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*Collect. Math.* **56**, 1 (2005), 57–83


# An application of Mellin-Barnes type integrals to the mean square of Lerch zeta-functions II

Masanori Katsurada

*Mathematics, Hiyoshi Campus, Keio University*
*4–1–1 Hiyoshi, Kouhoku-ku, Yokohama 223–8521, Japan*
E-mail: katsurad@hc.cc.keio.ac.jp; masanori@math.hc.keio.ac.jp

Received April 16, 2004. Revised September 27, 2004

Abstract

For the Lerch zeta-function $\phi(s,x,\lambda)$ defined below, the multiple mean square of the form (1.1), together with its discrete and hybrid analogues, (1.2) and (1.3), are investigated by means of Atkinson's [2] dissection method applied to the product $\phi(u,x,\lambda)\phi(v,x,-\lambda)$, where $u$ and $v$ are independent complex variables (see (4.2)). A complete asymptotic expansion of (1.1) as $\operatorname{Im} s \to \pm\infty$ is deduced from Theorem 1, while those of (1.2) and (1.3) as $q \to \infty$ and (at the same time) as $\operatorname{Im} s \to \pm\infty$ are deduced from Theorems 2 and 3 respectively. In the proofs, Atkinson's method above is enhanced by Mellin-Barnes type of integral formulae (see (4.1)), which further enable us systematic use of various properties of hypergeometric functions (see Section 5); especially in the proof of Theorem 1 crucial rôles are played by Lemmas 3 and 5.

## 1. Introduction

Let $s$ be a complex variable, and let $x$ and $\lambda$ be real parameters with $x>0$. We use the notation $e(\lambda)=e^{2\pi i\lambda}$ hereafter. The Lerch zeta-function $\phi(s,x,\lambda)$ is defined by

$$\phi(s,x,\lambda)=\sum_{n=0}^{\infty}e(\lambda n)(n+x)^{-s}\qquad(\operatorname{Re} s>1),$$

and its meromorphic continuation over the whole $s$-plane (cf. [24]). It is an entire function for $\lambda\in\mathbb{R}\setminus\mathbb{Z}$, while if $\lambda\in\mathbb{Z}$ it reduces to the Hurwitz zeta-function $\zeta(s,x)$, and so $\zeta(s,1)=\zeta(s)$ is the Riemann zeta-function. We remark that the order of

*Keywords:* Riemann zeta-function, Hurwitz zeta-function, Lerch zeta-function, Mellin-Barnes integral, mean square, asymptotic expansion.

*MSC2000:* Primary 11M35; Secondary 11M06.

Research supported in part by Grants-in-Aid for Scientific Research (No. 13640041), the Ministry of Education, Culture, Sports, Science and Technology of Japan.

the variables in $\phi$ above differs from the usual notation, in order to retain notational consistency with other terminology.

The present paper proceeds further with our previous study [9] of the mean squares of Lerch zeta-functions. We shall prove a general explicit formula for a multiple average of the product $\phi(u,a+x,\lambda)\phi(v,a+x,-\lambda)$, where $u$ and $v$ are independent complex variables and $a$ is a positive real number. This formula leads to a complete asymptotic expansion of the multiple mean square

$$\int_0^1\cdots\int_0^1|\phi(s,a+x_1+\cdots+x_m,\lambda)|^2dx_1\cdots dx_m \tag{1.1}$$

for any positive integer $m$, in the descending order of $\operatorname{Im}s$ as $\operatorname{Im}s\to\pm\infty$. The method of the present paper is also applicable to treat a discrete analogue of (1.1) in the form

$$q^{-m}\sum_{r_1=0}^{q-1}\cdots\sum_{r_m=0}^{q-1}\Bigl|\phi\Bigl(s,\frac{a}{q^m}+\frac{r_1}{q}+\cdots+\frac{r_m}{q^m},\lambda\Bigr)\Bigr|^2, \tag{1.2}$$

and their hybridization (with $m=2$)

$$\int_0^1 q^{-1}\sum_{r=0}^{q-1}\Bigl|\phi\Bigl(s,\frac{a+r+x}{q},\lambda\Bigr)\Bigr|^2dx \tag{1.3}$$

for any integer $q\geq 1$. We shall prove complete asymptotic expansions of these mean values in the descending order of $q$ as $q\to\infty$; a bonus here is that the expansion of (1.3) gives (at the same time) a complete asymptotic expansion in the descending order of $\operatorname{Im}s$ as $\operatorname{Im}s\to\pm\infty$. When $m=1$ and $a=1$, the existence of complete asymptotic expansions of (1.1) and (1.2) were shown in [9]; however, it is rather remarkable that similar asymptotic series still exists for more general multiple averages such as (1.1) and (1.2).

The paper is organized as follows. After a brief overview of the history of research, we state our first main result (Theorem 1), which implies the asymptotic expansion of (1.1), in the next section. The second and the third main results (Theorems 2 and 3), which imply the asymptotic expansions of (1.2) and (1.3) respectively, are stated in Section 3. A fundamental formula for the proofs is prepared in Section 4. Sections 5 and 6 are devoted to the proofs of Theorem 1 and its corollaries, while Theorems 2, 3 and their corollaries are proved in Section 7. A supplementary argument is given in the final section.

The author would like to thank the referee for valuable comments and refinements of the first version of the present paper.

## 2. History and the first main result

Let $u$ and $v$ be independent complex variables, and let $\Gamma(s)$ denote the gamma function. Mikolás [25] in 1956 proved the formula

$$\int_0^1 \zeta(u,x)\zeta(v,x)dx = 2(2\pi)^{u+v-2}\Gamma(1-u)\Gamma(1-v) \times \cos\left\{\frac{1}{2}\pi(u-v)\right\}\zeta(2-u-v) \tag{2.1}$$

if $\max(\operatorname{Re} u, \operatorname{Re} v, \operatorname{Re}(u+v)) < 1$; otherwise the integral diverges since $\zeta(s,x)$ has a singularity at $x=0$ (see also [26] for variants of (2.1)). It is hence natural to consider the function $\zeta_1(s,x) = \zeta(s,x) - x^{-s}$ for which the singularity in $x$ is removed. The mean square

$$I(s) = \int_0^1 |\zeta_1(s,x)|^2 dx$$

was already studied in 1952 by Koksma-Lekkerkerker [23], who proved that $I(1/2+it) = O(\log t)$ for $t \geq 2$. Improvements upon this result were due to Balasubramanian [3], Rane [29] and Sitaramachandrarao [30] up to the end of 1980's. (Sitaramachandrarao's result was announced on page 28 of Hardy-Ramanujan Journal **10** (1987), but it seems unpublished.) Next progress of the research were made in the first half of 1990's by several mathematicians, independently of each other. Zhang [33] obtained an asymptotic formula for $I(1/2+it)$ with the error term $O(t^{-1})$, while Andersson [1] proved an explicit formula for the mean value $\int_0^1 \zeta_1(u,x)\zeta_1(v,x)dx$, the remainder term of which involves the sequence $\zeta(u+n)$ and $\zeta(v+n)$ with $n = 0,1,\ldots$, and it leads to the same error estimate $O(t^{-1})$ for $I(s)$ as in [33]. Another explicit formula for the same mean value was derived by Katsurada-Matsumoto [17, 18]; a merit of this formula is that it implies a complete asymptotic expansion of $I(s)$ in the descending order of $\operatorname{Im} s$. It is also applied in [19] to study $I(s)$ especially when $s$ is an integer, and further in [20] to derive an asymptotic formula for the mean square of the derivatives $(\partial/\partial s)^h \zeta_1(s,x)$ for any $h \geq 1$. A discrete analogue of $I(s)$ was studied in [16]. Very recently the direction of Mikolás has been revisited and further pursued by Espinosa-Moll [6, 7].

As for asymptotic aspects of Lerch zeta-functions, hybrid-type mean value theorems for the weighted mean square

$$\int_0^\infty |\phi(\sigma+it,x,\lambda)|^2 e^{-2\delta t}dt$$

as $\delta \to +0$ were proved by Klusch [21, 22], while an asymptotic formula for the mean square

$$I(s;\lambda) = \int_0^1 |\phi_1(s,x,\lambda)|^2 dx,$$

where $\phi_1(s,x,\lambda) = \phi(s,x,\lambda) - x^{-s}$, with the error term $O(t^{-1})$ was derived by Zhang [32]. The author [9] obtained a complete asymptotic expansion of $I(s;\lambda)$ in the descending order of $\operatorname{Im} s$; this is again a consequence of an explicit formula for the mean value $\int_0^1 \phi_1(u,x,\lambda)\phi_1(v,x,-\lambda)dx$ proved in [9], where Mellin-Barnes type of integrals were used, combined with Atkinson's [2] method. This type of integrals were first applied by Motohashi to investigate higher power moments of zeta functions (see for e.g., [27, 28]). It is worth while noting that applications of the integrals have advantage over heuristic treatments, in studying asymptotic aspects and transformation properties of zeta and theta functions (see also [10, 15]). Egami-Matsumoto [4] recently applied this type of integrals to discrete analogues of higher power moments of $\zeta(s,x)$.

From the observation that $\phi_1(s,x,\lambda)=e(\lambda)\phi(s,1+x,\lambda)$, we are naturally led to extend the formulations above by Mikolás, Koksma-Lekkerkerker and others to a more general (multiple) average

$$I_m(u,v;a,\lambda)=\int_0^1\cdots\int_0^1\phi(u,a+x_1+\cdots+x_m,\lambda)\times\phi(v,a+x_1+\cdots+x_m,-\lambda)dx_1\cdots dx_m, \tag{2.2}$$

which seems to be interesting from a (multiple) statistical point of view. Let $\zeta_\lambda(s)=e(\lambda)\phi(s,1,\lambda)$ be the exponential zeta-function, and $(s)_n=\Gamma(s+n)/\Gamma(s)$ for any integer $n$ Pochhammer's symbol. Note in particular that $(s)_{-m}=1/(s-1)\cdots(s-m)$ for $m=1,2,\ldots$. Then our first main result can be stated as

**Theorem 1**

*Let $I_m(u,v;a,\lambda)$ be defined by (2.2), where $m$ is any positive integer, $u$ and $v$ are independent complex variables, and $a$ and $\lambda$ are real numbers with $a>0$. Define the set $\widetilde{E}_m\subset\mathbb{C}^2$ by*

$$\widetilde{E}_m=\{(u,v);\ u+v\in\mathbb{Z},\ u+v\le\max(2,m)\}\cup\{(u,v);\ u\in\mathbb{Z}\text{ or }v\in\mathbb{Z}\}. \tag{2.3}$$

*Then for any integer $N\ge1$, in the region $1-N<\operatorname{Re}u<m+N$ and $1-N<\operatorname{Re}v<m+N$ except the points of $\widetilde{E}_m$ the formula*

$$\begin{aligned}I_m(u,v;a,\lambda)&=-\frac{1}{(1-u-v)_m}\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}(a+j)^{m-u-v}\\&\quad+R(u,v;\lambda)+R(v,u;-\lambda)-\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}\\&\quad\times\{S_{m,N}(u,v;a+j,\lambda)+S_{m,N}(v,u;a+j,-\lambda)\\&\quad+T_{m,N}(u,v;a+j,\lambda)+T_{m,N}(v,u;a+j,-\lambda)\}\end{aligned} \tag{2.4}$$

*holds. Here*

$$R(u,v;\lambda)=\Gamma(u+v-1)\zeta_\lambda(u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)}, \tag{2.5}$$

$$S_{m,N}(u,v;x,\lambda)=\sum_{n=0}^{N-1}\frac{(u)_n(m)_n}{(1-v)_{m+n}n!}x^{m+n-v}e(\lambda)\phi(u+n,1+x,\lambda) \tag{2.6}$$

*and*

$$T_{m,N}(u,v;x,\lambda)=\frac{x^{m+N-v}}{(m-1)!}\frac{(u)_N}{(1-v)_{m+N-1}}\sum_{l=1}^{\infty}e(\lambda l)f_{0,l,m,N}(u,v;x), \tag{2.7}$$

*where* $f_{0,l,m,N}(u,v;x)$ *is defined by* (2.10) *below with* $K=0$. *Furthermore, for any integer* $K\geq 0$ *the expression*

(2.8)
$$T_{m,N}(u,v;x,\lambda)=\frac{x^{m+N-v}}{(m-1)!}\times\Biggl\{\sum_{k=1}^{K}\frac{(-1)^{k-1}(m+1-u-v)_{k-1}(u)_{N-k}}{(1-v)_{m+N-1}}\sum_{l=1}^{\infty}e(\lambda l)d_{k,l,m,N}(u;x)+\frac{(-1)^{K}(m+1-u-v)_{K}(u)_{N-K}}{(1-v)_{m+N-1}}\sum_{l=1}^{\infty}e(\lambda l)f_{K,l,m,N}(u,v;x)\Biggr\}$$

*follows in the same region of* $(u,v)$ *with*

$$d_{k,l,m,N}(u;x)=\frac{1}{(l+x)^{u+N-k}}\left(\frac{\partial}{\partial\xi}\right)^{m-1}\frac{(1+\xi)^{m+N-1}}{(l-x\xi)^{k}}\Bigg|_{\xi=0} \tag{2.9}$$

*and*

$$f_{K,l,m,N}(u,v;x)=\left(\frac{\partial}{\partial\xi}\right)^{m-1}\frac{(1+\xi)^{m+N-1}}{(l-x\xi)^{u+v-m}}\times\int_{l}^{\infty}\frac{(y-x\xi)^{u+v-m-K-1}}{(y+x)^{u+N-K}}dy\Bigg|_{\xi=0}, \tag{2.10}$$

*where the empty sums are to be regarded as null.*

*Remark 1.* The exceptional set $\widetilde{E}_m$ in (2.3) is defined by collecting all singular points of the factors on the right side of (2.4); formulae similar to (2.4) for the exceptional points of $\widetilde{E}_m$ can be deduced as the limiting cases of Theorem 1 (see Corollaries 1.4 and 1.5 below).

*Remark 2.* On the right side of (2.10), each integral (differentiated with respect to $\xi$) converges absolutely in $\operatorname{Re}v<m+N$, since the integrand is at most of order $O(y^{\operatorname{Re}v-m-N-1})$ as $y\to\infty$. It is further shown that $d_{k,l,m,N}(u;x)$ and $f_{K,l,m,N}(u,v;x)$ are both of order $O(l^{-\operatorname{Re}u-N})$ as $l\to\infty$. The expressions in (2.7) and (2.8) are hence valid for $\operatorname{Re}u>1-N$ and $\operatorname{Re}v<m+N$.

*Remark 3.* The case $m=1$ and $a=1$ of Theorem 1 reduces to [9, Theorem 3], and further to [18, Theorem] if $\lambda\in\mathbb{Z}$.

Theorem 1 yields various consequences, which are stated in the following Corollaries 1.1–1.7. The limiting case $N\to\infty$ of Theorem 1 is stated in Corollary 1.1, while Corollary 1.2 is the case $u=\sigma+it$ and $v=\sigma-it$ of Theorem 1, where the expression (2.8) particularly gives a complete asymptotic expansion as $t\to\pm\infty$. Corollary 1.3 is the limiting case $N\to\infty$ of Corollary 1.2. Two important exceptional cases $\sigma=1/2$ and $\sigma=1$ of Corollary 1.2 are supplemented in Corollaries 1.4 and 1.5 respectively. The remaining Corollaries 1.6 and 1.7 show that the direction of Mikolás [25] can be treated as the limiting case $a\to+0$ of Theorem 1.

We shall in fact prove in Section 6 that $\lim_{N\to\infty}T_{m,N}(u,v;x,\lambda)=0$ for any fixed $(u,v)\in\mathbb{C}^2\setminus\widetilde{E}_m$. The limiting case $N\to\infty$ of Theorem 1 therefore yields.

**Corollary 1.1**

*Let $m$, $u$, $v$, $a$, $\lambda$, $I_m$, $\widetilde{E}_m$ and $R$ be as in Theorem 1. Then for any $(u,v) \in \mathbb{C}^2 \setminus \widetilde{E}_m$ the formula*

$$
\begin{aligned}
I_m(u,v;a,\lambda) \\
&= -\frac{1}{(1-u-v)_m} \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} (a+j)^{m-u-v} \\
&\quad + R(u,v;\lambda) + R(v,u;-\lambda) - \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} \\
&\quad \times \{S_m^*(u,v;a+j,\lambda) + S_m^*(v,u;a+j,-\lambda)\}
\end{aligned} \tag{2.11}
$$

*holds, where*

$$
S_m^*(u,v;x,\lambda) = \sum_{n=0}^{\infty} \frac{(u)_n (m)_n}{(1-v)_{m+n} n!} x^{m+n-v} e(\lambda) \phi(u+n, 1+x, \lambda). \tag{2.12}
$$

*Remark.* The case $m = 1$ and $a = 1$ of this corollary reduces to [9, Corollary 5], and further to [18, Corollary 4] if $\lambda \in \mathbb{Z}$.

When $u = \sigma + it$ and $v = \sigma - it$, Theorem 1 particularly yields a complete asymptotic expansion of (1.1) in the descending order of $t$ as $t \to \pm\infty$:

**Corollary 1.2**

*Let $m$, $a$, $\lambda$, $R$, $S_{m,N}$ and $T_{m,N}$ be as in Theorem 1. Define the set $E_m \subset \mathbb{C}$ by*

$$
E_m = \{n/2 + it;\ n \in \mathbb{Z},\ n \le \max(1, m/2),\ t \in \mathbb{R}\} \cup \mathbb{Z}. \tag{2.13}
$$

*Then for any integer $N \ge 1$, in the region $1 - N < \sigma < m + N$ and $t \in \mathbb{R}$ except the points of $E_m$ the formula*

$$
\begin{aligned}
\int_0^1 \cdots \int_0^1 |\phi(\sigma+it, a+x_1+\cdots+x_m, \lambda)|^2 dx_1 \cdots dx_m \\
&= -\frac{1}{(1-2\sigma)_m} \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} (a+j)^{m-2\sigma} \\
&\quad + 2\,\mathrm{Re}\, R(\sigma+it, \sigma-it; \lambda) \\
&\quad - 2 \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} \{\mathrm{Re}\, S_{m,N}(\sigma+it, \sigma-it; a+j, \lambda) \\
&\quad + \mathrm{Re}\, T_{m,N}(\sigma+it, \sigma-it; a+j, \lambda)\}
\end{aligned} \tag{2.14}
$$

*holds. Furthermore, for any integer $K \geq 0$ the expression*

$$
\begin{aligned}
&T_{m,N}(\sigma+it,\sigma-it;x,\lambda) = \frac{x^{m+N-\sigma+it}}{(m-1)!} \\
&\times \Biggl\{ \sum_{k=1}^{K} \frac{(-1)^{k-1}(m+1-2\sigma)_{k-1}(\sigma+it)_{N-k}}{(1-\sigma+it)_{m+N-1}} \sum_{l=1}^{\infty} e(\lambda l) d_{k,l,m,N}(\sigma+it;x) \\
&+ \frac{(-1)^{K}(m+1-2\sigma)_{K}(\sigma+it)_{N-K}}{(1-\sigma+it)_{m+N-1}} \sum_{l=1}^{\infty} e(\lambda l) f_{K,l,m,N}(\sigma+it,\sigma-it;x) \Biggr\}
\end{aligned}
\tag{2.15}
$$

*follows in the same region of $\sigma+it$, and this gives a complete asymptotic expansion in the descending order of $t$ as $t \to \pm\infty$, where each term of the asymptotic series is estimated as*

$$
\begin{aligned}
\frac{(-1)^{k-1}(m+1-2\sigma)_{k-1}(\sigma+it)_{N-k}}{(1-\sigma+it)_{m+N-1}} &= O(|t|^{1-m-k}), \\
\sum_{l=1}^{\infty} e(\lambda l) d_{k,l,m,N}(\sigma+it,x) &= O(1), \\
\frac{(-1)^{K}(m+1-2\sigma)_{K}(\sigma+it)_{N-K}}{(1-\sigma+it)_{m+N-1}} &= O(|t|^{1-m-K}), \\
\sum_{l=1}^{\infty} e(\lambda l) f_{K,l,m,N}(\sigma+it,\sigma-it;x) &= O(|t|^{-1})
\end{aligned}
\tag{2.16}
$$

*for any $\sigma$ and $t$ with $1-N < \sigma < m+N$ and $|t| \geq 1$, and any $K \geq k \geq 1$. Here the implied $O$-constants depend at most on $m$, $N$, $K$, $x$, $\lambda$ and $\sigma$.*

Taking $u = \sigma+it$ and $v = \sigma-it$ in Corollary 1.1, or letting $N \to \infty$ in Corollary 1.2, we obtain

**Corollary 1.3**

*Let $m$, $a$ and $R$ be as in Theorem 1, and $E_m$ and $S_m^*$ defied by (2.13) and (2.12) respectively. Then for any $\sigma+it \in \mathbb{C} \setminus E_m$ the formula*

$$
\begin{aligned}
&\int_0^1 \cdots \int_0^1 |\phi(\sigma+it, a+x_1+\cdots+x_m,\lambda)|^2 dx_1 \cdots dx_m \\
&= -\frac{1}{(1-2\sigma)_m} \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} (a+j)^{m-2\sigma} \\
&+ 2\operatorname{Re} R(\sigma+it,\sigma-it;\lambda) - 2\sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} \\
&\times \operatorname{Re} S_m^*(\sigma+it,\sigma-it;a+j,\lambda)
\end{aligned}
\tag{2.17}
$$

*holds.*

We next supplement two exceptional (but important) cases of Theorem 1. For any integers $m \geq 1$ and $n \geq 0$, and any real $a > 0$, we define

$$C_{m,n}(a) = \begin{cases} \dfrac{1}{(n-1)!}\displaystyle\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}(a+j)^{n-1}\log(a+j) & \text{if } n \geq 1, \\ 1/a & \text{if } n = 0. \end{cases} \tag{2.18}$$

Let $\psi(s) = (\Gamma'/\Gamma)(s)$ be the digamma function, $\gamma_0 = -\psi(1)$ Euler's constant, and $\zeta'_\lambda(s) = (\partial/\partial s)\zeta_\lambda(s)$. We first state the limiting case $\sigma \to 1/2$ of Corollary 1.2:

**Corollary 1.4**

*Let $m$, $a$, $\lambda$, $S_{m,N}$ and $T_{m,N}$ be as in Theorem 1. Then for any integer $N \geq 1$ and any $t \in \mathbb{R}$ the formula*

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 \left|\phi\Big(\frac{1}{2}+it, a+x_1+\cdots+x_m, \lambda\Big)\right|^2 dx_1\cdots dx_m \\ &= \gamma_0 + \sum_{j=1}^{m-1}\frac{1}{j} - C_{m,m}(a) + 2\operatorname{Re}\Big\{\zeta'_\lambda(0) - \zeta_\lambda(0)\psi\Big(\frac{1}{2}+it\Big)\Big\} \\ &\quad - 2\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}\Big\{\operatorname{Re} S_{m,N}\Big(\frac{1}{2}+it, \frac{1}{2}-it; a+j, \lambda\Big) \\ &\quad + \operatorname{Re} T_{m,N}\Big(\frac{1}{2}+it, \frac{1}{2}-it; a+j, \lambda\Big)\Big\} \end{aligned} \tag{2.19}$$

*holds. Furthermore, for any integer $K \geq 0$ the expression* (2.15) *follows in particular for $T(1/2+it, 1/2-it; x, \lambda)$, and this gives a complete asymptotic expansion in the descending order of $t$ as $t \to \pm\infty$ (in the sense of* (2.16)*). Moreover, we may let $N \to \infty$ in* (2.19)*; the resulting formula holds with the limits $\lim_{N\to\infty} S_{m,N} = S^*_m$ and $\lim_{N\to\infty} T_{m,N} = 0$.*

Let $\{\lambda\} = \lambda - [\lambda]$ denote the fractional part of $\lambda$, and

$$\zeta_\lambda(s) = \frac{\gamma_{-1}(\lambda)}{s-1} + \gamma_0(\lambda) + O(s-1) \tag{2.20}$$

the Laurent series expansion at $s = 1$. From the definition it is seen that $\zeta_\lambda(s) = \zeta_{\{\lambda\}}(s)$,

$$\gamma_{-1}(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.21}$$

and

$$\gamma_0(\lambda) = \begin{cases} \gamma_0 & \text{if } \lambda \in \mathbb{Z}, \\ -\log(2\sin\pi\lambda) - \pi i(\{\lambda\} - 1/2) & \text{otherwise,} \end{cases} \tag{2.22}$$

(cf. [15, (2.7), (6.4) and (7.5)]). The limiting case $\sigma \to 1$ of Corollary 1.2 then asserts

**Corollary 1.5**

*Let $m$, $a$, $\lambda$, $S_{m,N}$ and $T_{m,N}$ be as in Theorem 1. Then for any integer $N \geq 1$ and any $t \in \mathbb{R} \setminus \{0\}$ the formula*

$$
\begin{aligned}
&\int_0^1 \cdots \int_0^1 |\phi(1+it, a+x_1+\cdots+x_m, \lambda)|^2 dx_1 \cdots dx_m \\
&= C_{m,m-1}(a) - \gamma_{-1}(\lambda)\Big\{2\operatorname{Re}\frac{\psi(1+it)}{it} + \frac{1}{t^2}\Big\} + \frac{2\operatorname{Im}\gamma_0(\lambda)}{t} \\
&\quad - 2\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}\{\operatorname{Re} S_{m,N}(1+it, 1-it; a+j, \lambda) \\
&\quad + \operatorname{Re} T_{m,N}(1+it, 1-it; a+j, \lambda)\}
\end{aligned}
\tag{2.23}
$$

*holds. Furthermore, for any integer $K \geq 0$ the expression* (2.15) *follows in particular for $T_{m,N}(1+it, 1-it; x, \lambda)$, and this gives a complete asymptotic expansion in the descending order of $t$ as $t \to \pm\infty$ (in the sense of* (2.16)*). Moreover, we may let $N \to \infty$ in* (2.23)*; the resulting formula holds with the limits* $\lim_{N\to\infty} S_{m,N} = S_m^*$ *and* $\lim_{N\to\infty} T_{m,N} = 0$.

*Remark.* The case $m = 1$ and $a = 1$ of this corollary reduces to [18, Corollary 3] if $\lambda \in \mathbb{Z}$, and to [9, Corollary 2] if $0 < \lambda < 1$.

We now turn to the direction of Mikolás [25]. Formula (2.1) suggests that the limit

$$\lim_{a\to+0} I_m(u, v; a, \lambda) = I_m(u, v; 0, \lambda)$$

exists in a certain restricted domain of $(u, v)$. This is in fact valid, and the following corollaries are proved.

**Corollary 1.6**

*Let $m$, $u$, $v$, $\lambda$, $I_m$, $\widetilde{E}_m$, $R$, $S_{m,N}$ and $T_{m,N}$ be as in Theorem 1. Then for any integer $N \geq 1$, in the region $1 - N < \operatorname{Re} u < m$, $1 - N < \operatorname{Re} v < m$ and $\operatorname{Re}(u+v) < m$ except the points of $\widetilde{E}_m$ the formula*

$$
\begin{aligned}
I_m(u, v; 0, \lambda) &= -\frac{1}{(1-u-v)_m}\sum_{j=1}^{m-1}(-1)^{m-1-j}\binom{m-1}{j} j^{m-u-v} \\
&\quad + R(u, v; \lambda) + R(v, u; -\lambda) - \sum_{j=1}^{m-1}(-1)^{m-1-j}\binom{m-1}{j} \\
&\quad \times \{S_{m,N}(u, v; j, \lambda) + S_{m,N}(v, u; j, -\lambda) \\
&\quad + T_{m,N}(u, v; j, \lambda) + T_{m,N}(v, u; j, -\lambda)\}
\end{aligned}
\tag{2.24}
$$

*holds. Moreover, we may let $N \to \infty$ in* (2.24)*; the resulting formula follows for all $(u, v) \in \mathbb{C}^2 \setminus \widetilde{E}_m$ with the limits* $\lim_{N\to\infty} S_{m,N} = S_m^*$ *and* $\lim_{N\to\infty} T_{m,N} = 0$.

*Remark.* The case $m = 1$ and $\lambda \in \mathbb{Z}$ of this corollary reduces to (2.1).

We lastly supplement two important exceptional cases of Corollary 1.6. For any integers $m \geq 2$ and $n \geq 2$, we define

$$C^*_{m,n} = \frac{1}{(n-1)!} \sum_{j=1}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} j^{n-1} \log j. \tag{2.25}$$

**Corollary 1.7**

Let $N$ be any positive integer, and $\lambda$, $S_{m,N}$ and $T_{m,N}$ as in Theorem 1. Then the following formulae hold: i) For any integer $m \geq 2$ and any $t \in \mathbb{R}$,

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \left|\phi\Big(\frac{1}{2}+it, x_1+\cdots+x_m, \lambda\Big)\right|^2 dx_1 \cdots dx_m \\
&= \gamma_0 + \sum_{j=1}^{m-1} \frac{1}{j} - C^*_{m,m} + 2\operatorname{Re}\Big\{\zeta'_\lambda(0) - \zeta_\lambda(0)\psi\Big(\frac{1}{2}+it\Big)\Big\} \\
&\quad - \sum_{j=1}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} \Big\{ \operatorname{Re} S_{m,N}\Big(\frac{1}{2}+it, \frac{1}{2}-it; j, \lambda\Big) \\
&\quad + \operatorname{Re} T_{m,N}\Big(\frac{1}{2}+it, \frac{1}{2}-it; j, \lambda\Big)\Big\}.
\end{aligned} \tag{2.26}$$

ii) For any integer $m \geq 3$ and any $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 |\phi(1+it, x_1+\cdots+x_m, \lambda)|^2 dx_1 \cdots dx_m \\
&= C^*_{m,m-1} - \gamma_{-1}(\lambda)\Big\{2\operatorname{Re}\frac{\psi(1+it)}{it} + \frac{1}{t^2}\Big\} + \frac{2\operatorname{Im}\gamma_0(\lambda)}{t} \\
&\quad - 2\sum_{j=1}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} \{\operatorname{Re} S_{m,N}(1+it, 1-it; j, \lambda) \\
&\quad + \operatorname{Re} T_{m,N}(1+it, 1-it; j, \lambda)\}.
\end{aligned} \tag{2.27}$$

## 3. Results on discrete and hybrid analogues

In this section, the results on (1.2) and (1.3) are stated in a more extended manner. For this purpose we introduce discrete and hybrid analogues of (2.2) in the form

$$\begin{aligned}
\widehat{I}_m(u,v;a,\lambda;q) &= q^{-m} \sum_{r_1=0}^{q-1} \cdots \sum_{r_m=0}^{q-1} \phi\Big(u, \frac{a}{q^m} + \frac{r_1}{q} + \cdots + \frac{r_m}{q^m}, \lambda\Big) \\
&\quad \times \phi\Big(v, \frac{a}{q^m} + \frac{r_1}{q} + \cdots + \frac{r_m}{q^m}, -\lambda\Big),
\end{aligned} \tag{3.1}$$

and

$$H(u,v;a,\lambda;q) = \int_0^1 q^{-1} \sum_{r=0}^{q-1} \phi\Big(u, \frac{a+r+x}{q}, \lambda\Big)\phi\Big(v, \frac{a+r+x}{q}, -\lambda\Big) dx. \tag{3.2}$$

Our second main result is then stated as

**Theorem 2**

*Let $\widehat{I}_m(u,v;a,\lambda;q)$ be defined by (3.1), where $m$ and $q$ are any positive integers, $u$ and $v$ are independent complex variables, and $a$ and $\lambda$ are real numbers with $a>0$. Let $\widetilde{E}_1$ be defined by (2.3) with $m=1$, and $R$ by (2.5). Then for any integer $N\geq 1$, in the region $1-N<\operatorname{Re}u<1+N$ and $1-N<\operatorname{Re}v<1+N$ except the points of $\widetilde{E}_1$ the formula*

$$\begin{aligned}(3.3)\qquad &\widehat{I}_m(u,v;a,\lambda;q)\\ &\quad = q^{m(u+v-1)}\zeta(u+v,a)+R(u,v;\lambda)+R(v,u;-\lambda)\\ &\qquad +\widehat{S}_N(u,v;a,\lambda;q^m)+\widehat{S}_N(v,u;a,-\lambda;q^m)\\ &\qquad +\widehat{T}_N(u,v;a,\lambda;q^m)+\widehat{T}_N(v,u;a,-\lambda;q^m)\end{aligned}$$

*holds, where*

$$(3.4)\qquad \widehat{S}_N(u,v;a,\lambda;q^m)=\sum_{n=0}^{N-1}\frac{(-1)^n(u)_n}{n!}\zeta_\lambda(u+n)\zeta(v-n,a)q^{m(v-n-1)},$$

*$\widehat{T}_N(u,v;a,\lambda;q)$ is expressed as the vertical integral (7.5) below, and satisfies the estimate*

$$(3.5)\qquad \widehat{T}_N(u,v;a,\lambda;q^m)=O(q^{m(\operatorname{Re}v-N-1)})$$

*for any $q\geq 1$ with the implied $O$-constants depending at most on $m$, $N$, $u$, $v$, $a$ and $\lambda$; the corresponding estimate also holds for $\widehat{T}_N(v,u;a,-\lambda;q^m)$. Here the right sides of (3.4) and (3.5) show that Formula (3.3) gives a complete asymptotic expansion in the descending order of $q$ as $q\to\infty$.*

Setting $u=\sigma+it$ and $v=\sigma-it$ in Theorem 2 we obtain

**Corollary 2.1**

*Let $m$, $a$, $\lambda$, $R$, $\widehat{S}_N$ and $\widehat{T}_N$ be as in Theorem 2, and $E_1$ defined by (2.13) with $m=1$. Then for any integer $N\geq 1$, in the region $1-N<\sigma<1+N$ and $t\in\mathbb{R}$ except the points of $E_1$ the formula*

$$\begin{aligned}(3.6)\qquad &q^{-m}\sum_{r_1=0}^{q-1}\cdots\sum_{r_m=0}^{q-1}\left|\phi\Big(\sigma+it,\frac{a}{q^m}+\frac{r_1}{q}+\cdots+\frac{r_m}{q^m},\lambda\Big)\right|^2\\ &\quad = q^{m(2\sigma-1)}\zeta(2\sigma,a)+2\operatorname{Re}R(\sigma+it,\sigma-it;\lambda)\\ &\qquad +2\operatorname{Re}\widehat{S}_N(\sigma+it,\sigma-it;a,\lambda;q^m)\\ &\qquad +2\operatorname{Re}\widehat{T}_N(\sigma+it,\sigma-it;a,\lambda;q^m)\end{aligned}$$

*holds, and $\widehat{T}_N$ satisfies the estimate*

$$(3.7)\qquad \widehat{T}_N(\sigma+it,\sigma-it;a,\lambda;q^m)=O\{q^{m(\sigma-N-1)}(|t|+1)^{\nu(\sigma,N,\varepsilon)}\}$$

*for any integer $q\geq 1$ and any real $\sigma$ and $t$ with $1-N<\sigma<1+N$, where the implied $O$-constant depends at most on $m$, $N$, $a$, $\lambda$ and $\sigma$. Here the exponent $\nu$ in (3.7) is* *given by*

$$\nu(\sigma,N,\varepsilon)=\begin{cases}(4N+1-2\sigma)/2 & \text{if } 1-N<\sigma<N,\\ (3N+1-\sigma+\varepsilon)/2 & \text{if } N\le\sigma<N+1\end{cases}$$

*with any small* $\varepsilon>0$.

*Remark 1.* It is reasonable that such a bound as in (3.7) holds, since the term with the index $n(\geq 1)$ in (3.4) for $\widehat{S}_N(\sigma+it,\sigma-it;a,\lambda;q)$ is of order $O\{q^{m(\sigma-n-1)}(|t|+1)^{\nu(\sigma,n,\varepsilon)}\}$ for $q\geq 1$, $1-n<\sigma<1+n$ and any $t\in\mathbb{R}$.

*Remark 2.* The case $m=1$ and $a=1$ of this corollary reduces to [9, Theorem 2], and further to [16, Theorem 2] if $\lambda\in\mathbb{Z}$.

We next supplement two important exceptional cases of this corollary.

**Corollary 2.2**

*Let $N$ be a positive integer, and $m$, $a$, $\lambda$, $q$, $\widehat{S}_N$ and $\widehat{T}_N$ as in Theorem 2. Then the following formulae hold*: i) *For any* $t\in\mathbb{R}$,

$$\begin{aligned}(3.8)\qquad & q^{-m}\sum_{r_1=0}^{q-1}\cdots\sum_{r_m=0}^{q-1}\Big|\phi\Big(\frac12+it,\frac{a}{q^m}+\frac{r_1}{q}+\cdots+\frac{r_m}{q^m},\lambda\Big)\Big|^2\\ &=m\log q-\psi(a)+\gamma_0+2\operatorname{Re}\Big\{\zeta_\lambda'(0)-\zeta_\lambda(0)\psi\Big(\frac12+it\Big)\Big\}\\ &\quad+2\operatorname{Re}\widehat{S}_N\Big(\frac12+it,\frac12-it;a,\lambda;q^m\Big)\\ &\quad+2\operatorname{Re}\widehat{T}_N\Big(\frac12+it,\frac12-it;a,\lambda;q^m\Big).\end{aligned}$$

ii) *For any* $t\in\mathbb{R}\setminus\{0\}$,

$$\begin{aligned}(3.9)\qquad & q^{-m}\sum_{r_1=0}^{q-1}\cdots\sum_{r_m=0}^{q-1}\Big|\phi\Big(1+it,\frac{a}{q^m}+\frac{r_1}{q}+\cdots+\frac{r_m}{q^m},\lambda\Big)\Big|^2\\ &=q^m\zeta(2,a)-\gamma_{-1}(\lambda)\Big\{2\operatorname{Re}\frac{\psi(1+it)}{it}+\frac{1}{t^2}\Big\}+\frac{2\operatorname{Im}\gamma_0(\lambda)}{t}\\ &\quad+2\operatorname{Re}\widehat{S}_N(1+it,1-it;a,\lambda;q^m)\\ &\quad+2\operatorname{Re}\widehat{T}_N(1+it,1-it;a,\lambda;q^m).\end{aligned}$$

*The estimate* (3.7) *in particular follows for* $\widehat{T}_N$ *in* (3.8) *and* (3.9).

*Remark.* The case $m=1$ and $a=1$ of (3.8) reduces to [9, Corollary 3], and further to [16, Theorem 1] if $\lambda\in\mathbb{Z}$, while the same case of (3.9) reduces to [9, Corollary 4] if $0<\lambda<1$.

We finally proceed to state the results on the hybrid analogue $H(u,v;a,\lambda;q)$.

**Theorem 3**

*Let $H(u,v;a,\lambda;q)$ be defined by* (3.2), *where $u$ and $v$ are independent complex variables, $a$ and $\lambda$ are real numbers with $a>0$, and $q$ is any positive integer. Let* $\widetilde{E}_1$

*defined by* (2.3) *with* $m = 1$, *and* $R$ *by* (2.5). *Then for any integer* $N \geq 1$, *in the region* $1 - N < \operatorname{Re} u < 1 + N$ *and* $1 - N < \operatorname{Re} v < 1 + N$ *except the points of* $\widetilde{E}_1$ *the formula*

$$\begin{aligned} H(u, v; a, \lambda; q) = &-\frac{(a/q)^{1-u-v}}{1-u-v} + R(u, v; \lambda) + R(v, u; -\lambda) \\ &- S_{1,N}\Big(u, v; \frac{a}{q}, \lambda\Big) - S_{1,N}\Big(v, u; \frac{a}{q}, -\lambda\Big) \\ &- T_{1,N}\Big(u, v; \frac{a}{q}, \lambda\Big) - T_{1,N}\Big(v, u; \frac{a}{q}, -\lambda\Big) \end{aligned} \tag{3.10}$$

*holds, where* $S_{1,N}$ *and* $T_{1,N}$ *are defined by* (2.6) *and* (2.7) *with* $m = 1$ *respectively, and* $T_{1,N}$ *satisfies the estimate*

$$T_{1,N}\Big(u, v; \frac{a}{q}, \lambda\Big) = O\Big\{\Big(\frac{a}{q}\Big)^{1+N-\operatorname{Re} v}\Big\} \tag{3.11}$$

*for any real* $a > 0$ *and any integer* $q \geq 1$ *with the implied* $O$*-constant depending at most on* $N$, $u$, $v$ *and* $\lambda$. *Here the right sides of* (2.6) (*with* $m = 1$) *and* (3.11) *show that Formula* (3.10) *gives an asymptotic expansion in the ascending powers of* $a/q$ *as* $a/q \to +0$. *Furthermore, for any integer* $K \geq 0$ *the expression* (2.8) *follows in particular for* $T_{1,N}(u, v; a/q, \lambda)$.

Setting $u = \sigma + it$ and $v = \sigma - it$ in Theorem 3 we obtain

**Corollary 3.1**

*Let* $a$, $\lambda$, $q$, $R$, $S_{1,N}$ *and* $T_{1,N}$ *be as in Theorem 3, and* $E_1$ *defined by* (2.13) *with* $m = 1$. *Then for any integer* $N \geq 1$, *in the region* $1 - N < \sigma < 1 + N$ *and* $t \in \mathbb{R}$ *except the points of* $E_1$ *the formula*

$$\begin{aligned} &\int_0^1 q^{-1} \sum_{r=0}^{q-1} \Big|\phi\Big(\sigma + it, \frac{a + r + x}{q}, \lambda\Big)\Big|^2 dx \\ &= -\frac{(a/q)^{1-2\sigma}}{1-2\sigma} + 2 \operatorname{Re} R(\sigma + it, \sigma - it; \lambda) \\ &\quad - 2 \operatorname{Re} S_{1,N}\Big(\sigma + it, \sigma - it; \frac{a}{q}, \lambda\Big) \\ &\quad - 2 \operatorname{Re} T_{1,N}\Big(\sigma + it, \sigma - it; \frac{a}{q}, \lambda\Big) \end{aligned} \tag{3.12}$$

*holds, and the estimate* (3.11) *follows in particular for* $T_{1,N}(\sigma + it, \sigma - it; a/q, \lambda)$. *Here the right sides of* (2.6) (*with* $m = 1$) *and* (3.11) *show that Formula* (3.12) *gives an asymptotic expansion in the ascending powers of* $a/q$ *as* $a/q \to +0$. *Furthermore, for any integer* $K \geq 0$ *the expression* (2.8) *follows in particular for* $T_{1,N}(\sigma + it, \sigma - it; a/q, \lambda)$, *and this gives a complete asymptotic expansion in the descending order of* $t$ *as* $t \to \pm\infty$ (*in the sense of* (2.16)). *Moreover, the limiting cases* $\sigma \to 1/2$ *and* $\sigma \to 1$ *of* (3.12) *are also valid; the results are precisely the same as those given on the right sides of* (2.19) *and* (2.23), *respectively, with* $m = 1$ *and* $a/q$ *in place of* $a$.

## 4. A fundamental formula

We first prepare the formula (4.2) below, which is fundamental in proving Theorems 1, 2 and 3. For this we suppose temporarily that $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$, and define

$$g(u, v; x, \lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(u+s)\Gamma(-s)}{\Gamma(u)} \zeta_\lambda(-s)\zeta(u+v+s, x) ds \tag{4.1}$$

for $x > 0$ and any real $\lambda$, where $\mathcal{C}$ is a vertical path which is directed upward and suitably indented so as to separate the (possible) poles of $\Gamma(-s)\zeta_\lambda(-s)\zeta(u+v+s,x)$ at $s = 1-u-v$, $-1+n$ $(n = 0, 1, \ldots)$, from the poles of $\Gamma(u+s)$ at $s = -u-n$ $(n = 0, 1, \ldots)$. Here, and in the sequel, we suppose that the poles which occur are at most simple poles, since otherwise the results can be deduced by taking the limits. Then from Atkinson's [2] dissection method applied to the product $\phi(u, x, \lambda)\phi(v, x, -\lambda)$, we have shown

**Lemma 1** ([9, Lemma 1]).

In the region $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$ the formula

$$\begin{aligned} \phi(u, x, \lambda)\phi(v, x, -\lambda) &= \zeta(u+v, x) + R(u, v, \lambda) + R(v, u; -\lambda) \\ &\quad + g(u, v; x, \lambda) + g(v, u; x, -\lambda) \end{aligned} \tag{4.2}$$

holds, where $R$ is given by (2.5), and $g(v, u; x, -\lambda)$ is defined similarly to (4.1).

*Remark.* Formula (4.2) remains valid if the paths of the integral expressions of $g(u, v; x, \lambda)$ and $g(v, u; x, -\lambda)$ are modified according to the location of $(u, v)$, unless they move across the poles of the corresponding integrands.

## 5. Proof of Theorem 1

Our previous frame of the proof (see [9], Section 4) is appropriately extended to apply the present situation. For the purpose, two new key devices (Lemmas 3 and 5) are needed.

We replace $x$ by $a + x_1 + \cdots + x_m$ in (4.2), and integrate both sides with respect to $x_1, \ldots, x_m$; in the process we use

**Lemma 2**

*Let $m \geq 1$ be an integer, and $a > 0$ any real number. Then the formula*

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 \zeta(w, a + x_1 + \cdots + x_m) dx_1 \cdots dx_m \\ &= -\frac{1}{(1-w)_m} \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} (a+j)^{m-w} \end{aligned} \tag{5.1}$$

*holds for any complex $w \neq 1, 2, \ldots, m$.*

*Proof.* We have shown in [9, Lemma 2] that

$$\int_0^1 \zeta(w, a + x_1) dx_1 = -\frac{a^{1-w}}{1-w}$$

for $w \neq 1$, which is obtained by integrating the series expression of $\zeta(w, a + x_1)$, and then by the analytic continuation. Replacing $a$ by $a + x_2 + \cdots + x_m$ in this equality, and then integrating both sides repeatedly, we obtain the assertion by induction on $m$. □

We therefore find from (4.1), (4.2) and this lemma that

$$
\begin{aligned}
I_m(u, v; a, \lambda) = &-\frac{1}{(1-u-v)_m} \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} (a+j)^{m-u-v} \\
&+ R(u, v; \lambda) + R(v, u; -\lambda) - \sum_{j=0}^{m-1} (-1)^{m-1-j} \binom{m-1}{j} \\
&\times \{\widetilde{g}_m(u, v; a+j, \lambda) + \widetilde{g}_m(v, u; a+j, -\lambda)\},
\end{aligned}
\tag{5.2}
$$

where

$$
\widetilde{g}_m(u, v; x, \lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(u+s)\Gamma(-s)}{\Gamma(u)} \zeta_\lambda(-s) \frac{x^{m-u-v-s}}{(1-u-v-s)_m} ds.
\tag{5.3}
$$

Here the inversion of the $s$-integral and the $x$-integral is justified by the fact that the integral on the right side of (4.1) converges uniformly in $x$ belonging to any compact subset in the range $(0, +\infty)$, since the estimate $\Gamma(u+s)\Gamma(-s) = O(|\operatorname{Im} s|^C e^{-\pi|\operatorname{Im} s|})$ as $\operatorname{Im} s \to \pm\infty$ holds with some constant $C > 0$ (cf. [8, p. 492, (A. 34)]).

To transform the integral in (5.3) we apply

**Lemma 3**

*Let $z$ and $w$ be complex variables with $0 < \operatorname{Re} z < \operatorname{Re} w$. Then the formula*

$$
\frac{1}{(w-z)_m} = \frac{1}{2\pi i} \int_{(\rho_0)} \frac{\Gamma(z+r)\Gamma(m+r)\Gamma(w)\Gamma(-r)}{\Gamma(z)\Gamma(m)\Gamma(w+m+r)} e^{\pi i r} dr
\tag{5.4}
$$

*holds for any integer $m \geq 1$, where $\rho_0$ is a constant satisfying $\max(-\operatorname{Re} z, -m) < \rho_0 < 0$ and $(\rho_0)$ denotes the vertical straight line from $\rho_0 - i\infty$ to $\rho_0 + i\infty$.*

*Proof.* Let $\theta$ be a real number with $|\theta| < \pi$. Then a Mellin-Barnes formula for Gauss' hypergeometric function ${}_2F_1$ (cf. [5, p. 62, 2.1.3(15)]) gives

$$
{}_2F_1(z, m; w+m; -e^{i\theta}) = \frac{1}{2\pi i} \int_{(\rho_0)} \frac{\Gamma(z+r)\Gamma(m+r)\Gamma(w+m)\Gamma(-r)}{\Gamma(z)\Gamma(m)\Gamma(w+m+r)} e^{i\theta r} dr.
$$

By continuity we may let $\theta \to \pi - 0$ in this equality, provided $\operatorname{Re} z < \operatorname{Re} w$, since the integrand is of order $O(e^{-(\pi-|\theta|)|\operatorname{Im} r|}|\operatorname{Im} r|^{\operatorname{Re} z - \operatorname{Re} w - 1})$ as $\operatorname{Im} r \to \pm\infty$. The limit of the left side can be evaluated by Gauss' summation formula

$$
{}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}
$$

for $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$ and $\operatorname{Re}(\gamma - \alpha - \beta) > 0$ (cf. [5, p. 61, 2.1.3(14)] ), which implies

$$
{}_2F_1(z, m; w+m; 1) = \frac{(w)_m}{(w-z)_m},
$$

by $\Gamma(s+m) = (s)_m \Gamma(s)$. The assertion therefore follows. □

We suppose at this stage that $\operatorname{Re} u > 1$ and $\operatorname{Re} v < 1$. Then the path $\mathcal{C}$ in (5.3) can be taken as a straight line $(c_0)$, where $c_0$ is a constant satisfying $-\operatorname{Re} u < c_0 < \min(-1, 1 - \operatorname{Re}(u+v))$. Under this choice of $c_0$, it is possible to take $b_0$ such as $\max(-\operatorname{Re} u - c_0, -m) < b_0 < 0$. We substitute (5.4) with $z = u + s$, $w = 1 - v$ and $\rho_0 = b_0$ into the right side of (5.3) to obtain

$$\begin{aligned} &\widetilde{g}_m(u, v; x, \lambda) \\ &\quad = \frac{1}{2\pi i} \int_{(c_0)} \frac{\Gamma(-s)}{\Gamma(u)} \zeta_\lambda(-s) x^{m-u-v-s} \\ &\quad \times \frac{1}{2\pi i} \int_{(b_0)} \frac{\Gamma(u+s+r)\Gamma(m+r)\Gamma(1-v)\Gamma(-r)}{\Gamma(m)\Gamma(1-v+m+r)} e^{\pi i r} dr ds, \end{aligned} \tag{5.5}$$

where, by the choice of $c_0$, the condition $0 < \operatorname{Re}(u+s) < \operatorname{Re}(1-v)$ of Lemma 3 is fulfilled on the path $\operatorname{Re} s = c_0$. To invert the order of the integrals on the right side of (5.5), we temporarily restrict ourselves to the case $\operatorname{Re}(u+v) < 1/2$; Fubini's theorem can then be applied (see Section 8 for the details) to give

$$\begin{aligned} \widetilde{g}_m(u, v; x, \lambda) = \frac{1}{2\pi i} \int_{(b_0)} &\frac{\Gamma(u+r)\Gamma(m+r)\Gamma(1-v)\Gamma(-r)}{\Gamma(u)\Gamma(m)\Gamma(1-v+m+r)} e^{\pi i r} \\ &\times x^{m+r-v} e(\lambda)\phi(u+r, 1+x, \lambda) dr, \end{aligned} \tag{5.6}$$

where the resulting inner $s$-integral is evaluated by Lemma 4 below. We note that (5.6) is now valid for $\operatorname{Re} u > 1$, $\operatorname{Re} v < 1$ and $\operatorname{Re}(u+v) < 1$ by analytic continuation, since the integrand is of order $O(|\operatorname{Im} r|^{\operatorname{Re}(u+v)-2})$ as $\operatorname{Im} r \to \pm\infty$.

**Lemma 4**

*For any complex $w$ with $\operatorname{Re} w > 1$, and any real $x$ and $\lambda$ with $x > 0$ it follows that*

$$\frac{1}{2\pi i} \int_{(\sigma_0)} \frac{\Gamma(w+s)\Gamma(-s)}{\Gamma(w)} \zeta_\lambda(-s) x^{-s} ds = x^w e(\lambda)\phi(w, 1+x, \lambda), \tag{5.7}$$

*where $\sigma_0$ is a constant satisfying $-\operatorname{Re} w < \sigma_0 < -1$.*

*Proof.* We substitute the series expression $\zeta_\lambda(-s) = \sum_{l=1}^{\infty} e(\lambda l) l^s$ for $\operatorname{Re} s = c_0 (< -1)$ into the left side of (5.7), and then perform the term-by-term integration, which is legitimate by absolute convergence. Each resulting term can be evaluated by the Mellin-Barnes formula

$$(1-z)^{-\alpha} = \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\alpha)} (-z)^s ds$$

for $-\operatorname{Re} \alpha < \sigma < 0$ and $|\arg(-z)| < \pi$ (cf. [31, p. 289, 14.51, Corollary]), which shows that the left side of (5.7) equals to $x^w$ times

$$\sum_{l=1}^{\infty} e(\lambda l)(l+x)^{-w} = e(\lambda)\phi(w, 1+x, \lambda). \tag{5.8}$$

Lemma 4 is proved. □

Let $N$ be any positive integer, and $b_N$ the constant satisfying $N - 1 < b_N < N$. Then we can move the path of integration in (5.6) from $(b_0)$ to $(b_N)$, provided

$\mathrm{Re}(u+v) < 1$, since the integrand for $\mathrm{Re}\, r \geq b_0$ is of order $O(|\operatorname{Im} r|^{\mathrm{Re}(u+v)-2})$ as $\operatorname{Im} r \to \pm\infty$. We collect the residues of the poles at $r = n$ $(n = 0, 1, \ldots, N-1)$ to obtain

$$\widetilde{g}_m(u,v;x,\lambda) = S_{m,N}(u,v;x,\lambda) + T_{m,N}(u,v;x,\lambda), \tag{5.9}$$

where $S_{m,N}(u,v;x,\lambda)$ is defined by (2.6) and

$$\begin{aligned} T_{m,N}(u,v;x,\lambda) = \frac{1}{2\pi i}\int_{(b_N)} &\frac{\Gamma(u+r)\Gamma(m+r)\Gamma(1-v)\Gamma(-r)}{\Gamma(u)\Gamma(m)\Gamma(1-v+m+r)} e^{\pi i r} \\ &\times x^{m+r-v} e(\lambda)\phi(u+r, 1+x, \lambda) dr. \end{aligned}$$

The last integral is further transformed by substituting the series expression (5.8) and integrating term-by-term, provided $\mathrm{Re}(u+v) < 1$. Then each term in the resulting expression can be evaluated by changing the variable $r = N + s$, upon noting $\Gamma(1+N+s)\Gamma(-N-s) = (-1)^N \Gamma(1+s)\Gamma(-s)$, and then by using

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(b_N-N)} \frac{\Gamma(u+N+s)\Gamma(m+N+s)(-1)^N\Gamma(1+s)\Gamma(-s)}{\Gamma(1-v+m+N+s)\Gamma(1+N+s)} \Big(\frac{e^{\pi i}x}{l+x}\Big)^{N+s} ds \\ &= \Big(\frac{x}{l+x}\Big)^N \frac{\Gamma(u+N)\Gamma(m+N)}{\Gamma(1-v+m+N)\Gamma(1+N)} \\ &\quad \times {}_3F_2\Big(u+N, 1, m+N; 1-v+m+N, 1+N; \frac{x}{l+x}\Big), \end{aligned}$$

where ${}_3F_2$ denotes the generalized hypergeometric function (cf. [5, p. 202, 5.1(2)]). The last equality is derived by letting $-z \to e^{\pi i}x/(l+x)$ in the Mellin-Barnes formula

$$\begin{aligned} &\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\varepsilon)} {}_3F_2(\alpha,\beta,\gamma;\delta,\varepsilon;z) \\ &= \frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma+s)\Gamma(-s)}{\Gamma(\delta+s)\Gamma(\varepsilon+s)} (-z)^s ds \end{aligned}$$

for $|\arg(-z)| < \pi$ and $\max(-\mathrm{Re}\,\alpha, -\mathrm{Re}\,\beta, -\mathrm{Re}\,\gamma) < \sigma < 0$ (cf. [5, p. 207, 5.3(1); p. 215, 5.6(1)]). We therefore obtain

$$\begin{aligned} &T_{m,N}(u,v;x,\lambda) \\ &= x^{m+N-v} \frac{(u)_N (m)_N}{(1-v)_{m+N} N!} \sum_{l=1}^{\infty} \frac{e(\lambda l)}{(l+x)^{u+N}} \\ &\quad \times {}_3F_2\Big(u+N, 1, m+N; 1-v+m+N, 1+N; \frac{x}{l+x}\Big). \end{aligned} \tag{5.10}$$

This is continued to a meromorphic function of $(u,v)$ in the region $\mathrm{Re}\, u > 1-N$ and any $v$, because the last infinite series converges absolutely for $\mathrm{Re}\, u > 1-N$ and any $v \neq 1+m+N+n$ $(n = 0, 1, \ldots)$ by the fact that $x/(l+x) \to 0$ as $l \to \infty$, and so that

$${}_3F_2\Big(u+N, 1, m+N; 1-v+m+N, 1+N; \frac{x}{l+x}\Big) \sim 1$$

by the defining series of ${}_3F_2$ (see [5, p. 202, 5.1(2)]).

It is in fact possible to reduce ${}_3F_2$ in (5.10) to a simpler ${}_2F_1$ by the following lemma.

**Lemma 5**

For any complex $\alpha$, $\beta$, $\gamma$ and $\delta$ with $\gamma, \delta \neq -n$ $(n = 0, 1, \ldots)$, and for any integer $k \geq 0$ we have

$$
{}_3F_2(\alpha, \beta, \delta + k; \gamma, \delta; z) = \frac{1}{(\delta)_k} \Big(\frac{\partial}{\partial w}\Big)^k w^{\delta+k-1} {}_2F_1(\alpha, \beta; \gamma; zw)\Big|_{w=1}. \tag{5.11}
$$

*Proof.* For all non-negative integers $k$ and $n$ it is seen that

$$
\frac{(\delta + k)_n}{(\delta)_n} = \frac{\Gamma(\delta + k + n)\Gamma(\delta)}{\Gamma(\delta + k)\Gamma(\delta + n)} = \frac{(\delta + n)_k}{(\delta)_k},
$$

and hence from $(\partial/\partial w)^k w^{\delta+k-1+n} = (\delta + n)_k w^{\delta-1+n}$ the identity

$$
\frac{w^{1-\delta}}{(\delta)_k} \Big(\frac{\partial}{\partial w}\Big)^k w^{\delta+k-1} (zw)^n = \frac{(\delta + k)_n}{(\delta)_n} (zw)^n
$$

is valid for any $k, n \geq 0$. Multiplying both sides by $(\alpha)_n (\beta)_n / (\gamma)_n n!$ and summing up over $n = 0, 1, \ldots$, we obtain

$$
\frac{w^{1-\delta}}{(\delta)_k} \Big(\frac{\partial}{\partial w}\Big)^k w^{\delta+k-1} {}_2F_1(\alpha, \beta; \gamma; zw) = {}_3F_2(\alpha, \beta, \delta + k; \gamma, \delta; zw)
$$

for $|zw| < 1$, and the assertion follows. $\square$

We apply (5.11) with $k = m - 1$, $\delta = 1 + N$, $z = x/(l + x)$ and $w = 1 + \xi$ to rewrite (5.10) as

$$
\begin{aligned}
&T_{m,N}(u, v; x, \lambda) \\
&= \frac{x^{m+N-v}}{(m - 1)!} \frac{(u)_N}{(1 - v)_{m+N}} \sum_{l=1}^{\infty} \frac{e(\lambda l)}{(l + x)^{u+N}} \Big(\frac{\partial}{\partial \xi}\Big)^{m-1} (1 + \xi)^{m+N-1} \\
&\quad \times {}_2F_1\Big(u + N, 1; 1 - v + m + N; \frac{x(1 + \xi)}{l + x}\Big)\Big|_{\xi=0}.
\end{aligned} \tag{5.12}
$$

The right side is further transformed by substituting

$$
\begin{aligned}
&{}_2F_1\Big(u + N, 1; 1 - v + m + N; \frac{x(1 + \xi)}{l + x}\Big) \\
&= (-v + m + N) \int_0^1 (1 - \eta)^{-v+m+N-1} \Big(1 - \frac{x(1 + \xi)\eta}{l + x}\Big)^{-u-N} d\eta \\
&= (-v + m + N) \frac{(l + x)^{u+N}}{(l - x\xi)^{u+v-m}} \int_l^{\infty} \frac{(y - x\xi)^{u+v-m-1}}{(y + x)^{u+N}} dy
\end{aligned}
$$

for any $x > 0$, $l \geq 1$ and $|\xi| < l/x$, where the integrals converge absolutely for $\operatorname{Re} v < m + N$. Here the first equality is derived by Euler's formula

$$
{}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 \eta^{\beta-1} (1 - \eta)^{\gamma-\beta-1} (1 - z\eta)^{-\alpha} d\eta
$$

for $\operatorname{Re} \gamma > \operatorname{Re} \beta > 0$ and $|z| < 1$, while the second by changing the variable $\eta = (y - l)/(y - x\xi)$ in the first, upon noting the relations $1 - \eta = (l - x\xi)/(y - x\xi)$ and

$l+x-x(1+\xi)\eta=(1-\eta)(y+x)$. We therefore obtain

$$
\begin{aligned}
T_{m,N}(u,v;x,\lambda) &= \frac{x^{m+N-v}}{(m-1)!}\frac{(u)_N}{(1-v)_{m+N-1}} \\
&\quad\times\sum_{l=1}^{\infty}e(\lambda l)\left(\frac{\partial}{\partial\xi}\right)^{m-1}\frac{(1+\xi)^{m+N-1}}{(l-x\xi)^{u+v-m}}\widetilde{f}_{l,m,N}(u,v;x,\xi)\bigg|_{\xi=0}
\end{aligned}
\tag{5.13}
$$

with

$$
\widetilde{f}_{l,m,N}(u,v;x,\xi)=\int_l^{\infty}\frac{(y-x\xi)^{u+v-m-1}}{(y+x)^{u+N}}dy,
\tag{5.14}
$$

where the infinite series in (5.13) converges absolutely for $\operatorname{Re}u>1-N$ and $\operatorname{Re}v<m+N$, since the estimate

$$
\begin{aligned}
&\left(\frac{\partial}{\partial\xi}\right)^{j_1}\frac{1}{(l-x\xi)^{u+v-m}}\left(\frac{\partial}{\partial\xi}\right)^{j_2}\widetilde{f}_{l,m,N}(u,v;x,\xi)\bigg|_{\xi=0} \\
&\qquad\ll l^{m-\operatorname{Re}(u+v)-j_1}\int_l^{\infty}y^{-\operatorname{Re}v-m-N-j_2-1}dy\ll l^{-\operatorname{Re}u-N-j_1-j_2}
\end{aligned}
$$

as $l\to\infty$ holds for all $j_1,j_2\geq 0$. The assertions (2.4)–(2.7) of Theorem 1 are thus concluded from (5.2), (5.9), (5.13) and the corresponding results for $\widetilde{g}_m(v,u;x,-\lambda)$, upon noting that $f_{0,l,m,N}$ is defined by (5.16) below with $K=0$. Furthermore, it is seen from the repeated integration by parts ($K$-times) that

$$
\begin{aligned}
&\widetilde{f}_{l,m,N}(u,v;x,\xi) \\
&\quad=\sum_{k=1}^{K}\frac{(-1)^{k-1}(m+1-u-v)_{k-1}}{(u+N-1)\cdots(n+N-k)}\frac{(l-x\xi)^{u+v-m-k}}{(l+x)^{u+N-k}} \\
&\qquad+\frac{(-1)^{K}(m+1-u-v)_{K}}{(u+N-1)\cdots(u+N-K)}\widetilde{f}_{l,m,N}(u-K,v;x,\xi),
\end{aligned}
\tag{5.15}
$$

and this with (5.13) yields the assertions (2.8)–(2.10), upon noting $(u)_N/(u+N-1)\cdots(u+N-k)=(u)_{N-k}$ for $k\geq 0$, defining $d_{k,l,m,N}(u;x)$ by (2.9) and

$$
f_{K,l,m,N}(u,v;x)=\left(\frac{\partial}{\partial\xi}\right)^{m-1}\frac{(1+\xi)^{m+N-1}}{(l-x\xi)^{u+v-m}}\widetilde{f}_{l,m,N}(u-K,v;x,\xi)\bigg|_{\xi=0}.
\tag{5.16}
$$

It is in fact possible to interpret the integration process above from a point of view of hypergeometric functions. The repeated use of a contiguity relation

$$
(\beta-\alpha)(1-z)_2F_1(\alpha,\beta;\gamma;z)=(\gamma-\alpha)_2F_1(\alpha-1,\beta;\gamma;z)-(\gamma-\beta)_2F_1(\alpha,\beta-1;\gamma;z)
$$

(cf. [5, p. 103, 2.8(37)]) yields the identity

$$
\begin{aligned}
&\frac{(u)_N}{(1-v)_{m+N}}\,{}_2F_1\Big(u+N,1;1-v+m+N;\frac{x(1+\xi)}{l+x}\Big)\\
&\quad=\sum_{k=1}^{K}\frac{(-1)^{k-1}(m+1-u-v)_{k-1}(u)_{N-k}}{(1-v)_{m+N-1}}\Big(\frac{l+x}{l-x\xi}\Big)^{k}\\
&\qquad+\frac{(-1)^{K}(m+1-u-v)_{K}(u)_{N-K}}{(1-v)_{m+N}}\Big(\frac{l+x}{l-x\xi}\Big)^{K}\\
&\qquad\times{}_2F_1\Big(u+N-K,1;1-v+m+N;\frac{x(1+\xi)}{l+x}\Big),
\end{aligned}\tag{5.17}
$$

which is equivalent to (5.15). A direct substitution of (5.17) into the right side of (5.12) again implies the assertions (2.8) and (2.9). The proof of Theorem 1 is complete. □

## 6. Proof of Corollaries 1.1–1.7

We proceed to prove Corollaries 1.1–1.7.

*Proof of Corollary 1.1.* For the proof we first show the estimate

$$
T_{m,N}(u,v;x,\lambda)\ll N^{\mathrm{Re}(u+v)-1}\Big(\frac{x}{1+x}\Big)^{N}\tag{6.1}
$$

for $\mathrm{Re}\,u>1-N$, $\mathrm{Re}\,v<m+N$, $(0<)a\le x\le a+m-1$ and any real $\lambda$, where, also in the proof of Corollary 1, the implied $\ll$-constants are irrelevant to $l$ and $N$. This with the estimate for $T_{m,N}(v,u;x,-\lambda)$ immediately yields the assertion by letting $N\to\infty$ in (2.4).

It follows from (5.14) and (5.16) with $K=0$ that

$$
\begin{aligned}
f_{0,l,m,N}(u,v;x)&\ll\sum_{\substack{j_1+j_2+j_3=m-1\\ j_1,j_2,j_3\ge0}}(m+N-1)\cdots(m+N-j_1)x^{j_2}l^{m-\mathrm{Re}(u+v)-j_2}\\
&\quad\times\Big(\frac{\partial}{\partial\xi}\Big)^{j_3}\widetilde{f}_{l,m,N}(u,v;x,0)\\
&\ll N^{m-1}l^{m-\mathrm{Re}(u+v)}(l+x)^{\mathrm{Re}\,v-m-N},
\end{aligned}
$$

where the term under differentiation above was estimated as

$$
\ll\int_l^\infty\frac{x^{j_3}y^{\mathrm{Re}(u+v)-m-1-j_3}}{(y+x)^{\mathrm{Re}\,u+N}}\ll\int_l^\infty(y+x)^{\mathrm{Re}\,v-m-N-1-j_3}dy\ll(l+x)^{\mathrm{Re}\,v-m-N-j_3},
$$

upon noting that $1\ll a\le x\le a+m-1\ll1$ and $y+x\ll y\ll y+x$ for $1\le l\le y<\infty$. We hence obtain

$$
\begin{aligned}
\sum_{l=1}^{\infty}e(\lambda l)f_{0,l,m,N}(u,v;x)&\ll\sum_{l=1}^{\infty}N^{m-1}l^{m-\mathrm{Re}(u+v)}(l+x)^{\mathrm{Re}\,v-m-N}\\
&\ll N^{m-1}\sum_{l=1}^{\infty}(l+x)^{-\mathrm{Re}\,u-N}\ll N^{m-1}(1+x)^{1-\mathrm{Re}\,u-N},
\end{aligned}
$$

since $l+x \ll l \ll l+x$ for $l \geq 1$. The required estimate (6.1) therefore follows from (2.7), by combining this bound with $(u)_N/(1-v)_{m+N-1} \ll N^{\mathrm{Re}(u+v)-m}$ for $N \geq 1$. Corollary 1.1 is proved. □

*Proofs of Corollaries 1.2 and 1.3.* We first set $u=\sigma+it$ and $v=\sigma-it$ in (2.4) and (2.8) to obtain the assertions (2.14) and (2.15) respectively. Then it remains to show (2.16) to prove Corollary 1.2. The first and the third estimates are immediate by the definition of Pochhammer's symbol, while the second follows from, by (2.9),

$$d_{k,l,m,N}(\sigma+it;x) \ll (l+x)^{-\sigma-N+k}l^{-k} \ll l^{-\sigma-N} \tag{6.2}$$

for $l \geq 1$, where, also in the proof of Corollary 1.2, the implied $\ll$-constants are irrelevant to $l$ and $t$. Moreover, the case $K=1$ of (5.15) with $u$ replaced by $u-K$ gives

$$\begin{aligned}\widetilde{f}_{l,m,N}(u-K,v;x,\xi) &= \frac{1}{u+N-K-1}\frac{(l-x\xi)^{u+v-m-K-1}}{(l+x)^{u+N-K-1}}\\ &\quad - \frac{m+1-u-v+K}{u+N-K-1}\widetilde{f}_{l,m,N}(u-K-1,v;x,\xi),\end{aligned}$$

from which the equality

$$\begin{aligned}f_{K,l,m,N}(\sigma+it,\sigma-it;x) &= \frac{1}{\sigma+it+N-K-1}d_{K+1,l,m,N}(\sigma+it;x)\\ &- \frac{m+1-2\sigma+K}{\sigma+it+N-K-1}f_{K+1,l,m,N}(\sigma+it,\sigma-it;x)\end{aligned}$$

follows by (2.9) and (5.16). Here the first term on the right side is estimated as $\ll |t|^{-1}l^{-\sigma-N}$ by (6.2), while $f_{K+1,l,m,N}$ in the second term, by (5.14) and (5.16), as

$$\begin{aligned}&\ll \sum_{\substack{j_1+j_2+j_3=m-1\\ j_1,j_2,j_3\geq 0}} |(m-2\sigma)\cdots(m-2\sigma-j_2+1)|l^{m-2\sigma-j_2}\\ &\quad\times \int_l^\infty \frac{|(2\sigma-m-K-2)\cdots(2\sigma-m-K-1-j_3)|y^{2\sigma-m-K-2-j_3}}{(y+x)^{\sigma+N-K-1}}dy\\ &\ll l^{m-2\sigma}\int_l^\infty y^{\sigma-m-N-1}dy \ll l^{-\sigma-N}\end{aligned}$$

for $\sigma < m+N$ and $l \geq 1$; these bounds show that $f_{K,l,m,N}(\sigma+it,\sigma-it;x) \ll |t|^{-1}l^{-\sigma-N}$, and this implies the fourth estimate in (2.16) for $1-N<\sigma<m+N$. Corollary 1.2 is thus proved. Corollary 1.3 is derived by letting $N\to\infty$ in (2.14). The proofs are complete. □

*Proofs of Corollaries 1.4 and 1.5.* For the proofs we first prepare

**Lemma 6**

*For any integer $m \geq 1$ and any $a>0$ we have*

$$\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}(a+j)^{\nu+m-1} = \begin{cases}0 & \textit{if } \nu=-1,\dots,-m+1,\\ (m-1)! & \textit{if } \nu=0.\end{cases} \tag{6.3}$$

*Proof.* The induction on $m$ shows that the left side of (6.3) is equal to

$$(\nu+1)\cdots(\nu+m-1)\int_0^1\cdots\int_0^1(a+x_1+\cdots+x_{m-1})^\nu dx_1\cdots dx_{m-1},$$

which immediately implies the assertion. □

To prove Corollary 1.4, we set $\sigma=1/2+\delta$ with a small $\delta$ in (2.14), and compute its limiting form as $\delta\to0$. Since

$$\frac{1}{(-2\delta)_m}=\frac{\Gamma(1-2\delta)}{\Gamma(m-2\delta)(-2\delta)}=\Gamma(m)^{-1}\Big\{-\frac12\delta^{-1}-\gamma_0-\psi(m)+O(\delta)\Big\},$$

and

$$\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}(a+j)^{m-1-2\delta}=(m-1)!\{1-2C_{m,m}(a)\delta+O(\delta^2)\}$$

by (6.3) and (2.18), the first term on the light side of (2.14) is

$$\frac12\delta^{-1}+\gamma_0+\psi(m)-C_{m,m}(a)+O(\delta)\qquad(\delta\to0). \tag{6.4}$$

On the other hand, noting

$$\frac{\Gamma(1/2+it-\delta)}{\Gamma(1/2+it+\delta)}=1-2\psi\Big(\frac12+it\Big)\delta+O(\delta^2),$$

form (2.5) we have

$$\begin{aligned}R\Big(\frac12+it+\delta,\frac12-it+\delta;\lambda\Big)&=\frac12\zeta_\lambda(0)\delta^{-1}-\zeta_\lambda(0)\Big\{\gamma_0+\psi\Big(\frac12+it\Big)\Big\}\\&\quad+\zeta_\lambda'(0)+O(\delta).\end{aligned} \tag{6.5}$$

The assertion (2.19) of Corollary 1.4 is thus concluded by combining (6.4) with (6.5), because $\psi(m)=\sum_{j=1}^{m-1}1/j-\gamma_0$,

$$\zeta_\lambda(0)=\begin{cases}-1/2 & \text{if } \lambda\in\mathbb{Z},\\ (-1+i\cot\pi\lambda)/2 & \text{otherwise}\end{cases}$$

(cf. [15, (2.8) and (7.6)]), and the terms $S_{m,N}$ and $T_{m,N}$ are continuous when $1-N<\sigma<m+N$ and $|t|\ge1$.

We next set $\sigma=1+\delta$ with a small $\delta$ in (2.14) to prove Corollary 1.5. If $m\ge2$, we have

$$\frac{1}{(-1-2\delta)_m}=\frac{\Gamma(1-2\delta)}{\Gamma(m-1-2\delta)(-2\delta)(-1-2\delta)}=\frac12\Gamma(m-1)^{-1}\{\delta^{-1}+O(1)\}$$

and

$$\sum_{j=0}^{m-1}(-1)^{m-1-j}\binom{m-1}{j}(a+j)^{m-2-2\delta}=-2(m-2)!C_{m,m-1}(a)\{\delta+O(\delta^2)\}$$

by (6.3) and (2.18), and hence the first term on the right side of (2.14) is

$$C_{m,m-1}(a)+O(\delta)\qquad(\delta\to0), \tag{6.6}$$

with the excluded case $m=1$ being incorporated. On the other hand, noting (2.20) and

$$\frac{\Gamma(it-\delta)}{\Gamma(1+it+\delta)}=\frac{1}{it}\Big\{1-\Big(2\psi(1+it)-\frac{1}{it}\Big)\delta+O(\delta^2)\Big\},$$

from (2.5) we have

$$(6.7)\qquad \begin{aligned} R(1+it+\delta, 1-it+\delta; \lambda) = \frac{1}{it}\Big\{\frac{1}{2}\gamma_{-1}(\lambda)\delta^{-1} + \gamma_0(\lambda) - \gamma_0\gamma_{-1}(\lambda) \\ - \gamma_{-1}(\lambda)\Big(\psi(1+it) - \frac{1}{2it}\Big) + O(\delta)\Big\}. \end{aligned}$$

The assertion (2.23) is thus concluded by combining (6.6) with (6.7), upon noting (2.21) and (2.22). Corollaries 1.4 and 1.5 are proved. □

*Proofs of Corollaries 1.6 and 1.7.* We first prove Corollary 1.6 by letting $a \to +0$ in (2.4). On the right side of (2.4), the term with $j = 0$ in the first sum vanishes when $a \to +0$ by the condition $\mathrm{Re}(u+v) < m$. Next since $\phi(u+n, 1+x, \lambda) = O(1)$ and $f_{K,l,m,N}(u, v; x) = O(l^{-\mathrm{Re}\,v-N})$, we see that the term with the index $n$ in (2.6) is of order $O(x^{m+n-\mathrm{Re}\,v})$ and $T_{m,N}(u, v; x, \lambda) = O(x^{m+N-\mathrm{Re}\,v})$ as $x \to +0$. Hence the terms with $j = 0$ in the second sum on the right side of (2.4) also vanish by the conditions $\mathrm{Re}\,u < m$ and $\mathrm{Re}\,v < m$. We therefore conclude (2.24). Formulae (2.26) and (2.27) are immediate by letting $a \to +0$ in (2.19) and (2.23) respectively, upon noting that $\lim_{a\to+0} C_{m,n}(a) = C^*_{m,n}$ for $m, n \geq 2$. The proofs are complete. □

## 7. Proofs of Theorems 2, 3 and their corollaries

We first proceed to prove Theorem 2 and their corollaries. In order to take multiple summation of both sides of (4.2), we need

**Lemma 7**

*Let $a$ and $\lambda$ be real parameters with $a > 0$. Then for any positive integers $m$ and $q$ the relation*

$$(7.1)\qquad \sum_{r_1=0}^{q-1} \cdots \sum_{r_m=0}^{q-1} \zeta\Big(w, \frac{a}{q^m} + \frac{r_1}{q} + \cdots + \frac{r_m}{q^m}\Big) = q^{mw}\zeta(w, a)$$

*holds for any complex $w \neq 1$.*

*Proof.* Since $r = \sum_{j=1}^m r_j q^{m-j}$ with $r_j \in \{0, 1, \ldots, q-1\}$ is the unique $q^m$-adic representation of an integer $r \in \{0, 1, \ldots, q^m - 1\}$, we find that the left side of (7.1) reduces to

$$\sum_{r=0}^{q^m-1} \zeta\Big(w, \frac{a+r}{q^m}\Big) = \sum_{r=0}^{q^m-1} \sum_{n=0}^{\infty} \Big(n + \frac{a+r}{q^m}\Big)^{-w} \quad (\mathrm{Re}\,w > 1),$$

which is further modified into the right side of (7.1). The assertion therefore follows from the analytic continuation. □

Applying this lemma to (4.1) and (4.2), from (3.1) we obtain

$$(7.2)\qquad \begin{aligned} \widehat{I}_m(u, v; a, \lambda; q) = q^{m(u+v-1)}\zeta(u+v, a) + R(u, v; \lambda) + R(v, u; -\lambda) \\ + \widehat{g}(u, v; a, \lambda; q^m) + \widehat{g}(v, u; a, -\lambda; q^m) \end{aligned}$$

for $\mathrm{Re}\,u > 1$ and $\mathrm{Re}\,v > 1$, where

$$(7.3)\qquad \widehat{g}(u, v; a, \lambda; q^m) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(u+s)\Gamma(-s)}{\Gamma(u)}\zeta_\lambda(-s)\zeta(u+v+s, a)q^{m(u+v+s-1)}ds,$$

and $\widehat{g}(v,u;a,-\lambda;q^m)$ is similarly defined. Note that the case $m=1$ and $a=1$ of (7.3) was studied in [9, Section 5]; the same treatment as there by path moving is applied to the present case, and this yields that

$$\widehat{g}(u,v;a,\lambda;q^m)=\widehat{S}_N(u,v;a,\lambda;q^m)+\widehat{T}_N(u,v;a,\lambda;q^m), \tag{7.4}$$

where $\widehat{S}_N$ is defined by (3.4) and

$$\begin{aligned}&\widehat{T}_N(u,v;a,\lambda;q^m)\\ &=\frac{1}{2\pi i}\int_{(c_N)}\frac{\Gamma(u+s)\Gamma(-s)}{\Gamma(u)}\zeta_\lambda(-s)\zeta(u+v+s,a)q^{m(u+v+s-1)}ds.\end{aligned} \tag{7.5}$$

Here the restriction on $(u,v)$ above can now be relaxed to $\operatorname{Re}u>1-N$ and $\operatorname{Re}v<1+N$, and then the constant $c_N$ in (7.5) is taken with $-\operatorname{Re}u-N<c_N<\min(-1,-\operatorname{Re}u-N+1,1-\operatorname{Re}(u+v))$, by which the path $(c_N)$ separates the (possible) poles of the integrand at $s=-u-n$ $(n=N,N+1,\ldots)$ from the poles at $s=1-u-v$, $-1+n$ $(n=0,1,\ldots)$ and $-u-n$ $(n=0,1,\ldots,N-1)$. The assertions (3.3) and (3.4) thus follow from (7.2)–(7.4). A further path moving in (7.5) form $(c_N)$ to $(c_{N+1})$ shows

$$\begin{aligned}\widehat{T}_N(u,v;a,\lambda;q^m)=&\frac{(-1)^N(u)_N}{N!}\zeta_\lambda(u+N)\zeta(v-N,a)q^{m(v-N-1)}\\ &+\widehat{T}_{N+1}(u,v;a,\lambda;q^m),\end{aligned}$$

which implies (3.5), since the estimate $\widehat{T}_{N+1}(u,v;a,\lambda;q^m)\ll q^{m\{\operatorname{Re}(u+v)+c_{N+1}-1\}}$ holds with $c_{N+1}<-\operatorname{Re}u-N$. The proof of Theorem 2 is complete. □

*Proof of Corollary 2.1.* Formula (3.6) follows by setting $u=\sigma+it$ and $v=\sigma-it$ in (3.3). The same method as in [11, Section 4], where the case $m=1$, $a=1$ and $\lambda\in\mathbb{Z}$ of (7.5) is treated, is also applicable to the present case; a suitable modification of the path $(c_N)$ have to be made according to the vertical bounds of the integrand (see [11, Lemma]), and the estimate (3.7) is concluded. Corollary 2.1 is thus proved. □

*Proof of Corollary 2.2.* Formula (3.8) is obtained by letting $\sigma\to1/2$ in (3.6), upon noting (6.5) and

$$\zeta(1+2\delta,a)=\frac{1}{2}\delta^{-1}-\psi(a)+O(\delta)$$

as $\delta\to0$ (cf. [5, p. 26, 1.10(9)]), while Formula (3.9) by letting $\sigma\to1$ in (3.6), upon noting (6.7), (2.21) and (2.22). □

We finally proceed to prove Theorem 3 and its corollaries. For the proof of Theorem 3, note first from (3.2) that

$$H(u,v;a,\lambda;q)=\int_0^1\widehat{I}_1(u,v;a+x,\lambda;q)dx.$$

We therefore replace $a$ by $a+x$ in (7.2) and (7.3) with $m=1$, and integrate both sides with respect to $x$ over $[0,1]$ to obtain, by (5.1) and (5.3) with $m=1$, that

$$\begin{aligned}H(u,v;a,\lambda;q)=&-\frac{(a/q)^{1-u-v}}{1-u-v}+R(u,v;\lambda)+R(v,u;-\lambda)\\ &-\widetilde{g}_1\Big(u,v;\frac{a}{q},\lambda\Big)-\widetilde{g}_1\Big(v,u;\frac{a}{q},-\lambda\Big)\end{aligned}$$

for $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$. This formula shows that the remaining analysis reduces precisely to that of the case $m = 1$ and $x = a/q$ of (5.3). The assertion (3.10) therefore follows. Furthermore, as shown in the proof of Corollary 1.6, the term with the index $n$ on the right side of (2.6) (with $m = 1$ and $x = a/q$) is of order $O\{(a/q)^{1+n-\operatorname{Re} v}\}$ as $a/q \to +0$, and also the estimate (3.11) holds. The proof of Theorem 3 is complete. □

*Proof of Corollary 3.1.* Formula (3.12) is immediate by setting $u = \sigma + it$ and $v = \sigma - it$ in (3.10). The computations of the limiting forms as $\sigma \to 1/2$ and $\sigma \to 1$ of (3.12) reduce precisely to those for Corollaries 1.4 and 1.5, respectively, with $m = 1$ and $a/q$ in place of $a$. □

## 8. A supplementary argument

The purpose of this section is to show that the inversion of the order of the integrals in (5.5) is justified by Fubini's theorem under the conditions $\operatorname{Re} u > 1$, $\operatorname{Re} v < 1$ and $\operatorname{Re}(u + v) < 1/2$. For this the following lemma plays a key rôle.

**Lemma 8**

*Let $\alpha$, $\beta$ and $T$ be real parameters with $\alpha > 0$ and $\beta > 0$. Then the estimate*

$$\int_{-\infty}^{\infty} |\Gamma(\alpha + i(T+t))\Gamma(\beta - it)|dt \ll (|T| + 1)^{\alpha+\beta} e^{-\pi|T|/2} \tag{8.1}$$

*holds for any $T$, where the implied $\ll$-constant depends on $\alpha$ and $\beta$.*

*Proof.* We first note that the vertical estimates

$$(|t| + 1)^{\sigma-1/2} e^{-\pi|t|/2} \ll \Gamma(\sigma + it) \ll (|t| + 1)^{\sigma-1/2} e^{-\pi|t|/2} \tag{8.2}$$

hold for $\sigma > 0$ and any $t \in \mathbb{R}$, with the implied $\ll$-constants depend on $\sigma$ (cf. [8, p. 492, (A.34)]). Suppose temporarily that $T \geq 2$. We estimate the integral $I$ in (8.1), say, by dividing $I = \sum_{j=1}^{5} I_j$, where

$$I_1 = \int_{-\infty}^{-T-1}, \qquad I_2 = \int_{-T-1}^{-T+1}, \qquad I_3 = \int_{-T+1}^{-1}, \qquad I_4 = \int_{-1}^{1}, \qquad I_5 = \int_{1}^{\infty}.$$

Using the upper bound in (8.2), we can show that

$$I_1, I_2 \ll T^{\beta-1/2} e^{-\pi T/2}; \qquad I_3 \ll T^{\alpha+\beta} e^{-\pi T/2}; \qquad I_4, I_5 \ll T^{\alpha-1/2} e^{-\pi T/2}.$$

The assertion therefore follows for $T \geq 2$, while the case $T \leq -2$ is reduced to the preceding case by the reflection principle. The remaining case $|T| \leq 2$ is trivial. □

To justify the inversion, it suffices to show the absolute convergence of the double integral

$$\int_{(b_0)} \frac{\Gamma(m + r)\Gamma(-r)e^{\pi ir}}{\Gamma(1 - v + m + r)} \int_{(c_0)} \Gamma(u + r + s)\Gamma(-s)\zeta_\lambda(-s)x^{m-u-v-s} ds dr. \tag{8.3}$$

We write $r = b_0 + i\tau$ and $s = c_0 + it$. Noting the inequality $\zeta_\lambda(-s)x^{m-u-v-s} \ll 1$ on the path $\operatorname{Re} s = c_0(< -1)$, we see that the inner $s$-integral is estimated, by (8.1), as

$$\ll (|\operatorname{Im} u + \tau| + 1)^{\operatorname{Re} u + b_0} e^{-\pi|\operatorname{Im} u + \tau|/2} \ll (|\tau| + 1)^{\operatorname{Re} u + b_0} e^{-\pi|\tau|/2},$$

for any $\tau \in \mathbb{R}$, upon noting $\operatorname{Re} u+b_0+c_0 > 0$ and $-c_0 > 0$. Moreover, the gamma factors which are irrelevant to $s$ are estimated, by (8.2), as $\ll (|\tau|+1)^{\operatorname{Re} v-b_0-3/2} e^{-\pi|\tau|/2-\pi\tau}$ for any $\tau \in \mathbb{R}$, upon noting $m+b_0 > 0$, $-b_0 > 0$ and $1-\operatorname{Re} v+m+b_0 > 0$. The double integral in (8.3) is therefore bounded as

$$\ll \int_{-\infty}^{\infty} (|\tau|+1)^{\operatorname{Re}(u+v)-3/2} e^{-\pi(|\tau|+\tau)} d\tau < +\infty,$$

provided $\operatorname{Re}(u+v) < 1/2$. This establishes the required change.

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