned}
\|F_{k,j}(f)\|_p &= \int_{\mathbb{R}^n\times\mathbb{R}^m} \sum_{k,j\in\mathbb{Z}} \int_{I_{k,a}\times I_{j,b}} \left(\sigma_{t,s} * (\Phi_{k+l} \otimes \Psi_{j+r}) * f\right)(x,y) \\
&\quad \times g_{k,j,t,s}(x,y) \frac{dtds}{ts} dxdy.
\end{aligned}$$

By changing variables, Proposition 4.2, Littlewood-Paley theory and Hölder's inequality we get (4.5). The proof of Lemma 4.1 is completed. □

5. Proof of Theorem 1.1 (a)

Assume that $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q > 1$ and satisfies (1.1). Thus Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu$, where $\lambda_\mu \in \mathbb{C}$, b_μ is a q-block supported on an interval I_μ on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ and $M_q^{(0,0)}(\{\lambda_\mu\}) < \infty$. To each block function $b_\mu(\cdot,\cdot)$, let $\tilde{b}_\mu(\cdot,\cdot)$ be a function defined by

$$\begin{aligned}\tilde{b}_\mu(x,y) &= b_\mu(x,y) - \int_{\mathbb{S}^{n-1}} b_\mu(u,y)d\sigma(u) - \int_{\mathbb{S}^{m-1}} b_\mu(x,v)d\sigma(v) \\ &\quad + \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}} b_\mu(u,v)d\sigma(u)d\sigma(v). \qquad (5.1)\end{aligned}$$

Let $\mathbb{J} = \{\mu \in \mathbb{N} : |I_\mu| < e^{-1}\}$. Let $\tilde{b}_0 = \Omega - \sum_{\mu\in\mathbb{J}} \lambda_\mu \tilde{b}_\mu$. Then it is easy to verify that the following hold:

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \tilde{b}_\mu(u,\cdot)\ d\sigma(u) &= \int_{\mathbb{S}^{m-1}} \tilde{b}_\mu(\cdot,v)\ d\sigma(v) = 0; && (5.2)\\ \left\|\tilde{b}_\mu\right\|_q &\le C\left|I_\mu\right|^{-1/q'}; && (5.3)\\ \left\|\tilde{b}_\mu\right\|_1 &\le C; && (5.4)\\ \Omega &= \sum_{\mu\in\mathbb{J}\cup\{0\}} \lambda_\mu \tilde{b}_\mu && (5.5)\end{aligned}$$

for all $\mu \in \mathbb{J} \cup \{0\}$, where I_0 is an interval on $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ with $|I_0| = e^{-2}$ and C is a positive constant independent of μ.

Define the family of measures $\sigma^{(\mu)} = \{\sigma_{t,s,\mu} : t,s \in \mathbb{R}_+\}$ and the corresponding maximal function on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^m} f d\sigma_{t,s,\mu} &= \frac{1}{ts}\int_{\{|x|\le t,|y|\le s\}} \frac{\tilde{b}_\mu(x,y)h(|x|,|y|)}{|x|^{n-1}|y|^{m-1}} f(x,y)dxdy\ ;\\ \sigma_\mu^*(f) &= \sup_{t,s\in\mathbb{R}_+} ||\sigma_{t,s,\mu}| * f|.\end{aligned}$$

For $\mu \in \mathbb{J} \cup \{0\}$, let $B_\mu = \log(|I_\mu|^{-1})$. Then it is easy to see that

$$\|\sigma_{t,s,\mu}\| \le C \quad \text{ for } t,s \in \mathbb{R}_+, \qquad (5.6)$$

which in turn implies

$$\int_{2^{kB_\mu}}^{2^{(k+1)B_\mu}} \int_{2^{jB_\mu}}^{2^{(j+1)B_\mu}} |\hat{\sigma}_{t,s,\mu}(\xi,\eta)|^2 \frac{dtds}{ts} \le CB_\mu^2 \qquad (5.7)$$

for some positive constant independent of μ.

By the cancellation properties of $\tilde{b}_\mu$, we have

$$\begin{aligned}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)| &\le \frac{1}{ts}\int_{\{|x|\le t,|y|\le s\}}\left|e^{-i\xi\cdot x}-1\right|\frac{\left|\tilde{b}_\mu(x,y)\right|h(|x|,|y|)}{|x|^{n-1}|y|^{m-1}}dxdy\\ &\le C\,|\xi t|\,.\end{aligned}$$

Thus,

$$\int_{2^{kB_\mu}}^{2^{(k+1)B_\mu}}\int_{2^{jB_\mu}}^{2^{(j+1)B_\mu}}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)|^2\frac{dtds}{ts}\le C2^{2B_\mu}\left|2^{kB_\mu}\xi\right|^2.$$

By combing this estimate with (5.7), we get

$$\int_{2^{kB_\mu}}^{2^{(k+1)B_\mu}}\int_{2^{jB_\mu}}^{2^{(j+1)B_\mu}}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)|^2\frac{dtds}{ts}\le CB_\mu^2\left|2^{kB_\mu}\xi\right|^{1/B_\mu}. \tag{5.8}$$

Similarly,

$$\int_{2^{kB_\mu}}^{2^{(k+1)B_\mu}}\int_{2^{jB_\mu}}^{2^{(j+1)B_\mu}}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)|^2\frac{dtds}{ts}\le CB_\mu^2\left|2^{jB_\mu}\eta\right|^{1/B_\mu}. \tag{5.9}$$

On the other hand, by the proof of Corollary 4.1 of [13],

$$\left|\frac{1}{t}\int_{|x|\le t}e^{-i\xi\cdot x}\frac{\tilde{b}_\mu(x,y)h(|x|,|y|)}{|x|^{n-1}}dx\right|\le C\,|t\xi|^{-\alpha/2}\left(\int_{\mathbb{S}^{n-1}}\left|\tilde{b}_\mu(x,y)\right|^q d\sigma(x)\right)^{1/q}$$

for some positive constant C and α with $\alpha q'<1$. Thus,

$$\begin{aligned}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)| &\le \frac{1}{s}\int_{|y|\le s}\frac{1}{|y|^{m-1}}\left|\frac{1}{t}\int_{|x|\le t}e^{-i\xi\cdot x}\frac{\tilde{b}_\mu(x,y)h(|x|,|y|)}{|x|^{n-1}}dx\right|dy\\ &\le C\,|t\xi|^{-\alpha/2}\left\|\tilde{b}_\mu\right\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})}\\ &\le C\,|t\xi|^{-\alpha/2}\left|I_\mu\right|^{-\frac{1}{q'}}\end{aligned}$$

which easily implies

$$\int_{2^{kB_\mu}}^{2^{(k+1)B_\mu}}\int_{2^{jB_\mu}}^{2^{(j+1)B_\mu}}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)|^2\frac{dtds}{ts}\le CB_\mu^2\left|2^{kB_\mu}\xi\right|^{-\alpha}\left|I_\mu\right|^{-2/q'}.$$

Therefore, by combining the last estimate with the trivial estimate (5.7) we obtain

$$\int_{2^{kB_\mu}}^{2^{(k+1)B_\mu}}\int_{2^{jB_\mu}}^{2^{(j+1)B_\mu}}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)|^2\frac{dtds}{ts}\le CB_\mu^2\left|2^{kB_\mu}\xi\right|^{-\alpha/B_\mu}. \tag{5.10}$$

Similarly,

$$\int_{2^{kB_\mu}}^{2^{(k+1)B_\mu}}\int_{2^{jB_\mu}}^{2^{(j+1)B_\mu}}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)|^2\frac{dtds}{ts}\le CB_\mu^2\left|2^{jB_\mu}\eta\right|^{-\alpha/B_\mu}. \tag{5.11}$$

By (5.7)–(5.11) we obtain

$$\int_{2^{kB_\mu}}^{2^{(k+1)B_\mu}}\int_{2^{jB_\mu}}^{2^{(j+1)B_\mu}}|\hat{\sigma}_{t,s,\mu}(\xi,\eta)|^2\frac{dtds}{ts}\le CB_\mu^2\left|2^{kB_\mu}\xi\right|^{\pm\alpha/B_\mu}\left|2^{jB_\mu}\eta\right|^{\pm\alpha/B_\mu}. \tag{5.12}$$

By the boundedness of the strong maximal operator on $\mathbb{R} \times \mathbb{R}$ we obtain

$$\left\|\sigma_\mu^*(f)\right\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_q \|f\|_{L^q(\mathbb{R}^n\times\mathbb{R}^m)} \quad \text{for } 1 < q \leq \infty, \tag{5.13}$$

where C_q is independent of μ.

By (5.6), (5.12)–(5.13) and invoking Lemma 4.1, we obtain

$$\left\|\mathcal{M}_{\tilde{b}_\mu,h}(f)\right\|_p \leq C_p B_\mu \|f\|_p \text{ for } 1 < p < \infty. \tag{5.14}$$

Finally, by (5.5), Minkowiski's inequality and (5.14), we have

$$\begin{aligned}
\left\|\mathcal{M}_{\Omega,h}(f)\right\|_p &\leq \sum_{\mu\in\mathbb{J}\cup\{0\}} |\lambda_\mu| \left\|\mathcal{M}_{\tilde{b}_\mu,h}(f)\right\|_p & (5.15)\\
&\leq C_p \sum_{\mu\in\mathbb{J}\cup\{0\}} |\lambda_\mu| B_\mu & (5.16)\\
&\leq C_p \|\Omega\|_{B_q^{(0,0)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \|f\|_p & (5.17)
\end{aligned}$$

for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n \times \mathbb{R}^m)$ which ends the proof of Theorem 1.1 (a). $\square$

References

1. H.M. Al-Qassem and A.J. Al-Salman, A note on Marcinkiewicz integral operators, *J. Math. Anal. Appl.* **282** (2003), 698–710.
2. H.M. Al-Qassem and Y. Pan, L^p boundedness for singular integrals with rough kernels on product domains, *Hokkaido Math. J.* **31** (2002), 555–613.
3. A. Al-Salman, H.M. Al-Qassem, L.C. Cheng, and Y. Pan, L^p bounds for the function of Marcinkiewicz, *Math. Res. Lett.* **9** (2002), 697–700.
4. A. Al-Salman, H.M. Al-Qassem, and Y. Pan, Singular integrals associated to homogeneous mappings with rough kernels, *Hokkaido Math. J.* **33** (2004), 551–569.
5. A. Benedek, A.-P. Calderón, and R. Panzone, Convolution operators on Banach space valued functions, *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962), 356–365.
6. J. Chen, Y. Ding, and D. Fan, L^p boundedness of the rough Marcinkiewicz integral on product domains, *Chinese J. Contemp. Math.* **21** (2000), 47–54.
7. J. Chen, D. Fan, and Y. Ying, Rough Marcinkiewicz integrals with $L(\log^+ L)^2$ kernels on product spaces, *Adv. Math. (China)* **30** (2001), 179–181.
8. Y. Choi, Marcinkiewicz integrals with rough homogeneous kernels of degree zero in product domains, *J. Math. Anal. Appl.* **261** (2001), 53–60.
9. Y. Ding, L^2-boundedness of Marcinkiewicz integral with rough kernel, *Hokkaido Math. J.* **27** (1998), 105–115.
10. Y. Ding, On Marcinkiewicz integral, *Proc. of the Conference Singular Integrals and Related Topics, III,* Oska, Japan (2001), 28–38.
11. Y. Ding, D. Fan, and Y. Pan, L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernels, *Acta Math. Sin. (Engl. Ser.)* **16** (2000), 593–600.
12. Y. Ding, D. Fan, and Y. Pan, Weighted boundedness for a class of rough Marcinkiewicz integrals, *Indiana Univ. Math. J.* **48** (1999), 1037–1055.
13. J. Duoandikoetxea, Multiple singular integrals and maximal functions along hypersurfaces, *Ann. Inst. Fourier (Grenoble)* **36** (1986), 185–206.
14. J. Duoandikoetxea and J.L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, *Invent. Math.* **84** (1986), 541–561.

15. L. Hörmander, Estimates for translation invariant operators in L^p spaces, *Acta Math.* **104** (1960), 93–140.
16. Y. Jiang and S. Lu, A class of singular integral operators with rough kernel on product domains, *Hokkaido Math. J.* **24** (1995), 1–7.
17. M. Keitoku and E. Sato, Block spaces on the unit sphere in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* **119** (1993), 453–455.
18. S. Lu, M.H. Taibleson, and G. Weiss, *Spaces Generated by Blocks*, Beijing Normal University Press, 1989, Beijing.
19. Y. Meyer, M.H. Taibleson, and G. Weiss, Some functional analytic properties of the spaces B_q generated by blocks, *Indiana Univ. Math. J.* **34** (1985), 493–515.
20. S. Sato, Remarks on square functions in the Littlewood-Paley theory, *Bull. Austral. Math. Soc.* **58** (1998), 199–211.
21. E.M. Stein, On the functions of Littlewood-Paley, Lusin and Marcinkiewicz, *Trans. Amer. Math. Soc.* **88** (1958), 430–466.
22. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.
23. M.H. Taibleson and G. Weiss, Certain function spaces connected with almost everywhere convergence of Fourier series, *Conference on Harmonic Analysis in Honor of Antoni Zygmund,* Wadsworth Math. Ser. (1983), 95–113.
24. T. Walsh, On the function of Marcinkiewicz, *Studia Math.* **44** (1972), 203–217.
25. A. Zygmund, On certain integrals, *Trans. Amer. Math. Soc.* **55** (1944), 170–204.