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# Bounds on Castelnuovo-Mumford regularity for divisors on rational normal scrolls

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Abstract

The Castelnuovo-Mumford regularity is one of the most important invariants in studying the minimal free resolution of the defining ideals of the projective varieties. There are some bounds on the Castelnuovo-Mumford regularity of the projective variety in terms of the other basic invariants such as dimension, codimension and degree. This paper studies a bound on the regularity conjectured by Hoa, and shows this bound and extremal examples in the case of divisors on rational normal scrolls.

## 1. Introduction

Let $X$ be a projective scheme of $\mathbb{P}^N_K$ over an algebraic closed field $K$. Let $S = K[x_0, \cdots, x_N]$ be the polynomial ring and $\mathfrak{m} = (x_0, \cdots, x_N)$ be the irrelevant ideal. Then we put $\mathbb{P}^N_K = \mathrm{Proj}(S)$. We denote by $\mathcal{I}_X$ the ideal sheaf of $X$. Let $m$ be an integer. Then $X$ is said to be $m$-regular if $\mathrm{H}^i(\mathbb{P}^N_K, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}^N_K$, introduced by Mumford by generalizing ideas of Castelnuovo, is the least such integer $m$ and is denoted by $\mathrm{reg}(X)$. The interest in this concept stems partly from the well-known fact that $X$ is $m$-regular if and only if for every $p \geq 0$ the minimal generators of the $p^{th}$ syzygy module of the defining ideal $I$ of $X \subseteq \mathbb{P}^N_K$ occur in degree $\leq m+p$, see, e.g., [4]. It is important to study

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upper bounds on the Castelnuovo-Mumford regularity for projective schemes in order to describe the minimal free resolutions of the defining ideals.

In what follows, for a rational number $\ell \in \mathbb{Q}$, we write $\lceil \ell \rceil$ for the minimal integer which is larger than or equal to $\ell$, and $\lfloor \ell \rfloor$ for the maximal integer which is smaller than or equal to $\ell$.

The starting point of our research on the Castelnuovo-Mumford regularity is an inequality $\operatorname{reg}(X) \leq \lceil (\deg(X)-1)/\operatorname{codim}(X) \rceil + 1$ for the ACM, that is, arithmetically Cohen-Macaulay, nondegenerate projective variety $X \subseteq \mathbb{P}_K^N$, which is a consequence of the Uniform Position Lemma for the generic hyperplane section of the projective curve for the characteristic zero case and the corresponding weaker result due to Ballico for the positive characteristic case, see [1, 2]. Moreover, the extremal ACM variety for the bound have been shown to be a variety of minimal degree in [14, 18] if its degree is large enough.

In order to study the regularity bounds for the non-ACM projective variety, we introduce the $k$-Buchsbaum property. Let $k$ be a nonnegative integer. Then $X$ is called $k$-Buchsbaum if the graded $S$-module $\mathrm{M}^i(X) = \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^i(\mathbb{P}_K^N, \mathcal{I}_X(\ell))$, which is called the deficiency module or the Hartshorne-Rao module of $X$, is annihilated by $\mathfrak{m}^k$ for $1 \leq i \leq \dim(X)$, see, e.g., [9, 10]. We call the minimal nonnegative integer $n$, if it exists, such that $X$ is $n$-Buchsbaum, as the Ellia-Migliore-Miró Roig number of $X$ and denote it by $k(X)$, see [3, 12]. Further we define $\tilde{k}(X)$ as the maximal integer $k$ such that all successive hyperplane sections of $X$, that is, $X \cap L$ with $\operatorname{codim}(X \cap L) = \operatorname{codim}(X) + \operatorname{codim}(L)$ for any linear space $L$ of $\mathbb{P}_K^N$, have the $k$-Buchsbaum property, see [5]. Note that $k(X) < \infty$ if and only if $\tilde{k}(X) < \infty$, which is equivalent to saying that $X$ is locally Cohen-Macaulay and equi-dimensional. In recent years upper bounds on the Castelnuovo-Mumford regularity of a projective variety $X$ have been given by several authors in terms of $\dim(X)$, $\deg(X)$, $\operatorname{codim}(X)$ and $k(X)$, see, e.g., [6, 7, 13, 16]. The following bound is the most optimal among the known results. Also, the extremal cases are classified, see, e.g., [3, 12].

**Proposition 1.1** (See [3, 16]).

*Let $X$ be a nondegenerate irreducible reduced projective variety in $\mathbb{P}_K^N$ over an algebraically closed field $K$. Then we have*

$$\operatorname{reg}(X) \leq \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + \max\{k(X) \dim(X), 1\}.$$

*Assume that $X$ is not ACM and that $\deg(X) \geq 2\operatorname{codim}(X)^2 + \operatorname{codim}(X) + 2$. Then the equality holds only if $X$ is a curve on a rational ruled surface.*

This motivates us to state a variation of Hoa's conjecture.

**Conjecture 1.2** ([12]).

*Let $X$ be a nondegenerate projective variety in $\mathbb{P}_K^N$ over an algebraically closed field $K$. Then we have $\operatorname{reg}(X) \leq \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + \max\{\tilde{k}(X), 1\}$. Furthermore, assume that $X$ is not ACM and that $\deg(X)$ is large enough. Then the equality holds only if $X$ is a divisor on a rational normal scroll.*

We remark that the original Hoa's conjecture takes $\bar{k}(X)$ instead of $\tilde{k}(X)$, where $\bar{k}(X)$ is the maximal integer $k$ such that all successive *hypersurface* sections of $X$ have the $k$-Buchsbaum property. The Buchsbaum case, that is, $\tilde{k}(X) = 1$, has been proved in [15, 17, 19].

The purpose of this paper is to prove the conjecture for divisors on rational normal scrolls and to give extremal varieties for all dimensions.

**Theorem 1.3**

*Let $X$ be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ of dimension $r$ over an algebraically closed field $K$. Put $k = \tilde{k}(X)$. Assume that $X$ is a divisor on a rational normal scroll. Then we have* $\operatorname{reg}(X) \le \lceil(\deg(X)-1)/\operatorname{codim}(X)\rceil + \max\{k, 1\}$. *Furthermore, there exist extremal examples for all $r$ and $k$.*

Before proving the inequality and describing the extremal cases for divisors on rational normal scrolls, we prepare the following notations. Let $r \ge 2$ be an integer. Let $\pi : Y = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1_K$ be a projective bundle, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e_r)$ for some $0 \le e_1 \le \cdots \le e_r$. Let $Z$ and $F$ be a minimal section and a fibre respectively. Now we have an embedding of $Y$ in $\mathbb{P}^N_K$ by a very ample divisor $H = Z + nF$ $(n > e_r)$, where $N = rn + r + n - e_1 - \cdots - e_r$. Then $Y$ is called a rational normal scroll. Let $X$ be a divisor on $Y$ linearly equivalent to $aZ + bF$. If $X$ is nondegenerate, then $\Gamma(Y, \mathcal{I}_{X/Y}(1)) = \Gamma(Y, \mathcal{O}_Y((1-a)Z + (n-b)F)) = 0$. In this case we see that either $a = 1$ and $b \ge n+1$, or $a \ge 2$ and $b \ge 1$. Also, we have $\operatorname{codim}(X) = rn + n - e_1 - \cdots - e_r$ and $\deg(X) = (aZ + bF)\cdot(Z + nF)^r = a(rn - e_1 - \cdots - e_r) + b$, because $Z^{r+1} = -e_1 - \cdots - e_r$, $Z^r \cdot F = 1$ and $Z^i \cdot F^{r+1-i} = 0$ for $0 \le i \le r-1$. Under the above conditions, we obtain the following classification of the divisor on a rational normal scroll with its Castelnuovo-Mumford regularity having such upper bound.

**Theorem 1.4**

*Let $X$ be a nondegenerate irreducible reduced divisor on a rational normal scroll in $\mathbb{P}^N_K$ of dimension $r$ constructed as above. Then we have*

$$\operatorname{reg}(X) \le \lceil(\deg(X) - 1)/\operatorname{codim}(X)\rceil + \max\{\tilde{k}(X), 1\}.$$

*Furthermore, assume that $X$ is not ACM. Then the equality holds if and only if $a \ge 1$ and $an + 2 \le b \le an + 1 - (r+1)n - e_1 - \cdots - e_r$.*

This result extends that of [12, Theorem 1.3] and give sharp examples for the conjecture. More precisely, the extremal variety $X$ satisfies $\operatorname{codim}(X) = (r+1)n - e_1 - \cdots - e_r$, $\deg(X) = a(rn - e_1 - \cdots - e_r) + b$, $\tilde{k}(X) = k(X) = \lfloor(b - a_r - 2)/(n - e_r)\rfloor - a + 1$ and $\operatorname{reg}(X) = \lfloor(b - ae_r - 2)/(n - e_r)\rfloor + 2$.

## 2. Proof of main Theorem

This section is devoted to the proof of the Theorem stated in §1.

Notations being as in (1.4), our proof starts with calculating the Castelnuovo-Mumford regularity and the Ellia-Migliore-Miró Roig number of the projective variety.

Let $S$ be the polynomial ring $\Gamma(Y,\mathcal{O}_Y(1))$. Note that $\Gamma(Y,\mathcal{O}_Y(1)) \cong \Gamma(\mathbb{P}^1_K,\mathcal{E}(n))$. Since $Y$ is ACM, the deficiency module $\mathrm{M}^i(X)$ of $X$ in $\mathbb{P}^N_K = \mathrm{Proj}(S)$, $1 \le i \le r$, is isomorphic to $\oplus_{\ell\in\mathbb{Z}}\mathrm{H}^i(Y,\mathcal{I}_{X/Y}(\ell))$ as graded $S$-modules. Thus we have

$$\mathrm{M}^i(X) \cong \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^i(Y,\mathcal{O}_Y((-a+\ell)Z+(-b+\ell n)F)),$$

for $1 \le i \le r$. Let us calculate the intermediate cohomologies.

**Lemma 2.1**

*Under the above condition, we have*

(i) $\mathrm{M}^1(X) \cong \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^1(\mathbb{P}^1_K,\mathrm{Sym}^{\ell-a}(\mathcal{E})\otimes\mathcal{O}_{\mathbb{P}^1_K}(n\ell-b))$,

(ii) $\mathrm{M}^i(X)=0$ *for* $1<i<r$, *and*

(iii) $\mathrm{M}^r(X) \cong \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^0(\mathbb{P}^1_K,(\mathrm{Sym}^{-\ell+a-r-1}(\mathcal{E}))'\otimes\mathcal{O}_{\mathbb{P}^1_K}(n\ell-b+e_1+\cdots+e_r))$.

*Proof.* The assertions immediately follow from [12, (2.13) and (2.14)] and their proofs. □

**Corollary 2.2**

*Under the above condition, we have*

(i) $\mathrm{M}^1(X)_\ell \ne 0$ *if and only if* $a \le \ell \le \lfloor (b-ae_r-2)/(n-e_r)\rfloor$. *In particular,* $\mathrm{M}^1(X)\ne 0$ *if and only if* $b \ge an+2$. *Furthermore,*

(ii) $\mathrm{M}^r(X)_\ell \ne 0$ *if and only if* $\lceil (b-e_1-\cdots-e_{r-1}+(r-a)e_r)/(n-e_r)\rceil \le \ell \le a-r-1$. *In particular,* $\mathrm{M}^r(X)\ne 0$ *if and only if* $b \le (a-r-1)n+e_1+\cdots+e_r$.

*Proof.* Note that $n > e_r$. By (2.1)(i), $\mathrm{M}^1(X)_\ell \ne 0$ if and only if $-a+\ell \ge 0$ and $-b+\ell n \le e_r(-a+\ell)-2$. By (2.1)(iii), $\mathrm{M}^r(X)_\ell \ne 0$ if and only if $-a+\ell \le -r-1$ and $-b+\ell n \ge e_r(-a+\ell)+re_r-e_1-\cdots-e_{r-1}$. □

*Remark 2.3.* From (2.2), $X$ is ACM if and only if $(a-r-1)n+e_1+\cdots+e_r+1 \le b \le an+1$. If $b \ge an+2$, then $\mathrm{M}^j(X)=0$ for $j\ne 1$, and if $b \le (a-r-1)n+e_1+\cdots+e_r$, then $\mathrm{M}^j(X)=0$ for $j \ne r$. But both cases are not ACM.

**Lemma 2.4**

*Under the above condition,* $\mathrm{H}^{r+1}(\mathbb{P}^N_K,\mathcal{I}_X(\ell))\ne 0$ *if and only if* $\ell \le a-r-1$ *and* $\ell \le \lfloor (b-2-e_1-\cdots-e_r)/n\rfloor$.

*Proof.* From the short exact sequence

$$0 \to H^{r+1}_*(\mathcal{I}_X) \to H^{r+1}_*(\mathcal{I}_{X/Y}) \to H^{r+2}_*(\mathcal{I}_Y) \to 0,$$

we see that $H^{r+1}_*(\mathcal{I}_X)$ is the kernel of the homomorphism $H^1(\mathbb{P}^1_K,\mathrm{Sym}^{-\ell+a-r-1}(\mathcal{E})'\otimes\mathcal{O}_{\mathbb{P}^1_K}(n\ell-b+e_1+\cdots+e_r)) \to H^1(\mathbb{P}^1_K,\mathrm{Sym}^{-\ell-r-1}(\mathcal{E})'\otimes\mathcal{O}_{\mathbb{P}^1_K}(n\ell+e_1+\cdots+e_r))$. Thus $\mathrm{H}^{r+1}(\mathcal{I}_X(\ell))\ne 0$ if and only if $-\ell+a-r-1\ge 0$ and $n\ell-b+e_1+\cdots+e_r \le -2$. □

*Remark 2.5.* The $a$-invariant of the coordinate ring $R$ of $X$ is defined as $a(R) = \max\{\ell \mid [\mathrm{H}^{\dim R}_{R_+}(R)]_\ell \ne 0\}$. Note that $\mathrm{H}^{r+1}_{R_+}(R) \cong \mathrm{H}^{r+1}_*(\mathbb{P}^N_K,\mathcal{I}_X)$. Therefore we have $a(R)=\min\{a-r-1,\lfloor (b-2-e_1-\cdots-e_r)/n\rfloor\}$.

From now on, we assume that $X$ is not ACM.

**Corollary 2.6**

*Under the above conditions,* $k(X) = \lfloor (b - ae_r - 2)/(n - e_r) \rfloor - a + 1$ *and* $\operatorname{reg}(X) = \lfloor (b - ae_r - 2)/(n - e_r) \rfloor + 2$ *if* $b \geq an + 2$, *and*

$$k(X) = a - r - 1 - \lceil (b - e_1 - \cdots - e_{r-1} + (r-1)e_r)/(n - e_r) \rceil + 1,$$

*and* $\operatorname{reg}(X) = a, a + 1$ *if* $b \leq (a - r - 1)n + e_1 + \cdots + e_r$.

*Proof.* It immediately follows from (2.1), (2.2), (2.3) and (2.4). □

**Lemma 2.7**

*Under the above conditions, we have* $\tilde{k}(X) = k(X)$.

*Proof.* It immediately follows from [8, (2.4)] and (2.1). □

Before proving the main theorem, we state a basic fact on the regularity bound.

**Proposition 2.8** ([16]).

*Let* $X$ *be a nondegenerate projective variety of dimension* $r$ *with the coordinate ring* $R$. *Let* $s$ *be a fixed integer with* $1 \leq s \leq r$. *Assume that* $X$ *is not ACM and that the deficiency module* $\mathrm{M}^i(X)$ *vanishes for any* $i \neq s$. *Then we have*

$$\operatorname{reg}(X) \leq a(R/hR) + r + 1 + k(X) \leq \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + k(X),$$

*where* $h$ *is a general linear form of* $R$.

*Proof of Theorem 1.4.* The inequality $\operatorname{reg}(X) \leq \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + \tilde{k}(X)$ follows straightforward from (2.3), (2.7) and (2.8).

First, in order to describe when the equality holds, we consider the case $b \leq (a - r - 1)n + e_1 + \cdots + e_r$. In this case, the intermediate cohomologies appear only in $\mathrm{M}^r(X)$, and we note that $\max\{\ell \mid [\mathrm{M}^r(X)]_\ell \neq 0\} = a - r - 1$ by (2.2). Also, we see that $a(R) \leq a - r - 1$ by (2.5). If $a(R) = a - r - 1$, then $\operatorname{reg}(X) = (a - r - 1) + 1 + r + 1 = a + 1$ and $a(R/hR) = a - r$. If $a(R) < a - r - 1$, then $\operatorname{reg}(X) = (a - r - 1) + 1 + r = a$ and $a(R/hR) = a - r - 1$. In fact, by the structure of $\mathrm{M}^r(X)$, see (2.1), we have $[\mathrm{M}^r(X)/h\mathrm{M}^r(X)]_{a-r-1} \neq 0$. In any case, we have

$$\begin{aligned}\operatorname{reg}(X) = a(R/hR) + r + 1 &\leq \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + 1 \\ &\leq \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + \tilde{k}(X),\end{aligned}$$

and the equality holds only if $\tilde{k}(X) = 1$, which is the Buchsbaum case and is classified by [15].

Next, for the case $b \geq an + 2$, we see that $\operatorname{reg}(X) = \lfloor (b - ae_r - 2)/(n - e_r) \rfloor + 2$ and $\tilde{k}(X) = \lfloor (b - ae_r - 2)/(n - e_r) \rfloor - a + 1$ by (2.6) and (2.7). Thus the equality holds if and only if $\lceil (a(rn - e_1 - \cdots - e_r) + b - 1)/((r+1)n - e_1 - \cdots - e_r) \rceil = a + 1$, which is equivalent to saying that $-(rn + n - e_1 - \cdots - e_r) + 1 \leq -na - (rn + n - e_1 - \cdots - e_r) + b + 1 \leq 0$. Hence the assertion is proved. □

EXAMPLE 2.9 ([11]): Let $Y = \mathbb{P}^1_K \times \mathbb{P}^1_K \times \mathbb{P}^1_K$ be the Segre embedding in $\mathbb{P}^9_K$. Let $X$ be an irreducible reduced divisor linearly equivalent to

$$p_1^*\mathcal{O}_{\mathbb{P}^1_K}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1_K}(a+b) \otimes p_3^*\mathcal{O}_{\mathbb{P}^1_K}(a+2b),$$

where $a \geq 1$ and $b \geq 2$. Then $k(X) = b$ and $\tilde{k}(X) > b$.

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