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Spaces of Lipschitz and Hölder functions and their applications

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Abstract

We study the structure of Lipschitz and Hölder-type spaces and their preduals on general metric spaces, and give applications to the uniform structure of Banach spaces. In particular we resolve a problem of Weaver who asks whether if *M* is a compact metric space and $0 < \alpha < 1$, it is always true the space of Hölder continuous functions of class α is isomorphic to ℓ_∞. We show that, on the contrary, if *M* is a compact convex subset of a Hilbert space this isomorphism holds if and only if M is finite-dimensional. We also study the (related) problem of when a quotient map $Q{:}Y \longrightarrow X$ between two Banach spaces admits a section which is uniformly continuous on the unit ball of *X*.

1. Introduction and description of results

This paper continues the ideas developed in [19]. Let $M = (M, d)$ be a metric space, with a designated origin (or special point) 0. We denote by $\text{Lip}(M)$ the space of all real-valued Lipschitz functions on M for which $f(0) = 0$ under the standard Lipschitz norm,

$$\|f\|_{\text{Lip}} = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : \qquad x \neq y\right\}.$$

The underlying idea of [19] is to study the Lipschitz structure of a metric space M (in particular the Lipschitz structure of a Banach space) by understanding the Banach space geometry of the associated space $\text{Lip}(M)$ of real-valued Lipschitz functions and of its canonical predual $\mathcal{F}(M)$, which we termed the *free-Lipschitz space* on M. This space is also known as the *Arens-Eells space* in [48]. The key property of the free-Lipschitz space $\mathcal{F}(M)$ is that a Lipschitz map $L : M_1 \to M_2$ admits a linearization

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$\hat{L} : \mathcal{F}(M_1) \to \mathcal{F}(M_2)$ (whose adjoint is the natural composition $f \to f \circ L$ ($\mathrm{Lip}(M_2) \to \mathrm{Lip}(M_1)$)). For precise definitions we refer to § 3.

In this paper we are motivated by the problem of understanding uniform homeomorphisms between Banach spaces or subsets of Banach spaces. In order to consider problems of this type it is necessary to consider changes of metric on the underlying space M. Thus if (M, d) is a metric space (with d as the given metric space) we consider $(M, \omega \circ d)$ where ω is any subadditive function $\omega : [0, \infty) \to [0, \infty)$ which satisfies $\lim_{t \to 0} \omega(t) = \omega(0) = 0$. In this case we write $\mathrm{Lip}_\omega(M) = \mathrm{Lip}(M, \omega \circ d)$ and $\mathcal{F}_\omega(M) = \mathcal{F}(M, \omega \circ d)$. Of particular importance is the choice $\omega(t) = t^\alpha$ where $0 < \alpha < 1$, which is related to the behavior of Hölder continuous maps on M. Replacing (M, d) by (M, d^α) is sometimes called *snowflaking* (see e.g. [20]). We denote the corresponding free space $\mathcal{F}(M, d^\alpha)$ by $\mathcal{F}^{(\alpha)}(M)$, and the corresponding Lipschitz space $\mathrm{Lip}^{(\alpha)}(M)$.

As it turns out, we feel that there is an interesting interplay, between nonlinear Banach space theory ([7]) and the linear theory of Lipschitz spaces and their preduals. Let us give an example. The basic theory of spaces of Lipschitz functions and their preduals is treated in the recent book of Weaver [48]. Although Weaver's motivation is to study Lipschitz algebras, he also treats the basic linear structure in some detail. Of particular interest is the situation when (M, d) is a compact metric space. In this case, if $0 < \alpha < 1$, $\mathcal{F}^{(\alpha)}(M)$ is itself the dual of the so-called *little Lipschitz space* of all $f \in \mathrm{Lip}^\alpha(M)$ such that

$$\lim_{\epsilon \to 0} \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : \qquad 0 < d(x, y) < \epsilon \right\} = 0.$$

In the special case when M is subspace of $\mathbb{R}^n$ (with the standard Euclidean metric) then an old result of Bonic, Frampton and Tromba [9] (corrected in [48]) asserts that $\mathrm{lip}^{(\alpha)}(M)$ is isomorphic to c_0. Weaver asks whether such a result holds for all compact metric spaces. We will answer this question negatively; in fact, for example, if M is a compact convex subset of a Hilbert space, then $\mathrm{lip}^{(\alpha)}(M)$ is isomorphic to c_0 if and only if M is finite-dimensional (and, indeed, much more general results of this type are obtained). The technique used to obtain these theorems itself leads to new questions about the nonlinear structure of Banach spaces, as we explain below.

Let us now describe the paper and its contents. In § 2 we simply gather together some Banach space preliminaries. In § 3 we similarly gather together known results about metric spaces and their associated free spaces. In § 4 we study some basic properties of $\mathcal{F}(M)$ and $\mathcal{F}_\omega(M)$ for arbitrary metric spaces. The most important result here is that if $\lim_{t \to 0} \omega(t)/t = \infty$ then $\mathcal{F}_\omega(M)$ is a Schur space (Theorem 4.6).

In § 5 we apply these results to the study of uniform homeomorphisms between Banach spaces, in a very similar spirit to the results of [19] for Lipschitz homeomorphisms. The key idea is that if X is a Banach space and $\omega(t) = t$ for $t \geq 1$ then there is a natural quotient map (the barycentric map) $\beta : \mathcal{F}_\omega(X) \to X$. Thus one has a short exact sequence

$$0 \longrightarrow \ker \beta \longrightarrow \mathcal{F}_\omega(X) \longrightarrow X \longrightarrow 0. \tag{1.1}$$

This sequence splits in a nonlinear sense. Precisely, there is a section $\delta : X \to \mathcal{F}_\omega(X)$ of β which assigns to each $x \in X$ the corresponding point-evaluation $\delta(x) \in \mathcal{F}_\omega(X) \subset \mathrm{Lip}_\omega(X)^*$. This section has modulus of continuity ω. Using this idea it is easy to construct many examples of pairs of non-isomorphic separable Banach spaces (X, Y) which are uniformly homeomorphic with modulus of continuity of the homeomorphism and its inverse controlled by ω (subject only to $\lim_{t\to 0} \omega(t)/t = \infty$). There are many known examples of such pairs ([7], [45], [23]) but the approach here is quite different. We also give applications to nets in Banach spaces.

In § 6, we consider compact metric spaces and Weaver's problem as described above, and review and extend the known results. The main new results are that if M is compact and $0 < \alpha < 1$ then $\mathrm{lip}^{(\alpha)}(M)$ embeds almost isometrically into c_0 (Theorem 6.6) and an example of a compact metric space failing finite Assouad dimension for which $\mathrm{lip}^{(\alpha)}(M)$ is isomorphic to c_0.

We now observe that if $\omega(t) = t$ for $t \geq 1$ then (1.1) can be modified to

$$0 \longrightarrow \ker \beta \longrightarrow \mathcal{F}_\omega(B_X) \longrightarrow X \longrightarrow 0, \tag{1.2}$$

where B_X is the unit ball of X. In this case the section δ is uniformly continuous on B_X. Thus to understand the structure of $\mathcal{F}_\omega(B_X)$ it becomes useful to understand when a quotient map $Q : Y \to X$ can admit a section which is uniformly continuous on the ball. (Let us remark here that if Q admits a section which is homogeneous and uniformly continuous on the entire space then the section is already Lipschitz and this reduces to the problem considered in [19].) In § 7 this problem is considered for the case when Y is an $\mathcal{L}_1-$space and $X = \ell_2$; it is shown that in this case no such uniformly continuous section exists. An immediate deduction is that $\mathcal{F}_\omega(B_{\ell_2})$ cannot be a $\mathcal{L}_1-$space (and, in particular, is not isomorphic to ℓ_1.)

In § 8 we answer Weaver's question by showing that if K is a compact convex subset of ℓ_2 and is infinite-dimensional then $\mathcal{F}^{(\alpha)}(K)$ cannot be isomorphic to ℓ_1; the idea is to show such an isomorphism would imply that $\mathcal{F}^{(\alpha)}(B_{\ell_2})$ is a $\mathcal{L}_1-$space and use the results of the previous section. In fact we prove much more general results. We conjecture that if K is an infinite-dimensional compact convex subset of any Banach space X then $\mathcal{F}^{(\alpha)}(K)$ cannot be isomorphic to ℓ_1 (and hence $\mathrm{lip}^{(\alpha)}(K)$ is not isomorphic to c_0 and $\mathrm{Lip}^{(\alpha)}(K)$ is not isomorphic to ℓ_∞.) We prove this if $0 < \alpha \leq \frac{1}{2}$ or if X has nontrivial Rademacher type or under certain approximation assumptions (see Theorems 8.4, 8.5 and 8.8). There are several interesting questions we could not resolve related to completing this theorem; for example it would be nice to have a good estimate of the extension constant for functions of Hölder class α when $\frac{1}{2} < \alpha < 1$ from a metric space into a Euclidean space of dimension n. See § 11.

In § 9 we introduce some terminology and study an approximation problem which seems interesting (but which raises questions we are unable to resolve). If M is a metric space, we say that M has the *uniform compact approximation property (ucap)* if there is an equi-uniformly continuous sequence of maps $\varphi_n : M \to M$ each with compact range so that $\lim_{n\to\infty} \varphi_n(x) = x$ for $x \in M$. We are particularly interested in the case when M is a closed bounded convex subset of a Banach space, especially the case of the closed unit ball. We do not know of any example of a separable Banach space X for which B_X fails (ucap).

In § 10 we return to the question of the existence of uniformly continuous sections relative to the ball for a quotient map $Q : Y \to X$. This is motivated by the results of § 7; it is natural to ask for which separable Banach spaces X the quotient $Q : \ell_1 \to X$ (which is essentially unique) admits a uniformly continuous section on the ball. Our main result concerns the construction of a global section when sections exists locally; in particular, we have in mind the situation when a quotient map Q splits locally (i.e. $\ker Q$ is locally complemented). A rather complete result which characterizes those spaces where such a construction is always possible, is given in Theorem 10.5. We should then that if $X = L_1$ or if X is the quotient of L_1 by a reflexive subspace then the quotient map $Q : \ell_1 \to X$ admits a uniformly continuous selection on the ball. We give applications to the uniform classification of the unit balls of Banach spaces. Similar results could be given for spheres although we note that Problem 9.14 of [7] asks whether the sphere is always uniformly homeomorphic to the ball.

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2. Banach space preliminaries

In this section we will gather together some basic facts from classical Banach space theory which will be used later. Most of the material can be found in [32], [49] or [15].

In this paper all Banach spaces will be real. If X is a Banach space we denote its closed unit ball by B_X and the surface of the ball by ∂B_X. Recall that the Banach-Mazur distance $d_{BM}(X, Y)$ between two Banach spaces is defined by

$$d_{BM}(X, Y) = \inf \left\{ \|T\|\|T^{-1}\| : T \text{ is an isomorphism of } X \text{ onto } Y \right\}.$$

If X and Y are linearly isomorphic we write $X \approx Y$.

Let us recall that a separable Banach space X is a $\mathcal{L}_p$−space where $1 \le p \le \infty$ if there is an increasing sequence of finite-dimensional subspaces (E_n) whose union is dense and such that the Banach-Mazur distances $d_{BM}(E_n, \ell_p^{\dim E_n})$ are bounded, i.e.

$$\sup d_{BM}(E_n, \ell_p^{\dim E_n}) < \infty.$$

Let us recall that a separable Banach space X has bounded approximation property (BAP) if there is a sequence of finite-rank operators $T_n : X \to X$ such that $\lim_{n\to\infty} T_n x = x$ for every $x \in X$; if we can take $\|T_n\| \le \lambda$ we say X has λ-(BAP). If $\lambda = 1$ then we say that X has the metric approximation property (MAP). If $S : X \to Y$ is any operator we shall say that S is *approximable* if there is a sequence of finite-rank operators T_n so that $\lim_{n\to\infty} T_n x = Sx$ for $x \in X$.

It is well-known that if X is a separable dual space and X has (BAP) (or even just the approximation property) then X has (MAP). We require a routine generalization of this fact.

Proposition 2.1

Suppose X and Y are separable dual spaces. If $S : X \to Y$ is approximable and is weak*-continuous then there is a sequence of finite-rank operators $\tilde{T}_n : X \to Y$ with $\|\tilde{T}_n\| \le \|S\|$ and $\lim_{n\to\infty} \tilde{T}_n x = Sx$ for $x \in X$.

Proof. Denote by X_* and Y_* the preduals of X and Y. Let $T_n : X \to Y$ be a sequence of bounded finite-rank operators such that $\lim_{n\to\infty} T_n x = Sx$ for $x \in X$. Let F_n be an increasing sequence of finite-dimensional subspaces of X whose union is dense. By the Principal of Local Reflexivity (see [49]) for each n there is a linear operator $V_n : T_n^*(Y^*) \to X_*$ with $\|V_n\| \le 2$ and $\langle x, T_n^* y^*\rangle = \langle x, V_n T_n^* y^*\rangle$ for $x \in F_n$ and $y^* \in Y^*$. Since $V_n T_n^* : Y_* \to X_*$ is weak*-continuous and $V_n T_n^*(Y^*) \subset X_*$ it follows that if we let $\tilde{R}_n : Y_* \to X_*$ be its restriction then $\tilde{R}_n^{**} = V_n T_n^*$. Then $\langle \tilde{R}_n^* x, y^*\rangle = \langle T_n x, y^*\rangle$ if $x \in F_n$ and $y \in Y^*$. It follows that $\tilde{R}_n^* x \to Sx$ weakly for $x \in X$; by passing to sequence of convex combinations (using Mazur's theorem and the separability of X) we can find a sequence $R_n : Y_* \to X_*$ such that $R_n^* x \to Sx$ strongly for $x \in X$.

Consider the space $\mathcal{K}(Y_*, X_*)$ of all compact operators from Y_* to X_*. Then $\mathcal{K}(Y_*, X_*)$ embeds isometrically into the space $C(B_X \times B_{Y^*})$ where B_X and B_{Y^*} have the weak*-topologies via the embedding $K \to f_K$ where $f_K(x, y^*) = \langle K^* x, y^*\rangle$. Then f_{R_n} is a bounded pointwise convergent sequence which converges to f_S. Hence f_{R_n} is weakly Cauchy and converges in $C(B_X \times B_{Y^*})^{**}$ to the Borel function $f_S(x, y^*) = \langle Sx, y^*\rangle$ regarded as an element of the bidual. Hence by using Goldstine's theorem we can find a sequence of convex combinations W_n so that $W_n^* x \to Sx$ strongly for $x \in X$ and $\|f_{W_n}\| \le \|f_S\| + n^{-1}$. Thus $\|W_n\| \le \|S\| + n^{-1}$ and the Proposition follows, by taking $\tilde{T}_n = c_n W_n$ for a suitable sequence $c_n \to 1$. $\square$

Let us recall by the Open Mapping Theorem that any bounded linear operator $S : Y \to X$ which is surjective is open and hence one can equip Y with an equivalent norm so that S becomes a quotient map.

Now suppose $Q : Y \to X$ is a quotient map. A *section* of Q is a map (not necessarily linear or continuous) $\varphi : X \to Y$ such that $Q \circ \varphi = Id_X$. Q induces a short exact sequence

$$0 \longrightarrow \ker Q \longrightarrow Y \longrightarrow X \longrightarrow 0.$$

This short exact sequence *splits* if $\ker Q$ is complemented in Y or (equivalently) there is a bounded linear section of Q; it is convenient to say then that Q splits.

We shall say that Q *locally splits* if the dual sequence

$$0 \longrightarrow X^* \longrightarrow Y^* \longrightarrow (\ker Q)^\perp \longrightarrow 0$$

splits. We recall that E is a *locally complemented* subspace of Y if $E^\perp$ is complemented in Y^* by some bounded projection P. Thus Q locally splits if $\ker Q$ is locally complemented.

The following is a simple consequence of the Principle of Local Reflexivity:

Lemma 2.2

Suppose E is a closed subspace of Y. The following are equivalent:

(1) *There is a bounded projection $P : Y^* \to E^\perp$ with $\|P\| \le \lambda$.*
(2) *If F is a finite-dimensional subspace of Y/E then for every $\epsilon > 0$ there is a bounded linear operator $T_{F,\epsilon} : F \to Y$ with $\|T\| < \lambda + \epsilon$ and $QT = Id_F$.*

Proof. (1) implies (2). One version of the Principle of Local Reflexivity ([14]) asserts that $\mathcal{L}(F, Y)^{**}$ can be identified with $\mathcal{L}(F, Y^{**})$. Let $T_0 : F \to Y$ satisfy $QT_0 = Id_F$. Let $\mathcal{H}$ be the closed subspace of all T so that $QT = 0$. Consider P^* as a linear operator from $(Y/E)^{**}$ to Y^{**}. Then it is easy to show by bipolars that $P^*|_F - T_0$ is in the weak*-closure of $\mathcal{H}$ and so P^* is in the weak*-closure of $T_0 + \mathcal{H}$. Since P^* is also in the weak*-closure in $\mathcal{L}(F, Y^{**})$ of $\|P\| B_{\mathcal{L}(F,Y)}$ it follows that there exists $T \in (T_0 + \mathcal{H}) \cap (\|P\| + \frac{1}{2}\epsilon) B_{\mathcal{L}(F,Y)}$.

In other direction consider $T^*_{F,\epsilon} : Y^* \to F^*$. By the Hahn-Banach theorem we can define a nonlinear map $\varphi_{F,\epsilon} : Y^* \to E^\perp$ so that $\varphi_{F,\epsilon}(y^*)(f) = \langle T_F f, y^* \rangle$ for $f \in F$ and $\|\varphi_F(y^*)\| \le \lambda \|y^*\|$. Regarding $(\varphi_{F,\epsilon})_{F,\epsilon}$ as a net in the space of all functions from Y^* to X^* (where X^* has the weak*-topology) we can find a cluster point P. It is easy to verify that P is bounded projection of Y^* onto $E^\perp$ with $\|P\| \le \lambda$. $\square$

It follows that if $Q : Y \to X$ locally splits then there exists λ so that for every finite-dimensional subspace F of X and $\epsilon > 0$ the quotient $Q|_{Q^{-1}(F)}$ admits a bounded linear section S with $\|S\| < \lambda + \epsilon$.

Any separable Banach space X is a quotient of ℓ_1. The elegant Lindenstrauss-Rosenthal theorem asserts that the surjection of ℓ_1 onto X is essentially unique:

Theorem 2.3 [31]

Let X be a separable Banach space not isomorphic to ℓ_1. Let $S_1, S_2 : \ell_1 \to X$ be two bounded linear surjections. Then there is a automorphism $T : \ell_1 \to \ell_1$ so that $S_2 = S_1 T$.

If X is infinite-dimensional and separable then the quotient map $Q : \ell_1 \to X$ splits if and only if X is isomorphic to ℓ_1 and locally splits if and only if X is a $\mathcal{L}_1-$space (the latter is clear since it is necessary and sufficient that X^* is isomorphic to ℓ_∞).

We will also need the Johnson-Zippin space C_1 [25]. This is defined by taking any sequence (E_n) of finite-dimensional Banach spaces which is dense for Banach-Mazur distance in the collection of all finite-dimensional Banach spaces and consider their ℓ_1-sum $\ell_1(E_n)$. The space C_1 is unique up to almost isometry (i.e. does not depend on the choice of (E_n)).

Theorem 2.4

Let X be a separable Banach space. The following conditions on X are equivalent.

1. *Whenever $Q : Y \to X$ is a quotient map which locally splits then Q splits.*
2. *X is isomorphic to a complemented subspace of C_1.*

Proof. If (1) holds let (F_n) be an increasing sequence of finite-dimensional subspaces of X whose union is dense. Consider the quotient map $Q : \ell_1(F_n) \to X$ defined by $Q(f_n)_{n=1}^\infty = \sum_{n=1}^\infty f_n$. It is trivial to check that Q locally splits. Hence X is isomorphic to a complemented subspace of $\ell_1(F_n)$ and hence to a complemented subspace of C_1.

Conversely let X be a complemented subspace of $\mathrm{C}_1 = \ell_1(E_n)$ by a projection P, and suppose $Q : Y \to X$ is a quotient map which locally splits. For each n we can find a bounded operator $T_n : E_n \to Y$ with $\sup_n \|T_n\| < \infty$ so that $QT_n e = P(\tilde{e})$ where $\tilde{e}$ is the sequence with e in the nth position and zero elsewhere. Then $T : \mathrm{C}_1 \to Y$ defined by $T(e_n) = \sum_{n=1}^\infty T_n e_n$ satisfies $QT = P$ and in particular $T|_X$ is a section of Q. $\square$

Let us also recall that a linear operator $T : X \to Y$ is called 2-absolutely summing if there is a constant C so that if $x_1, \cdots, x_n \in X$ we have

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^2 \Big)^{1/2} \leq C \sup_{\|x^*\|\leq 1} \Big(\sum_{k=1}^{n} |x^*(x_k)|^2 \Big)^{1/2}.$$

The least such constant C is denoted $\pi_2(T)$. Let us recall the Pietsch Factorization Theorem [15]:

Theorem 2.5

Let $T : X \to Y$ be 2-absolutely summing. Then there is a probability measure μ on B_{X^*} (with its weak*-topology) such that

$$\|Tx\| \leq \pi_2(T) \Big(\int_{B_{X^*}} |x^*(x)|^2 d\mu(x^*) \Big)^{1/2} \qquad x \in X.$$

The celebrated Grothendieck inequality gives the following [30], [15].

Theorem 2.6

Let X be an $\mathcal{L}_1$-space or an $\mathcal{L}_\infty$−space. Then every bounded operator $T : X \to \ell_2$ is 2-absolutely summing.

Let us also recall some facts from the isometric theory of Banach spaces. A Banach space X is called *stable* if whenever $(x_n), (y_n)$ are two sequences in X then, provided all limits exist,

$$\lim_{n\to\infty} \lim_{m\to\infty} \|x_n + y_m\| = \lim_{m\to\infty} \lim_{n\to\infty} \|x_n + y_m\|.$$

Finally we will need some facts from the so-called concentration of measure phenomenon and the local theory of Banach spaces. First let σ_n denoted normalized surface measure on $\partial B_{\ell_2^n}$.

Theorem 2.7 ([34] p. 5)

Let $A \subset \partial B_{\ell_2^n}$ satisfy $\sigma_n(A) \geq \frac{1}{2}$. Let $[A]_\epsilon = \{\xi \in \partial B_{\ell_2^n} : d(\xi, A) \leq \epsilon\}$. Then $\sigma_n([A]_\epsilon) \geq 1 - \sqrt{\frac{\pi}{8}} e^{-\frac{\epsilon^2 n}{2}}$.

The concentration of measure phenomenon is used to prove Dvoretzky's theorem which asserts that every infinite-dimensional Banach space X contains finite-dimensional subspaces E_n with $d_{BM}(E_n, \ell_2^n) \to 1$. For good Banach spaces these subspaces can be made well-complemented. We recall that X has nontrivial Rademacher type if for some $p > 1$ if there is a constant C so that

$$\Big(\mathbb{E} \,\Big\| \sum_{k=1}^{m} \epsilon_k x_k \Big\|^p \Big)^{1/p} \leq C \Big(\sum_{k=1}^{m} \|x_k\|^p \Big)^{1/p} \qquad x_1, \cdots, x_m \in X.$$

Here $(\epsilon_k)_{k=1}^m$ indicates a sequence of independent Rademacher random variables.

Theorem 2.8 [18], [34]

Suppose X is an infinite-dimensional Banach space with nontrivial type. Then X has a sequence of subspaces E_n and projections $P_n : X \to E_n$ so that $\sup_n d_{BM}(E_n, \ell_2^n) < \infty$ and $\sup_n \|P_n\| < \infty$.

We shall require a local quantitative version of this theorem, without assuming type.

Theorem 2.9

There are absolute constants c, C with the following property. Let X be an n-dimensional Banach space. Let b be the least constant so that we have both

$$\Big(\sum_{k=1}^n \|x_k\|^2\Big)^{1/2} \le b\Big(\mathbb{E}\,\Big\|\sum_{k=1}^n g_k x_k\Big\|^2\Big)^{1/2} \qquad x_1, \cdots, x_n \in X$$

and

$$\Big(\sum_{k=1}^n \|x_k^*\|^2\Big)^{1/2} \le b\Big(\mathbb{E}\,\Big\|\sum_{k=1}^n g_k x_k^*\Big\|^2\Big)^{1/2} \qquad x_1^*, \cdots, x_n^* \in X^*$$

where $g_1, \cdots, g_n$ is a sequence of independent normalized Gaussians. Then there exists $k \ge cb^{-2}n$ and linear operators $U : \ell_2^k \to X$ and $V : X \to \ell_2^k$ with $VU = Id_{\ell_2^k}$ and $\|U\|\|V\| \le C(1 + \log n)$.

Proof. Let us recall that if H is a finite-dimensional Hilbert space and $S : H \to X$ is any linear operator then we define the $\ell-$norm of S by

$$\ell(S) = \Big(\mathbb{E}\,\Big\|\sum_{k=1}^m g_k S e_k\Big\|^2\Big)^{1/2}$$

where $(e_k)_{k=1}^m$ is any orthonormal basis of H. We can then choose an isomorphism $S : \ell_2^n \to X$ with $\ell(S)\ell((S^{-1})^*) \le C_0 n(1 + \log n)$ where C_0 is an absolute constant (see [43] p. 37). Now by Corollary 15.8 (p. 111) of [34] we can find a subspace E of ℓ_2^n with $\dim E \ge cb^{-2}n$ and so that $\|S|_E\|\|(S^{-1})^*|_E\| \le 4C_0(1 + \log n)$. Let $U = S|_E$ and let $V : X \to E$ be such that $V^* = (S^{-1})^*|_E$. Then $VU = Id_E$ and we are done. □

3. Metric spaces

We now discuss metric spaces and their associated Lipschitz spaces. A good reference for much of this material is the recent book of Weaver [48], although his treatment is slightly different.

In this paper we will always consider *pointed metric spaces.* A pointed metric space is a metric space (M, d) with a distinguished point (the *origin*) which we always denote by 0. In most of our examples M is a subset of a Banach X and the origin is the origin of the Banach space. The assumption of an origin is a convenience to avoid considering spaces of Lipschitz functions modulo constants, and the particular choice of origin does not affect the theory substantially.

If A is a subset of M we denote by $[A]_\epsilon$ its $\epsilon-$neighborhood, i.e.

$$[A]_\epsilon = \{x \in M : \ d(x, A) \leq \epsilon\}.$$

The radius R of M is defined by

$$R = \sup\{d(x, 0) : \ x \in M\}.$$

We denote by $\mathrm{Lip}(M)$ the space of all real-valued Lipschitz functions $f : M \to \mathbb{R}$ with $f(0) = 0$ under the norm

$$\|f\|_{Lip} = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)}; \quad x, y \in X, \quad x \neq y\right\}.$$

The little Lipschitz space $\mathrm{lip}(M)$ is the subspace of $\mathrm{Lip}(M)$ of all functions such that

$$\lim_{\epsilon \to 0} \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)}; \quad x, y \in X, \quad x \neq y, \ d(x,y) < \epsilon\right\} = 0.$$

In general $\mathrm{lip}(M)$ can reduce to $\{0\}$; see Weaver [48] for a discussion of conditions so that $\mathrm{lip}(M)$ is sufficiently rich.

The Lipschitz-free space $\mathcal{F}(M) = \mathcal{F}(M, d)$ is defined to be the canonical predual of $\mathrm{Lip}(M)$ i.e. the closed linear span of the point evaluations

$$\delta_M(x)(f) = f(x) \qquad x \in M$$

in $\mathrm{Lip}(M)^*$. In [48] this space is called the Arens-Eells space; note however, that Weaver generally deals with bounded metrics or restricts Lipschitz functions to be bounded (which is essentially equivalent to adding a fictitious origin at distance one from every point). The map $\delta = \delta_M : M \to \mathcal{F}(M)$ is easily seen to be an isometric embedding. It is convenient to regard $\mathcal{F}(M)$ as the completion of the set of all measures μ of finite support under the norm

$$\|\mu\|_{\mathcal{F}} = \sup\left\{\int f \, d\mu : \ \|f\|_{\mathrm{Lip}} \leq 1\right\}.$$

Lemma 3.1

Let M_1 and M_2 be pointed metric spaces and suppose $L : M_1 \to M_2$ is a Lipschitz map such that $L(0) = 0$. There exists a unique linear map $\hat{L} : \mathcal{F}(M_1) \to \mathcal{F}(M_2)$ such that $\hat{L}\delta_{M_1} = \delta_{M_2} L$, i.e. so the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{L} & M_2 \\ \delta_{M_1}\Big\downarrow & & \Big\downarrow\delta_{M_2} \\ \mathcal{F}(M_1) & \xrightarrow{\hat{L}} & \mathcal{F}(M_2) \end{array}$$

Furthermore $\|\hat{L}\| = \|L\|_{Lip}$

In the special case where $M_1 \subset M_2$ (and both have the same origin) then the inclusion $\iota : M_1 \to M_2$ induces a linear isometric embedding $\hat{\iota} : \mathcal{F}(M_1) \to \mathcal{F}(M_2)$. The fact this is an isometry follows easily from the fact that Lipschitz function f on M_1 can be extended with preservation of norm to M_2 by the formula

$$\tilde{f}(y) = \inf \{ f(x) + \|f\|_{\mathrm{Lip}} d(x, y) : \ x \in M_1 \} .$$

It follows that $\mathcal{F}(M_1)$ can be regarded as a subspace of $\mathcal{F}(M_2)$.

If X is a Banach space then there is a natural linear operator $\beta = \beta_X : \mathcal{F}X \to X$ (the barycentric map) so that $\beta \circ \delta(x) = x$ for $x \in X$; see [19]. If M_2 in Lemma 3.1 is replaced by a Banach space X then composing $\hat{L}$ with β_X gives the following Lemma:

Lemma 3.2

Let X be any Banach space and let M be a pointed metric space. Let $L : M \to X$ be a Lipschitz map with $L(0) = 0$. Then there is a unique linear map $\overline{L} : \mathcal{F}(M) \to X$ so that $\overline{L}\delta_M = L$. Furthermore $\|\overline{L}\| = \|L\|_{Lip}$.

We define a *gauge* to be a function $\omega : [0, \infty) \to [0, \infty)$ which is a continuous increasing subadditive function with $\omega(0) = 0$ and $\omega(t) \geq t$ for $0 \leq t \leq 1$. We say ω is *normalized* if $\omega(1) = 1$ and *nontrivial* if $\lim_{t\to 0} \omega(t)/t = \infty$. The most natural examples of normalized nontrivial gauges are $\omega(t) = t^\alpha$ or $\omega(t) = \max(t, t^\alpha)$ when $0 < \alpha < 1$. For our purposes it will be useful to consider those gauges ω for which $\omega(t) = t$ for all $t \geq 1$, which we will term *strongly normalized*; such gauges do not distort the metric at large distances and are appropriate for the study of uniform homeomorphisms between Banach spaces. Any gauge ω is equivalent to a concave gauge ω_1.

If (M, d) is a pointed metric space then we can form a new metric by putting $d_\omega = \omega \circ d$. We remark that the procedure of replacing the metric d by a metric d^α where $0 < \alpha < 1$ is sometimes called *snowflaking* (see [20]). We then define $\mathcal{F}_\omega(M) = \mathcal{F}(M, \omega \circ d)$. If $\omega(t) = t^\alpha$ we write $\mathcal{F}^{(\alpha)}(M)$ and if $\omega(t) = \max(t, t^\alpha)$ we write $\mathcal{F}^{[\alpha]}(M)$. The dual of $\mathcal{F}_\omega(M)$ is the space $\mathrm{Lip}(M, \omega \circ d)$. If ω is non-trivial we refer to this as the *Hölder space* $\mathrm{Lip}_\omega(M)$. In the special case $\omega(t) = t^\alpha$ we write $\mathrm{Lip}^{(\alpha)}(M)$ and if $\omega(t) = \max(t, t^\alpha)$ we write $\mathrm{Lip}^{[\alpha]}(M)$. Note that if M has finite radius the spaces $\mathrm{Lip}^{(\alpha)}(M)$ and $\mathrm{Lip}^{[\alpha]}(M)$ coincide and have equivalent norms.

Let us note that if ω is strongly normalized then $\mathrm{Lip}(M) \subset \mathrm{Lip}_\omega(M)$ and the inclusion has norm one. It follows that there is a natural norm-decreasing one-one injection of $\mathcal{F}_\omega(M)$ into $\mathcal{F}(M)$ and it is clear that the range is dense. Hence we can and do regard $\mathcal{F}_\omega(M)$ as a dense subspace of $\mathcal{F}(M)$ when ω is strongly normalized.

If ω is nontrivial the little Lipschitz space associated to d_ω, $\mathrm{lip}(M, d_\omega) = \mathrm{lip}_\omega(M)$ is a nontrivial subspace of $\mathrm{Lip}_\omega(M)$ as it contains $\mathrm{Lip}(M, d)$. Let us recall that if X is a Banach space and E is a subspace of X^* then E is called *a-norming* where $a \geq 1$ if

$$\|x\| \leq a \sup_{x^* \in B_E} |x^*(x)| \qquad x \in X.$$

If $a = 1$ we call E norming. This is equivalent to requiring that B_{X^*} is contained in the weak*-closure of aB_E. It is clear that on bounded sets the weak*-topology on $\mathrm{Lip}(M)$ coincides with the topology of pointwise convergence on M. This remark implies easily:

Proposition 3.3

In order that a subspace E of $Lip(M)$ be $a-$norming it is necessary and sufficient that for any finite subset A of M containing the origin and any $f \in Lip(A)$ and $\epsilon > 0$ there exists $g \in E$ with $g(x) = f(x)$ for $x \in A$ and $\|g\|_{Lip(M)} \le (a+\epsilon)\|f\|_{Lip(A)}$.

The following Proposition is essentially contained in Weaver [48]:

Proposition 3.4

If E is a subspace of $Lip(M)$ which is also a sublattice then E is a-norming if and only if for every $x, y \in M$ and $\epsilon > 0$ there exists $f \in E$ with $\|f\|_{Lip} \le a + \epsilon$ and $|f(x) - f(y)| = d(x, y)$.

Proof. We verify Proposition 3.3. If $f \in \mathrm{Lip}(A)$ and $\|f\|_{\mathrm{Lip}} \le 1$ then we for every $x, y \in A$ we can find $f_{x,y} \in E$ with $f_{x,y}(x) = f(x)$ and $f_{x,y}(y) = f(y)$ and $\|f_{x,y}\|_{\mathrm{Lip}} \le a + \epsilon$. Let

$$g = \max_{x\in A} \min_{y\in A\setminus x} f_{x,y}.$$

Then $\|g\|_{\mathrm{Lip}} \le a + \epsilon$ and $g|_A = f$. □

Proposition 3.5

If ω is a nontrivial gauge then $lip_\omega(M)$ is a norming subspace of $Lip_\omega(M)$.

Proof. For each $n \in \mathbb{N}$ let

$$\omega_n(t) = \inf\{\omega(s) + n(t-s) : \ 0 \le s \le t\}.$$

Suppose $y \in M$. Then let $f_n(x) = d_{\omega_n}(x, y) - d_{\omega_n}(x, 0)$. Clearly $f_n \in \mathrm{Lip}(M) \subset \mathrm{lip}_\omega(M)$ and $\|f_n\|_{\mathrm{Lip}_\omega} \le 1$. For any $y \in M$ we have $f_n(y) - f_n(x) \to d_\omega(x, y)$. Now apply Proposition 3.4. □

Let us discuss the notion of a quotient space in the category of pointed metric spaces. Suppose M is a (pointed) metric space and A is a closed subset of M containing the origin. We define the quotient M/A as the space $M \setminus A \cup \{0\}$ with the metric d' given by

$$d'(x,y) = \begin{cases} \min(d(x,y), d(x,A) + d(y,A)) & x, y \neq 0 \\ d(x,A) & y = 0. \end{cases}$$

We refer to the discussion in Weaver [48] p. 12 (note that Weaver considers more general quotients but our restricted definition is a special case). It then follows that we have a natural short exact sequence

$$0 \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}(M) \longrightarrow \mathcal{F}(M/A) \longrightarrow 0.$$

Indeed the quotient map from $\mathcal{F}(M)$ to $\mathcal{F}(M/A)$ is induced by the Lipschitz map

$$\varphi(x) = \begin{cases} x & x \notin A \\ 0 & x \in A \end{cases}.$$

This follows from Proposition 1.4.3 p. 12 of [48] which identifies isometrically with $\mathrm{Lip}(M/A)$ with $\mathcal{F}(A)^\perp$ in $\mathrm{Lip}(M)$.

The quotient construction for pointed metric spaces behaves nicely with respect to gauges. To be precise if $d_\omega = \omega \circ d$ then it is clear that

$$\omega(d'(x,y)) \le d'_\omega(x,y) \le 2\omega(d'(x,y)) \qquad x, y \in M/A.$$

Thus, up to a constant 2, we have that $\mathcal{F}_\omega(M)/\mathcal{F}_\omega(A) \approx \mathcal{F}_\omega(M/A)$. Let us mention in this context a recent result of Brudnyi and Shvartsman [10], which we restate using Lemma 3.2:

Theorem 3.6

Suppose M is a pointed metric space and A is a closed subset containing the origin. Suppose that X is a Banach space with the property that every bounded operator $T : \mathcal{F}(A) \to X$ has a bounded extension $\tilde{T} : \mathcal{F}(M) \to X$. Then for every gauge ω, it is also true that every operator $T : \mathcal{F}_\omega(A) \to X$ has a bounded extension $\tilde{T} : \mathcal{F}_\omega(M) \to X$.

Remark. In [10] the gauge is assumed concave, but it is easy to see that any gauge is equivalent to a concave gauge.

Corollary 3.7

Suppose A is a closed subset of M containing the origin. Then if $\mathcal{F}(A)$ is complemented in $\mathcal{F}(M)$ it follows that for every gauge ω, we also have that $\mathcal{F}_\omega(A)$ is complemented in $\mathcal{F}_\omega(M)$.

Proof. Take $X = \mathcal{F}_\omega(A)$ and use Theorem 3.6. □

4. The structure of Lipschitz and Hölder spaces

We first study the Banach-space structure of Lipschitz and Hölder spaces. If M is an arbitrary metric space let $M_k = \{x \in M : \ d(x,0) \le 2^k\}$ for $k \in \mathbb{Z}$.

Lemma 4.1

Suppose $r_1, r_2, \cdots, r_n, s_1, \cdots, s_n \in \mathbb{Z}$ and $r_1 < s_1 < r_2 < s_2 < \cdots < r_n < s_n$. Suppose $\gamma_k \in \mathcal{F}(M_{s_k} \setminus M_{r_k})$ for $1 \le k \le n$. Let $\theta = \min_{1 \le k \le n}(r_{k+1} - s_k)$. Then

$$\|\gamma_1 + \cdots + \gamma_n\|_{\mathcal{F}} \ge \frac{2^\theta - 1}{2^\theta + 1} \sum_{k=1}^{n} \|\gamma_k\|_{\mathcal{F}}\,.$$

Proof. Pick $f_k \in \mathrm{Lip}(M)$ so that $\langle \gamma_k, f_k \rangle = \|\gamma_k\|_{\mathcal{F}}$ and $\|f_k\|_{\mathrm{Lip}} = 1$. We consider the map g defined on $\{0\} \cup \cup_{k=1}^{n}(M_{s_k} \setminus M_{r_k})$ by

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ f_k(x) & \text{if } x \in M_{s_k} \setminus M_{s_{k-1}}. \end{cases}$$

If $x \in M_{s_k} \setminus M_{r_k}$ and $y \in M_{s_j} \setminus M_{r_j}$ where $j < k$ then

$$\begin{aligned}|g(x) - g(y)| &\leq d(x,0) + d(y,0)\\ &\leq (1 + 2^{-\theta})d(x,0)\\ &\leq \frac{1+2^{-\theta}}{1-2^{-\theta}} d(x,y).\end{aligned}$$

It follows that we can extend g to some $f \in \mathrm{Lip}(M)$ with $\|f\|_{\mathrm{Lip}} \leq (2^\theta + 1)(2^\theta - 1)^{-1}$. Then

$$\|\gamma_1 + \cdots + \gamma_k\|_{\mathcal{F}} \geq \frac{2^\theta - 1}{2^\theta + 1} \sum_{k=1}^{n} \langle \gamma_k, f\rangle$$

and the result follows. □

Now for $n \in \mathbb{Z}$ let $T_n : \mathcal{F}(M) \to \mathcal{F}(M)$ be the linear operator such that

$$T_n\delta(x) = \begin{cases} 0 & \text{if } x \in M_{n-1} \\ (\log_2 d(x,0) - n + 1)\delta(x) & \text{if } x \in M_n \setminus M_{n-1} \\ (n + 1 - \log_2 d(x,0))\delta(x) & \text{if } x \in M_{n+1} \setminus M_n \\ 0 & \text{if } x \notin M_{n+1}. \end{cases}$$

Lemma 4.2

For every $\gamma \in \mathcal{F}(M)$ we have $\gamma = \sum_{n\in\mathbb{Z}} T_n\gamma$ unconditionally and

$$\sum_{n\in\mathbb{Z}} \|T_n\gamma\|_{\mathcal{F}} \leq 72\|\gamma\|_{\mathcal{F}}. \tag{4.1}$$

Proof. Let us write $T_n\delta(x) = \psi_n(x)\delta(x)$. Then we have (for $x, y \neq 0$),

$$|\psi_n(x) - \psi_n(y)| \leq \left|\log_2 \frac{d(x,0)}{d(y,0)}\right|.$$

Hence assuming $d(y,0) \geq d(x,0)$,

$$\begin{aligned}\|T_n\delta(x) - T_n\delta(y)\|_{\mathcal{F}} &\leq |\psi_n(y)|d(x,y) + |\psi_n(x) - \psi_n(y)|d(x,0)\\ &\leq d(x,y) + d(y,0) - d(x,0)\\ &\leq 2d(x,y).\end{aligned}$$

Let $(a_n)_{n\in\mathbb{Z}}$ be a finitely nonzero sequence and consider the operator $S = \sum_{n\in\mathbb{Z}} a_nT_n$. Then

$$\|S\delta(x) - S\delta(y)\| \leq \Bigg(\sum_{|\log_2 d(x,0)-n|<1} + \sum_{|\log_2 d(y,0)-n|<1}\Bigg)\|T_n\delta(x) - T_n\delta(y)\|$$

so that

$$\|S\delta(x) - S\delta(y)\| \leq 8d(x,y).$$

It follows that $\|S\| \leq 8$. It is easy to see that $\sum_{n\in\mathbb{Z}} T_n\delta(x)$ converges unconditionally for all $x \in M$ and hence it follows that $\sum_{n\in\mathbb{Z}} T_n\gamma$ converges unconditionally for all $\gamma \in \mathcal{F}(M)$.

Now by Lemma 4.1 if $r = 0, 1, 2$

$$\sum_{n\in\mathbb{Z}} \|T_{3n+r}\gamma\| \le 3\|\sum_{n\in\mathbb{Z}} T_{3n+r}\gamma\| \le 24\|\gamma\|_{\mathcal{F}}.$$

The Lemma follows quickly. □

The next Proposition uses arguments very similar to those employed in [27] to obtain almost isometries.

Proposition 4.3

If $\epsilon > 0$ then the space $\mathcal{F}(M)$ is $(1+\epsilon)$-isometric to a subspace of $\ell_1(\mathcal{F}(M_k))_{k\in\mathbb{Z}}$.

Proof. For each $\epsilon > 0$ we exhibit an operator $S = S_\epsilon : \mathcal{F}(M) \to \ell_1(\mathcal{F}(M_k))_{k\in\mathbb{Z}}$ so that $\|\gamma\|_{\mathcal{F}} \le \|S\gamma\| \le (1+\epsilon)\|\gamma\|_{\mathcal{F}}$.

Let $r, m \in \mathbb{Z}$ be chosen so that

$$\frac{2^{r-1}+1}{2^{r-1}-1}\Big(1+\frac{73}{m-1}\Big) < 1+\epsilon.$$

For $1 \le j \le m$, let $\mathbb{A}_{j,n}$ be the set $\{mnr+jr+1, mnr+jr+2, \cdots, mnr+(j+m-1)r\}$. Let $V_{jn} : \mathcal{F}(M) \to \mathcal{F}(M_{mnr+(j+m-1)r+1})$ be defined by

$$V_{jn}\gamma = \sum_{k\in\mathbb{A}_{j,n}} T_k\gamma.$$

We can thus induce a map $W_j : \mathcal{F}(M) \to \ell_1(\mathcal{F}(M_k))$ by setting $W_j\gamma$ to be sequence (ν_k) where $\nu_k = V_{jn}\gamma$ if $k = mnr + (j+m-1)r + 1$ and 0 otherwise. Then, applying Lemma 4.1 to the sequence $(V_{jn}\gamma)_{n\in\mathbb{Z}}$,

$$\|W_j\gamma\| = \sum_{n\in\mathbb{Z}} \|V_{jn}\gamma\|_{\mathcal{F}} \le \frac{2^{r-1}+1}{2^{r-1}-1}\Big\|\sum_{n\in\mathbb{Z}} V_{jn}\gamma\Big\|.$$

Let $\mathbb{B}_j$ be the complement of $\cup_{n\in\ Z}\mathbb{A}_{j,n}$. The sets $\mathbb{B}_j$ are pairwise disjoint. Then

$$\|W_j\gamma\| \le \frac{2^{r-1}+1}{2^{r-1}-1}\Big(\|\gamma\| + \sum_{n\in\mathbb{B}_j}\|T_n\gamma\|\Big).$$

Hence summing from $j = 1, 2, \cdots, m$,

$$\sum_{j=1}^{m} \|W_j\gamma\| \le \frac{2^{r-1}+1}{2^{r-1}-1}(m+72)\|\gamma\|.$$

On the other hand

$$\sum_{j=1}^{m}\sum_{n\in\mathbb{Z}} \|V_{jn}\gamma\| \ge (m-1)\|\gamma\|.$$

We thus define

$$S : \mathcal{F}(M) \to Y$$

where Y is ℓ_1-sum of m copies of $\ell_1(\mathcal{F}(M_k))$ by

$$S\gamma = \frac{1}{m-1}(W_j\gamma)_{j=1}^{m}$$

and then

$$\|\gamma\| \le \|S\gamma\| \le \frac{2^{r-1}+1}{2^{r-1}-1}\Big(1+\frac{73}{m-1}\Big)\|\gamma\|.$$

Since Y isometrically embeds into $\ell_1(\mathcal{F}(M_k))$ we are done. □

Proposition 4.4

Let M be uniformly discrete, i.e. suppose $\theta = \inf_{x\neq y} d(x,y) > 0$. Then $\mathcal{F}(M)$ is a Schur space with the Radon-Nikodym Property and the approximation property.

Proof. It is easy to see that for $f \in \mathrm{Lip}(M_k)$ we have, assuming M_k nonempty,

$$2^{-k}\|f\|_\infty \le \|f\|_{\mathrm{Lip}} \le 2\theta^{-1}\|f\|_\infty.$$

It follows that $\mathcal{F}(M_k)$ is isomorphic to ℓ_1. The facts that $\mathcal{F}(M)$ is has the Radon-Nikodym property and the Schur property now follow from Proposition 4.3. Note that each $\mathcal{F}(M_k)$ has the approximation property and hence Lemma 4.2 implies that $\mathcal{F}(M)$ has the approximation property. □

Remarks. Although $\mathcal{F}(M)$ embeds (in this case) in an ℓ_1-sum of spaces isomorphic to ℓ_1 it, of course does not follow that X embeds into ℓ_1. We will see examples below.

We also may ask if $\mathcal{F}(M)$ has the (BAP) or even (MAP). This is related to the unsolved problem of whether ℓ_1 in every renorming has the (MAP) (see [11]).

If M is not uniformly discrete then $\mathcal{F}(M)$ can fail to be a Schur space (cf. e.g. [19]). However the following Lemma shows that weakly null sequences are almost supported on "small" sets.

Let E be a subset of M. For $\gamma \in \mathcal{F}(M)$ let us define

$$D(\gamma, E) = \inf\{\|\gamma-\mu\| :\ \mu \in \mathcal{F}(E)\}.$$

Let us start by noting that for all $\gamma \in \mathcal{F}(M)$ we have

$$\inf\{D(\gamma,E) :\ |E| < \infty\} = 0.$$

It follows without difficulty that if $K \subset \mathcal{F}(M)$ is relatively compact then

$$\inf_{|E|<\infty}\sup_{\gamma\in K} D(\gamma,E) = 0.$$

Now if $E \subset M$ and $\delta > 0$ we define $[E]_\delta = \{y :\ d(x,E) \le \delta\}$.

Lemma 4.5

Suppose M has finite radius R. Suppose $(\gamma_n)_{n=1}^\infty$ is a weakly null sequence. Then, for $\delta > 0$,

$$\inf_{|E|<\infty}\sup_{n\in\mathbb{N}} D(\gamma_n,[E]_\delta) = 0.$$

Proof. We first note that it suffices to prove the Lemma under the hypothesis that each γ_n is a measure of finite support.

Let us assume the contrary statement, that is there exist $\delta, \epsilon > 0$ so that

$$\inf_{|E|<\infty} \sup_{n\in\mathbb{N}} D(\gamma_n, [E]_\delta) > \epsilon > 0.$$

We can assume $\delta < \frac{1}{R}$. We can then construct a subsequence $(\mu_n)_{n\in\mathbb{N}}$ and an increasing sequence $(E_n)_{n=0}^\infty$ of finite subsets of M with $E_0 = \{0\}$, so that

$$D(\mu_n, E_{n-1}) > \epsilon, \qquad n \geq 1$$

and

$$\text{supp } \mu_n \subset E_n.$$

Now by the Hahn-Banach theorem we can find $f_n \in \text{Lip}M$ with $\|f_n\|_{\text{Lip}} = 1$, $f_n([E_{n-1}]_\delta) = \{0\}$ and

$$\langle \mu_n, f_n\rangle > \epsilon.$$

Letting $g_n = \max(f_n, 0)$ or $\max(-f_n, 0)$ we have $\|g_n\|_{\text{Lip}} \leq 1$, $g_n \geq 0$, $g_n([E_{n-1}]_\delta) = \{0\}$ and

$$|\langle \mu_n, g_n\rangle| > \frac{1}{2}\epsilon.$$

Next let

$$h_n(x) = \max\{0, \sup_{y\in\text{supp } \mu_n} (g_n(y) - R\delta^{-1}d(x,y))\}.$$

Since $g_n(x) \leq d(x,0) \leq R$ for all $x \in E$ we have $h_n(x) = 0$ if $d(x, \text{supp } \mu_n) \geq \delta$. Clearly $0 \leq h_n \leq g_n$ so that $h_n([E_{n-1}]_\delta) = \{0\}$. Furthermore $\|h_n\|_{\text{Lip}} \leq R\delta^{-1}$. It follows that the sets $\{x : h_n(x) > 0\}$ are disjoint, and so $\sum_{n=1}^\infty h_n = \sup_n h_n = h \in \text{Lip}(M)$ and $\|h\|_{\text{Lip}} \leq R\delta^{-1}$. However,

$$|\langle \mu_n, h_n\rangle| = |\langle \mu_n, g_n\rangle| > \tfrac{1}{2}\epsilon$$

which gives a contradiction. □

Theorem 4.6

Suppose M is any metric space and ω is a non-trivial gauge. Then $\mathcal{F}_\omega(M)$ is a Schur space.

Proof. It will suffice to prove this on the assumption that M has finite radius; this follows from Proposition 4.3. Now suppose γ_n is a normalized weakly null sequence. Then by Lemma 4.5 we have

$$\inf_{|E|<\infty} \sup_{n\in\mathbb{N}} D_\omega(\gamma_n, [E]_\delta) = 0$$

whenever $\delta > 0$, where $D_\omega(\gamma, E)$ denotes the distance $D(\gamma, E)$ with respect to the metric space (M, d_ω), where $d_\omega = \omega \circ d$. We will show that for any $\gamma_0 \in \mathcal{F}_\omega(M)$ we have

$$\liminf_{n\to\infty} \|\gamma_0 + \gamma_n\| \geq \|\gamma_0\| + \tfrac{1}{2}. \tag{4.2}$$

Suppose $\epsilon > 0$. We pick $f_0 \in \text{lip}_\omega(M)$ with $\|f_0\|_{\text{Lip}_\omega} = 1$ and $\langle \gamma_0, f_0\rangle > \|\gamma_0\| - \epsilon$. Next pick $\theta > 0$ so that if $d(x,y) < \theta$ we have $|f_0(x) - f_0(y)| < \epsilon d_\omega(x,y)$. Pick $\delta < \theta$ so that $2\omega(\delta) < \epsilon\omega(\theta)$. We can then find a finite set E containing $\{0\}$ so that there exists $\mu_0 \in \mathcal{F}_\omega(E)$ with $\|\mu_0 - \gamma_0\| < \epsilon$ and for each $n \in \mathbb{N}$ we can find $\mu_n \in \mathcal{F}_\omega([E]_\delta)$ with $\|\mu_n - \gamma_n\|_{\mathcal{F}_\omega} < \epsilon$.

Notice, since E is finite, that we have

$$\liminf_{n\to\infty} D_\omega(\gamma_n, E) \geq \tfrac{1}{2}.$$

Hence we can find $f_n \in \mathrm{Lip}_\omega(M)$ with $f_n(E) = \{0\}$, $\|f_n\|_{\mathrm{Lip}_\omega} \leq 1$ and

$$\liminf_{n\to\infty} \langle \gamma_n, f_n\rangle > \tfrac{1}{2} - \epsilon.$$

Let g_n denote the restriction of $f_0 + f_n$ to $[E]_\delta$. If $x, y \in [E]_\delta$ and $d(x,y) < \theta$ then

$$|g_n(x) - g_n(y)| \leq |f_n(x) - f_n(y)| + \epsilon d_\omega(x,y) \leq (1+\epsilon) d_\omega(x,y).$$

If $d(x,y) \geq \theta$ then we can find $u, v \in E$ with $d(x,u) \leq \delta,\ d(y,v) \leq \delta$ and so

$$|f_n(x) - f_n(y)| \leq 2\omega(\delta) \leq \epsilon d_\omega(x,y).$$

Hence

$$|g_n(x) - g_n(y)| \leq (1+\epsilon) d_\omega(x,y) \qquad x, y \in E.$$

Thus, since $\langle \mu_0, f_n\rangle = 0$,

$$\|\mu_0 + \mu_n\| \geq (1+\epsilon)^{-1}(\langle \mu_0, f_0\rangle + \langle \mu_n, f_n\rangle + \langle \mu_n, f_0\rangle).$$

Now

$$\langle \mu_0, f_0\rangle > \|\gamma_0\| - 2\epsilon$$

and

$$\langle \mu_n, f_n\rangle > \tfrac{1}{2} - \epsilon.$$

We also have

$$\lim_{n\to\infty} \langle \gamma_n, f_0\rangle = 0$$

and so

$$\limsup_{n\to\infty} |\langle \mu_n, f_0\rangle| \leq \epsilon.$$

Combining we obtain

$$\liminf_{n\to\infty} \|\mu_0 + \mu_n\| \geq \|\gamma_0\| + \tfrac{1}{2} - 4\epsilon$$

and so

$$\liminf_{n\to\infty} \|\gamma_0 + \gamma_n\| \geq \|\gamma_0\| + \tfrac{1}{2} - 6\epsilon.$$

This proves (4.2).

Now (4.2) implies that the sequence (γ_n) has a subsequence equivalent to the unit vector basis of ℓ_1. This is a routine argument. Indeed we may pick a subsequence (γ'_n) so that

$$\Big\| \sum_{k=1}^N a_k \gamma'_k \Big\| \geq c_N \sum_{k=1}^N |a_k|$$

where (c_N) is any strictly descending sequence with $\frac{1}{2} > c_N > \frac{1}{3}$ for all N. If $(\gamma'_1, \cdots, \gamma'_N)$ have been chosen we observe that

$$\liminf_{n\to\infty} \Big\| \sum_{k=1}^N a_k \gamma'_k + t\gamma_n \Big\| \geq \Big\| \sum_{k=1}^N a_k \gamma'_k \Big\| + \tfrac{1}{2}|t|$$

uniformly for $|t| \leq 3$ and $|a_1| + \cdots + |a_N| \leq 1$. Hence we may find $\gamma'_{N+1} = \gamma_n$ for suitable n to continue the induction. This of course contradicts our initial hypothesis that (γ_n) is weakly null. □

5. Applications to Banach spaces; global results

We now consider applications of some of these ideas to the study of uniform homeomorphisms on Banach spaces. We follow the same basic idea as in [19]. If X is a Banach space, we can consider X as a metric space with metric $d(x, y) = \|x - y\|$. In this case, we recall there is a natural map (the barycentric map) $\beta = \beta_X : \mathcal{F}(X) \to X$ induced by the identity map $Id : X \to X$ using Lemma 3.2. It is clear that β is a quotient map of $\mathcal{F}(X)$ onto X. Furthermore $\delta : X \to \mathcal{F}(X)$ is a Lipschitz section of this map (i.e. $\beta_X \delta_X = Id_X$). If ω is a strongly normalized gauge it is also easy to see that the restriction of β to $\mathcal{F}_\omega(X)$ remains a quotient map and the map $\delta : X \to \mathcal{F}_\omega(X)$ is a uniformly continuous section which satisfies

$$\|\delta(x) - \delta(y)\| = \omega(\|x - y\|), \qquad x, y \in X.$$

Proposition 5.1

If ω is any strongly normalized gauge, there is a uniformly continuous homeomorphism $\varphi : \ker \beta \oplus_1 X \to \mathcal{F}_\omega(X)$ such that

$$\|\varphi(\xi_1) - \varphi(\xi_2)\| \leq 2\omega(\|\xi_1 - \xi_2\|) \tag{5.1}$$

and

$$\|\varphi^{-1}(\gamma_1) - \varphi^{-1}(\gamma_2)\| \leq 3\omega(\|\gamma_1 - \gamma_2\|). \tag{5.2}$$

Proof. We define $\varphi(\gamma, x) = \gamma + \delta(x)$ so that $\varphi^{-1}(\gamma) = (\gamma - \delta(\beta(\gamma)), \beta(\gamma))$. The calculations are immediate. □

Combining Theorem 4.6 and Proposition 5.1 gives:

Proposition 5.2

Let X be any Banach space. Then for any nontrivial strongly normalized gauge ω there is a Schur space Y and a Banach space Z which contains a complemented copy of X so that Y and Z are uniformly homeomorphic via a uniform homeomorphism $\varphi : Z \to Y$ such that both φ and φ^{-1} have modulus of continuity dominated by 3ω.

Of course there are many known examples of pairs of separable Banach spaces which are uniformly homeomorphic but not linearly isomorphic (see [7] pp. 244–253, [45], [23], [2]). Let us note also that it follows from Proposition 5.2 by taking $X = C[0, 1]$ that there is a Schur space U such that every separable Banach space is uniformly homeomorphic to a closed subset of U. We are grateful to the referee for this last remark.

We next discuss the uniform analogue of the problem considered in [19]. Let us say that a Banach space X has the *uniform lifting property* if whenever $Q : Y \to X$ is a quotient map admitting a uniformly continuous section $\varphi : X \to Y$ (with $Q\varphi = Id_X$)

then there is a bounded linear section $S : X \to Y$ (with $QS = Id_X$) (or, equivalently $\ker Q$ is complemented) . For the corresponding Lipschitz lifting property it is shown in [19] that every separable Banach space has the Lipschitz lifting property.

On the other hand, for any strongly normalized gauge ω the quotient map $\beta : \mathcal{F}_\omega(X) \to X$ admits a uniformly continuous section δ. Proposition 5.2 thus shows that a Banach space with the uniform lifting property must be a Schur space.

Let us define a property even stronger than the uniform lifting property. We recall that a subset G of a Banach space X, which we always assume to contain the origin, is called a *net* for X if there exist $0 < \epsilon_1 < \epsilon_2 < \infty$ so that if $x, y \in G$ we have $\|x - y\| \geq \epsilon_1$ while if $x \in X$ there exists $y \in G$ with $\|x - y\| \leq \epsilon_2$. We shall say that G is then a (ϵ_1, ϵ_2)-net. It is a result of Lindenstrauss, Matouskova and Preiss (see [29]) that, if X is infinite-dimensional, any two nets are Lipschitz isomorphic.

We shall say that a quotient map $Q : Y \to X$ has a net-lifting if there is a net $G \subset X$ and a Lipschitz section $\varphi : G \to Y$ (so that $Q\varphi = Id_G$). We shall say that X has the *net-lifting property* if whenever $Q : Y \to X$ is a quotient map with a net-lifting then there is a bounded linear section $S : X \to Y$ (i.e. so that $QS = Id_X$.) It is trivial to see that if Q has a uniformly continuous section, then the restriction to an arbitrary net is Lipschitz so the net-lifting property implies the uniform lifting property.

Let us remark first that if there is a Lipschitz section of a quotient map on any net G then there is also a Lipschitz section on any other net, so that in the above definition we could fix our choice of G.

Lemma 5.3

Let $Q : Y \to X$ be a quotient map. Suppose G, H are nets in X. Suppose there is a Lipschitz section $\varphi : G \to Y$ of Q. Then there is a section $\tilde{\varphi} : X \to Y$ satisfying an estimate

$$\|\tilde{\varphi}(x) - \tilde{\varphi}(x')\| \leq C(\|x - x'\| + 1)$$

and hence then there is a Lipschitz section $\psi : H \to Y$.

Proof. Suppose G is a $(\epsilon_1, \epsilon_2)-$net, and φ has Lipschitz constant B. If $x \in X$ we pick $x' \in G$ with $\|x - x'\| \leq \epsilon_2$. Then we can find $y \in Y$ with $\|y\| \leq 2\epsilon_2$ and $Qy = x - x'$. Define $\tilde{\varphi}(x) = \varphi(x') + y$.

If $x_1, x_2 \in X$ we define y_1, y_2, x_1', x_2' as above. Then

$$\begin{aligned}\|\tilde{\varphi}(x_1) - \tilde{\varphi}(x_2)\| &\leq \|y_1 - y_2\| + B\|x_1' - x_2'\| \\ &\leq 4\epsilon_2 + B(\|x_1 - x_2\| + 2\epsilon_2).\end{aligned}$$

This proves the first part of the statement; we obtain ψ by restricting to H. □

Proposition 5.4

Suppose $Q : Y \to X$ is a quotient map with a net-lifting. Then $\ker Q$ is locally complemented in Y.

Proof. We recall that there is a bounded projection P of $\mathrm{Lip}(X)$ onto its subspace X^*, given by $Pf(x) = \Lambda(f_x)$ where $f_x(\xi) = f(x + \xi) - f(\xi)$ and Λ is an invariant mean on the space $C_b(X)$ of bounded continuous functions on X. Let G be any net in X and let $R : \mathrm{Lip}(X) \to \mathrm{Lip}(G)$ be the restriction map (which is a quotient map). Note that if $Rf = 0$ then f is bounded and so the functions $\{f_x : x \in X\}$ are uniformly bounded in $C_b(X)$ and so Pf is a bounded function: thus $Pf = 0$. This means P factors to a contractive projection $P_G : \mathrm{Lip}(G) \to X^*$ (where X^* is identified as a subspace of $\mathrm{Lip}(G)$ by restriction).

Now suppose $\varphi : G \to Y$ is a Lipschitz section of Q. Define $T : Y^* \to \mathrm{Lip}(G)$ by $Ty^* = y^* \circ \varphi$. Then $P_G T : Y^* \to X^*$ satisfies $P_G T Q^* = Id_{X^*}$ so that $Q^*(X^*) = (\ker Q)^\perp$ is complemented in Y^*. $\square$

It follows that any complemented subspace of C_1 has the net-lifting property. However we shall see that there are other examples.

Theorem 5.5

In order that a Banach space X has the net-lifting property it is necessary and sufficient that X is isomorphic to a complemented subspace of a space $\mathcal{F}(M)$ where M is a uniformly discrete metric space.

In particular X is a Schur space with the Radon-Nikodym property and the approximation property.

Proof. We suppose that X is a subspace of $\mathcal{F}(M)$ and that $P : \mathcal{F}(M) \to X$ is a bounded projection. Let $Q : Y \to X$ be any quotient map with a net-lifting. By Lemma 5.3 we can find a section φ of Q with

$$\|\varphi(\xi_1) - \varphi(\xi_2)\| \leq C(\|\xi_1 - \xi_2\| + 1) \qquad \xi_1, \xi_2 \in X.$$

Then, since M is uniformly discrete, there is a Lipschitz map $\psi : M \to Y$ defined by $\psi(x) = \varphi(P\delta(x))$. By Lemma 3.1 there is a bounded linear map $L : \mathcal{F}(M) \to \mathcal{F}(Y)$ so that $L(\delta(x)) = \delta(\psi(x))$. Consider the quotient $\beta_Y : \mathcal{F}(Y) \to Y$. Then $\beta_Y L(\delta(x)) = \psi(x)$ and so $Q\beta_Y L(\delta(x)) = P\delta(x)$. Hence $Q\beta_Y L = P$ and so $\beta_Y L|_X$ is a bounded linear section of Q.

The converse is easy. Let G be any net in X; then the quotient $\beta_X : \mathcal{F}(G) \to X$ admits a net-lifting. Hence X is isomorphic to a complemented subspace of $\mathcal{F}(G)$.

The final remarks follow from Proposition 4.4. $\square$

We can now give a simple example of a space which has the net-lifting property but is not isomorphic to a subspace of the Johnson-Zippin space C_1. Our argument is related to ideas of [44].

EXAMPLE 5.6: Let G be the integer lattice net in c_0 (i.e. G is the space of all $(m_n)_{n=1}^\infty \subset \mathbb{Z}^{\mathbb{N}}$ which are finitely nonzero equipped with the standard sup-norm metric. We claim that $\mathcal{F}(G)$ is not isomorphic to a subspace of C_1. Indeed, the proof is standard since G cannot be Lipschitz embedded in a stable space. Let $f_n = e_1 + \cdots + e_n$ where (e_k) are the standard basis vectors. Then

$$\lim_{n_1\to\infty} \lim_{m_1\to\infty} \cdots \lim_{n_k\to\infty} \lim_{m_k\to\infty} \Big\| \sum_{j=1}^k f_{n_j} - \sum_{j=1}^k f_{m_j} \Big\|_\infty = 1$$

but

$$\lim_{n_1\to\infty} \lim_{n_2\to\infty} \cdots \lim_{n_k\to\infty} \lim_{m_1\to\infty} \cdots \lim_{m_k\to\infty} \Big\| \sum_{j=1}^k f_{n_j} - \sum_{j=1}^k f_{m_j} \Big\|_\infty = k.$$

Now suppose $\varphi : G \to Y$ is a Lipschitz map where Y is stable and $\varphi(0) = 0$. Then

$$\lim_{n_1\to\infty}\lim_{m_1\to\infty}\cdots\lim_{n_k\to\infty}\lim_{m_k\to\infty}\left\|\varphi\left(\sum_{j=1}^k f_{n_j}-\sum_{j=1}^k f_{m_j}\right)\right\|\le K$$

where K is the Lipschitz constant of φ. Thus

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty}\cdots\lim_{n_k\to\infty}\lim_{m_1\to\infty}\cdots\lim_{m_k\to\infty}\left\|\varphi\left(\sum_{j=1}^k f_{n_j}-\sum_{j=1}^k f_{m_j}\right)\right\|\le K.$$

In particular, φ cannot be a Lipschitz embedding.

We remark that it is not clear what conditions on a uniformly discrete metric space imply that it embeds into C_1.

6. Duality

In [48] (see also [47]), Weaver uses similar results to Propositions 3.4 and 3.5 to study conditions when, for M compact, one has $\mathrm{lip}(M)^* = \mathcal{F}(M)$. His results (Theorem 3.3.3 and Corollary 3.3.5 of [48]) extend earlier duality results for Hölder classes; see [4], [21] and [22]). This gives the following duality theorem:

Theorem 6.1

Let M be a compact pointed metric space. Then lip(M) is a predual of $\mathcal{F}(M)$ if and only if lip(M) is a-norming for some $a \ge 1$. In particular let ω be a nontrivial gauge. Then $\mathrm{lip}_\omega(M)^* = \mathcal{F}_\omega(M)$ and $\mathrm{lip}_\omega(M)^{**} = \mathrm{Lip}_\omega(M)$.

We will now give a generalization of this result. If τ is some topology on M weaker than the original topology we denote by $C_\tau(M)$ the $\tau-$continuous functions on M.

Theorem 6.2

Let M be a separable complete pointed metric space of finite radius R. Suppose τ is a metrizable topology on M so that (M,τ) is compact and for every $x, y \in M$ and $\epsilon > 0$ there exists $f \in \mathrm{lip}(M) \cap C_\tau(M)$ with $\|f\|_{\mathrm{Lip}} \le 1$ and $f(y) - f(x) \ge d(x, y) - \epsilon$. Then the space $\mathrm{lip}(M) \cap C_\tau(M)$ is a predual of $\mathcal{F}(M)$.

Proof. First we notice that by Proposition 3.4, the subspace $\mathrm{lip}(M) \cap C_\tau(M)$ is a norming subspace of $\mathcal{F}(M)^*$.

Suppose $\phi \in (\mathrm{lip}(M) \cap C_\tau(M))^*$. Let K be the subset of $(M,\tau) \times (M,\tau) \times [0, 2R]$ consisting of all (x, y, t) so that $t \ge d(x, y)$. We show that K is a closed and hence compact subset. Suppose $(x_n, y_n, t_n) \in K$ converges to (x, y, t). Suppose $\epsilon > 0$; pick $f \in \mathrm{lip}(M)\cap C_\tau(M)$ with $\|f\|_{\mathrm{Lip}} \le 1$ and $f(y)-f(x) > d(x, y)-\epsilon$. Then $f(y_n)-f(x_n) \le d(x_n, y_n) \le t_n$ and so $f(y) - f(x) \le t$. Since $\epsilon > 0$ is arbitrary this means that $t \ge d(x, y)$ and so $(x, y, t) \in K$.

Note that we have essentially proved that the metric d is lower semi-continuous with respect to the topology $\tau \times \tau$. It follows that each set of the form $[y]_\epsilon = \{x :$

$d(x,y) \leq \epsilon\}$ is $\tau-$closed. Since M is separable this implies that the identity map $i : (M,\tau) \to M$ is Borel.

We define a linear map $S : \mathrm{lip}(M) \cap C_\tau(M) \to C(K)$ by

$$Sf(x,y,t) = \begin{cases} t^{-1}(f(y) - f(x)) & t > 0 \\ 0 & t = 0. \end{cases}$$

One can easily see that S is an isometry. Hence there exists a Borel measure μ on K with $\|\mu\| = \|\phi\|$ and such that

$$\phi(f) = \int_K Sf \, d\mu.$$

Clearly $\mu(K_0) = 0$ if $K_0 = \{(x,y,0) \in K\}$. Now the map $(x,y,t) \to t^{-1}(\delta(y) - \delta(x))$ is Borel from $K \setminus K_0$ into $\mathcal{F}(M)$ (since the identity $i : (M,\tau) \to M$ is Borel). It follows that the integral

$$\int_{K \setminus K_0} \frac{\delta(y) - \delta(x)}{t} d\mu(x,y,t)$$

converges as a Bochner integral to some γ in $\mathcal{F}(M)$ Hence

$$\phi(f) = \langle \gamma, f \rangle \qquad f \in \mathrm{lip}(M) \cap C_\tau(M).$$

Since $\mathrm{lip}(M) \cap C_\tau(M)$ is norming this shows it is a predual. $\square$

The main application here is to the case when K is a weakly compact subset of a separable Banach space. The following Proposition is then immediate:

Proposition 6.3

Suppose ω is a nontrivial gauge. Suppose X is a Banach space which is a separable dual. Let K be a weak compact set, containing the origin. Let $\mathrm{lip}_{\omega,*}(K)$ denote the subspace of $\mathrm{lip}_\omega(K)$ of all weak*-continuous functions. Then $\mathrm{lip}_{\omega,*}(K)$ is a predual of $\mathcal{F}_\omega(K)$ and thus $\mathcal{F}_\omega(K)$ is a separable dual space.*

We may note here as an example that if X is a separable dual space then so is $\mathcal{F}_\omega(B_X)$ (when ω is a nontrivial gauge) and hence $\mathcal{F}_\omega(X)$ has the Radon-Nikodym property by applying Proposition 4.3.

In the special case when M is a compact subset of $\mathbb{R}^n$ (and hence if M is a compact subset of a finite-dimensional normed space) one can go much further and identify the space $\mathrm{lip}(M)$. This result is due to Bonic, Frampton and Tromba (see [9], [48]):

Theorem 6.4

Let M be a compact subset of a finite-dimensional normed space. Then for $0 < \alpha < 1$ the space $\mathrm{lip}^{(\alpha)}(M)$ is isomorphic to c_0 and hence $\mathrm{Lip}^{(\alpha)}(M)$ is isomorphic to ℓ_∞ and $\mathcal{F}^{(\alpha)}(M)$ is isomorphic to ℓ_1.

We remark that it is of some interest to determine good estimates on the Banach-Mazur distance $d_{BM}(\mathrm{lip}^{(\alpha)}(M), c_0)$ in terms of dimension and the geometry of the norm.

Let us recall that a metric space (M, d) satisfies the *doubling condition* ([20] p. 81) (or has finite *Assouad dimension*) if there is a integer N so that for any $\delta > 0$ so that any closed ball B of radius δ can be covered by most N balls of radius $\delta/2$. It is a theorem of Assouad [3] that if (M, d) satisfies the doubling condition then every snowflaking (M, d^α) with $0 < \alpha < 1$ Lipschitz embeds in $\mathbb{R}^n$ for some n. Hence we can restate Theorem 6.4 as:

Theorem 6.5

Let M be a compact metric space satisfying the doubling condition. Then for $0 < \alpha < 1$ the space $lip^{(\alpha)}(M)$ is isomorphic to c_0 and hence $Lip^{(\alpha)}(M)$ is isomorphic to ℓ_∞ and $\mathcal{F}^{(\alpha)}(M)$ is isomorphic to ℓ_1.

The question is raised in Weaver [48] p. 98 whether this isomorphism can be extended to any compact metric space. We will answer this question later (see e.g. Theorem 8.3). Let us first give a weaker conclusion. The following result is known and the author is grateful to Yoav Benyamini for showing us the simple proof. Note that in the statement we do not exclude the possibility that $\mathrm{lip}(M)$ reduces to $\{0\}$.

Theorem 6.6

Suppose M is a compact metric space. Then for any $\epsilon > 0$, $lip(M)$ is $(1+\epsilon)-$isometric to a subspace of c_0.

Proof. We may suppose $\epsilon < 1$. Consider $M \times M$ with the metric

$$d'((x_1, x_2), (y_1, y_2)) = \max(d(x_1, y_1), d(x_2, y_2)).$$

For $n \in \mathbb{Z}$ choose a finite $2^{n-3}\epsilon$-net F_n in the compact set $\{(x_1, x_2) : 2^n \le d(x_1, x_2) \le 2^{n+1}\}$ so that F_n is empty for large enough n. Then $F = \cup_{n\in\mathbb{Z}} F_n$ is countable. Define the map $T : X \to c_0(F)$ by

$$Tf(x_1, x_2) = \frac{f(x_1) - f(x_2)}{d(x_1, x_2)}.$$

Thus $\|T\| \le 1$. On the other hand if $y_1 \neq y_2$ we may find n so that $2^n \le d(y_1, y_2) \le 2^{n+1}$ and then $(x_1, x_2) \in F_n$ with $d(x_1, y_1), d(x_2, y_2) \le 2^{n-3}\epsilon$. Hence

$$d(x_1, x_2) \ge d(y_1, y_2) - 2^{n-2}\epsilon \ge d(y_1, y_2)\left(1 - \frac{1}{4}\epsilon\right).$$

Now for any $f \in \mathrm{lip}(M)$,

$$\begin{aligned}\left|\frac{f(y_1) - f(y_2)}{d(y_1, y_2)}\right| &\le \left|\frac{f(x_1) - f(x_2)}{d(y_1, y_2)}\right| + \frac{1}{4}\epsilon\|f\|_{\mathrm{Lip}} \\ &\le (1 + \frac{1}{2}\epsilon)\left|\frac{f(x_1) - f(x_2)}{d(x_1, x_2)}\right| + \frac{1}{4}\epsilon\|f\|_{\mathrm{Lip}} \\ &\le (1+\epsilon)\|Tf\|.\end{aligned}$$

Hence $\|f\|_{\mathrm{Lip}} \le (1+\epsilon)\|Tf\|$. □

Remark. In particular this implies that $\mathrm{lip}(M)$ is an M-ideal in $\mathrm{Lip}(M)$ when M is compact and $\mathrm{lip}(M)$ is a predual of $\mathcal{F}(M)$. See [8] for the case when $M = [0,1]$.

Corollary 6.7

If M is a compact pointed metric space and ω is a nontrivial gauge then the following are equivalent:
(1) $lip_\omega(M) \approx c_0$,
(2) $\mathcal{F}_\omega(M) \approx \ell_1$,
(3) $Lip_\omega(M) \approx \ell_\infty$.

Proof. Any of the three statements imply $\mathrm{lip}_\omega(M)$ is an $\mathcal{L}_\infty-$space and hence by Theorem 6.6 and [25] is isomorphic to c_0. The remainder follows by duality. □

Let us remark at this point that it might appear reasonable to conjecture that if M is compact then $\mathrm{Lip}^{(\alpha)}(M) \approx \ell_\infty$ if and only if M satisfies the doubling condition (cf. Theorem 6.5). However, this is false as our next result shows:

Proposition 6.8

There is a compact metric space (M,d) failing the doubling condition such that $lip^\alpha(M) \approx c_0$ (and $Lip^\alpha(M) \approx \ell_\infty$).

Proof. Let M_1 be the closed disk $2B_2$ of radius two in $\mathbb{R}^2$ (two-dimensional Euclidean space). Let $A = B_2$. Then since there is Lipschitz retraction of M_1 onto A the short exact sequence
$$0 \to \mathcal{F}^{(\alpha)}(A) \to \mathcal{F}^{(\alpha)}(M_1) \to \mathcal{F}^{(\alpha)}(M_1/A) \to 0$$
splits; furthermore since the induced projection is weak*-continuous the predual sequence
$$0 \to \mathrm{lip}^{(\alpha)}(M_1/A) \to \mathrm{lip}^{(\alpha)}(M_1) \to \mathrm{lip}^{(\alpha)}(A) \to 0$$
also splits. Hence using Theorem 6.4 and the fact that c_0 is prime, if $M = M_1/A$ then $\mathrm{lip}^{(\alpha)}(M) \approx c_0$. However M fails the doubling condition. Indeed consider the ball of radius δ around the origin in M; assume this can be covered by N balls of radius $\delta/2$. Then it follows easily that the annulus $(1+\delta)B_2 \setminus (1+\frac{1}{2}\delta)B_2$ can be covered by N balls of radius $\delta/2$ in M_1. However to cover the circle $\{x : \|x\| = 1+\delta\}$ requires at least $2\pi(1+\delta)/\delta$ such balls. □

7. Uniform sections of quotients relative to the ball

Let X be a Banach space and let B_X be its unit ball. If ω is a strongly normalized gauge then we can also consider the subspace $\mathcal{F}_\omega(B_X)$ of $\mathcal{F}_\omega(X)$. The barycentric map $\beta : \mathcal{F}_\omega(B_X) \to X$ is easily seen to be a quotient map as long as ω is strongly normalized. This time the section δ is defined only on B_X so the quotient map admits a uniformly continuous section on the ball.

We summarize these remarks as follows:

Proposition 7.1

If ω is a strongly normalized gauge the barycentric operator $\beta : \mathcal{F}_\omega(X) \to X$ is a quotient map and $\delta : B_X \to \mathcal{F}_\omega(X)$ is a (nonlinear) map satisfying $\beta\delta = Id_{B_X}$ and $\|\delta(x) - \delta(y)\| = \omega(\|x - y\|)$.

In order to answer the question of Weaver we must discuss general conditions when a quotient map has a uniformly continuous section on the ball.

Let us note first that if $Q : X \to Y$ is a quotient map which admits a uniformly continuous section $\varphi : B_Y \to X$ then it may be assumed that φ is homogeneous if $\varphi(ty) = t\varphi(y)$ when $y, ty \in B_Y$. Indeed, we can replace φ by a homogeneous function φ' by setting $\varphi'(x) = \frac{1}{2}(\varphi(x) - \varphi(-x))$ when $\|x\| = 1$ and extending by homogeneity. Then

$$\begin{aligned}
\|\varphi'(y_1) - \varphi'(y_2)\| &\le \|y_1 - y_2\| + \|y_1\|\omega\left(\left\|\frac{y_1}{\|y_1\|} - \frac{y_2}{\|y_2\|}\right\|\right) \\
&\le \|y_1 - y_2\| + 2\|y_1\|\omega\left(\frac{\|y_1 - y_2\|}{\|y_2\|}\right) \\
&\le \|y_1 - y_2\| + 4\omega(\|y_1 - y_2\|) \\
&\le C\omega(\|y_1 - y_2\|),
\end{aligned}$$

for a suitable constant C. Thus it will be convenient to assume each φ is already homogeneous. If necessary φ can then be extended to $\varphi : Y \to X$ in such a way that it remains homogeneous and then it is uniformly continuous on bounded sets.

Let us note one simple deduction from the existence of such a section, which will be useful later.

Proposition 7.2

Let X be a Banach space and let E be a closed subspace. If the quotient map $Q : X \to X/E$ admits a uniformly continuous section then B_X is uniformly homeomorphic to $B_E \times B_{X/E}$.

Proof. This principle is essentially used in Lemma 9.10 of [7] (for spheres rather than balls). Consider the map $x \to (x - \varphi(Qx), Qx)$ from X into $E \oplus_\infty X/E$ which is a homogeneous surjection is a bijection and induces a uniform homeomorphism between the unit balls. □

We shall particularly be concerned with the spaces $\mathcal{F}^{[\alpha]}(B_X)$ and $\mathrm{Lip}^{[\alpha]}(B_X)$. Note that is up to equivalence of norm these spaces coincide with $\mathcal{F}^{(\alpha)}(B_X)$ and $\mathrm{Lip}^{(\alpha)}(B_X)$ since B_X has radius one (and diameter two). It is however convenient to use a strongly normalized gauge so that the map β is an isometric quotient map.

Theorem 7.3

Let Y be a stable Banach space and suppose $Q : Y \to c_0$ is a quotient map. Then there is no uniformly continuous map $\varphi : B_{c_0} \to Y$ with $Q\varphi = Id_{B_{c_0}}$.

Proof. This is essentially due to Raynaud [44] (see also [7] p. 212–215). We simply observe that there is no uniform embedding of B_{c_0} into a stable Banach space. □

Remark. We will use this theorem later in the case when Y is a subspace of some L_1-space. For this application, it suffices to use the earlier result of Enflo [16] that B_{c_0} cannot be uniformly embedded in ℓ_2 (since bounded subsets of L_1 uniformly embed in ℓ_2, [7] Chapter 8).

Lemma 7.4

Let X be an arbitrary Banach space and suppose $T : X \to \ell_2^n$ where $n \geq 2$ be any surjective linear operator. Let $\varphi : B_{\ell_2^n} \to X$ be any continuous map such that $T\varphi = Id_{B_{\ell_2^n}}$. Then if

$$\omega_\varphi(\epsilon) = \sup\{\|\varphi(\xi) - \varphi(\eta)\| : \|\xi - \eta\| \leq \epsilon\}$$

and,

$$M = \sup\{\|\varphi(\xi)\| : \|\xi\| \leq 1\},$$

we have

$$\pi_2(T) \geq \frac{1}{2\omega(2\sqrt{\frac{\log(2Mn\|T\|)}{n}})}. \tag{7.1}$$

Proof. Let us put $\epsilon = 2\sqrt{\frac{\log(Mn\|T\|)}{n}}$ and $\sigma = \omega(\epsilon)$.

We first note that it suffices to consider the case when φ is homogeneous, as noted above. We use the Pietsch Factorization Theorem (see Theorem 2.5 above). There exists a probability measure μ on B_{X^*} such that

$$\|Tx\| \leq \pi_2(T)\Big(\int |x^*(x)|^2\, d\mu(x^*)\Big)^{1/2}.$$

Let λ denote normalized surface measure on the sphere $S = \{\xi : \|\xi\| = 1\}$. Then for fixed x^* the set $A = \{\xi : x^*(\varphi(\xi)) \leq 0\}$ has $\lambda-$ measure at least $\frac{1}{2}$. Let $B = \{\xi \in S : d(\xi, A) > \epsilon\}$. Then using the concentration of measure phenomenon (see Theorem 2.7 above),we have

$$\lambda(B) \leq \sqrt{\frac{\pi}{8}}e^{-(\epsilon^2 n)/2}.$$

Hence

$$\lambda(|x^*(\varphi)| \geq \sigma) \leq \sqrt{\frac{\pi}{2}}e^{-(\epsilon^2 n)/2}.$$

Thus

$$\int_S |x^*(\varphi(\xi))|^2 d\lambda \leq M^2\sqrt{\frac{\pi}{2}}e^{-(\epsilon^2 n)/2} + \sigma^2.$$

Hence

$$\int_\Omega\int_S |x^*(\varphi(\xi))|^2 d\lambda(\xi)d\mu(x^*) \leq M^2\sqrt{\frac{\pi}{2}}e^{-(\epsilon^2 n)/2} + \sigma^2.$$

Thus

$$1 \leq \pi_2(T)^2\Big(M^2\sqrt{\frac{\pi}{2}}e^{-(\epsilon^2 n)/2} + \sigma^2\Big).$$

Now let us recall that $\epsilon = 2n^{-1/2}\sqrt{\log(2M\|T\|n)}$. We obtain

$$1 \leq \pi_2(T)^2\left(\frac{1}{2}n^{-2}\|T\|^{-2} + \sigma^2\right).$$

Since $\pi_2(T) \leq n^{\frac{1}{2}}\|T\|$ this implies

$$1 \leq \frac{1}{2} + \pi_2(T)^2\sigma^2$$

and so $\pi_2(T) \geq \frac{1}{2}\sigma^{-1}$. □

The following Lemma is immediate:

Lemma 7.5

If we consider the quotient $\beta : \mathcal{F}_\omega(B_{\ell_2^n}) \to \ell_2^n$ then

$$\pi_2(\beta) \geq \frac{1}{2\omega\left(2\sqrt{\frac{\log(2n)}{n}}\right)}. \tag{7.2}$$

Theorem 7.6

Let X be a $\mathcal{L}_\infty$-space or a $\mathcal{L}_1$-space. Then any quotient map $Q : X \to \ell_2$ fails to have a uniformly continuous lift on the ball B_{ℓ_2}.

Proof. It is an immediate consequence of Grothendieck's theorem that Q is 2-absolutely summing and so the conclusion follows from Lemma 7.4. □

8. The structure of $\mathcal{F}^{(\alpha)}(K)$ and $\mathrm{Lip}^{(\alpha)}(K)$ when K is a closed bounded convex set.

Suppose $0 < \alpha < 1$ and K is a closed bounded convex subset of a separable Banach space. In this section we will show that $\mathcal{F}^{(\alpha)}(K)$ is an $\mathcal{L}_1-$space or, equivalently, $\mathrm{Lip}^{(\alpha)}(K)$ is isomorphic to ℓ_∞ if and only if K is finite-dimensional.

For an arbitrary Banach space X let define $\gamma_1(X)$ to be the infimum of all constants C so that if E is a finite-dimensional subspace of X then there are linear operators $S : E \to \ell_1$ and $T : \ell_1 \to X$ such that $TS = Id_E$ and $\|T\|\|S\| \leq C$. Then X is a $\mathcal{L}_1$-space if and only if $\gamma_1(X) < \infty$. Furthermore if $T : X \to \ell_2$ is bounded than $\pi_2(T) \leq K_G\gamma_1(X)\|T\|$ where K_G is the Grothendieck constant.

Based on the last section we can now state the following.

Proposition 8.1

(1) *For any gauge ω the space $\mathcal{F}_\omega(B_{c_0})$ and $\mathcal{F}_\omega(B_{\ell_2})$ fail to be $\mathcal{L}_1-$spaces.*

(2)

$$\gamma_1\left(\mathcal{F}^{(\alpha)}(B_{\ell_2^n})\right) \geq c\frac{n^{\alpha/2}}{(\log(2n))^{\alpha/2}}$$

for an absolute constant $c > 0$.

(3) $\lim_{n\to\infty}\gamma_1(\mathcal{F}^{(\alpha)}(\ell_\infty^n)) = \infty$.

Proof. (1) Consider the quotient map $\beta : \mathcal{F}_\omega(B_{\ell_2}) \to \ell_2$. Then $\delta : B_{\ell_2} \to \mathcal{F}_\omega(B_{\ell_2})$ is a uniformly continuous section. It follows from Theorem 7.6 that $\mathcal{F}_\omega(B_{\ell_2})$ cannot be a $\mathcal{L}_1-$space. The argument for c_0 is similar based on Theorem 7.3 and the fact that $\mathcal{L}_1-$spaces are isomorphic to subspaces of some $L_1(\mu)$ and are hence isomorphic to stable spaces.

(2) Note that $d_{BM}(\mathcal{F}^{(\alpha)}(B_{\ell_2^n}), \mathcal{F}^{[\alpha]}(B_{\ell_2^n})) \le 2$. Now by Lemma 7.5 (7.2) we have

$$\pi_2(\beta : \mathcal{F}^{[\alpha]}(B_{\ell_2^n}) \to \ell_2^n) \ge \frac{n^{\alpha/2}}{2(\log(2n))^{\alpha/2}}.$$

Hence

$$\gamma_1(\mathcal{F}^{[\alpha]}(B_{\ell_2^n})) \ge \frac{n^{\alpha/2}}{2K_G(\log(2n))^{\alpha/2}}.$$

(3) This is routine since $\cup_{n=1}^\infty \mathcal{F}^{(\alpha)}(B_{\ell_\infty^n})$ is dense in $\mathcal{F}^{(\alpha)}(B_{c_0})$ if we identify ℓ_∞^n as the span of the first n basis vectors. Hence $\infty = \gamma_1(\mathcal{F}^{(\alpha)}(B_{c_0})) \le \sup_{n_k} \gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_\infty^{n_k}}))$ for any increasing sequence n_k. $\square$

Before attempting to refine this result, let us use it to solve the problem posed by Weaver mentioned after Theorem 6.4. We need the following preparatory Lemma.

Lemma 8.2

Suppose K is a bounded closed convex set in a Banach space X; suppose 0 is an internal point of K and that the linear span of K is dense. Let E be a finite-dimensional Banach space and suppose $S : E \to X$ and $T : X \to E$ are bounded operators such that $TS = Id_E$. Then, if $0 < \alpha \le 1$, there are operators $U : \mathcal{F}^{(\alpha)}(B_E) \to \mathcal{F}^{(\alpha)}(K)$ and $V : \mathcal{F}^{(\alpha)}(K) \to \mathcal{F}^{(\alpha)}(B_E)$ so that $VU = Id_{\mathcal{F}^{(\alpha)}(B_E)}$ and $\|U\|\|V\| \le 4\|S\|^\alpha\|T\|^\alpha$.

In particular $\gamma_1(\mathcal{F}^{(\alpha)}(B_E)) \le 4\|S\|^\alpha\|T\|^\alpha\gamma_1(\mathcal{F}^{(\alpha)}(K))$.

Proof. Since the linear span of K is dense we can make a small perturbation of S, T, say S_1, T_1 so that $T_1S_1 = Id_E$, $\|S_1\|\|T_1\| \le 2\|S\|\|T\|$ and $S_1(E)$ is contained in the linear span of K. Now since 0 is internal there exists $b > 0$ so that $bS_1(B_E) \subset K$. We define U by $U\delta_E(e) = \delta_K(bS_1e)$. Then $\|U\| \le (b\|S_1\|)^\alpha$. Let r_E be the Lipschitz retraction of E onto B_E with Lipschitz constant at most 2. Define V by $V\delta_K(x) = r_E(b^{-1}T_1x)$. Then $\|V\| \le (2b^{-1}\|T_1\|)^\alpha$. The last part is clear. $\square$

Now Weaver [48] p. 98 asks whether it is true that for every compact metric space M one has that $\mathrm{lip}^{(\alpha)}(M)$ is isomorphic to c_0. This would require that $\mathcal{F}^{(\alpha)}(M)$ is isomorphic to ℓ_1. Let us give a very simple family of counterexamples:

Theorem 8.3

Suppose $0 < \alpha < 1$. Let K be any bounded closed convex subset of ℓ_2 containing 0. Then $\mathcal{F}^{(\alpha)}(K)$ is isomorphic to an $\mathcal{L}_1-$space if and only if K is finite-dimensional. In particular if K is compact then $lip^{(\alpha)}(K) \approx c_0$ if and only if K is finite-dimensional.

Proof. Lemma 8.2 and Proposition 8.1. □

We will push this result much further by using some Banach space theory. One application is:

Theorem 8.4

Suppose $0 < \alpha < 1$. *Let* K *be any bounded closed convex subset of a Banach space* X *containing* 0. *Suppose* X *has nontrivial Rademacher type. Then* $\mathcal{F}^{(\alpha)}(K)$ *is isomorphic to an* $\mathcal{L}_1-$*space (or equivalently* $Lip^{(\alpha)}(K)$ *is isomorphic to* ℓ_∞*) if and only if* K *is finite-dimensional. In particular, if* K *is compact,* $lip^{(\alpha)}(K) \approx c_0$ *if and only if* K *is finite-dimensional.*

Proof. We need only observe that if X has nontrivial type and is infinite-dimensional then X contains uniformly complemented ℓ_2^n's ([34], [18]) and then use Lemma 8.2 and Proposition 8.1. □

Remark. Assuming X is the closure of the linear span of K, this argument can be used to show that the result is valid if we merely have a sequence of subspaces E_n with $d_{BM}(E_n, \ell_2^{N_n})$ bounded where $N_n \to \infty$ and projections P_n onto E_n with $\|P_n\| \leq CN_n^a$ where $a < \frac{1}{2}$.

It is however likely that the conclusion of this theorem is true for every infinite-dimensional Banach space. We can prove this only in the case when $0 < \alpha \leq \frac{1}{2}$. We will present some fairly compelling evidence that it is true for all α later.

Theorem 8.5

Suppose $0 < \alpha \leq \frac{1}{2}$ *and that* K *is an infinite-dimensional closed bounded convex subset of a Banach space* X*; then* $\mathcal{F}^{(\alpha)}(K)$ *is not a* $\mathcal{L}_1-$*space, and* $Lip^{(\alpha)}(K)$ *is not isomorphic to* ℓ_∞. *If* K *is compact,* $lip^{(\alpha)}(K)$ *is not isomorphic to* c_0.

Proof. We can assume 0 is an internal point of K. Then by Dvoretzky's theorem for each n we can find an isomorphism $S_n : \ell_2^n \to X$ so that $S_n(B_{\ell_2^n}) \subset K$ and $\|S_n\xi\| \geq \frac{1}{2}\|S_n\|\|\xi\|$ for $\xi \in \ell_2^n$. Consider the map $S_n^{-1} : S_n(B_{\ell_2^n}) \to B_{\ell_2^n}$. Then

$$\|S_n^{-1}x_1 - S_n^{-1}x_2\| \leq 2(\min(\|S_n\|^{-1}\|x_1 - x_2\|, 1) \leq 2\|S_n\|^{-\alpha}\|x_1 - x_2\|^\alpha.$$

Hence using Corollary 1.15, p. 21 of [7] (or [35]) we can find a map $\varphi_n : K \to B_{\ell_2^n}$ with $\varphi_n \circ S_n = Id_{B_{\ell_2^n}}$ and

$$\|\varphi(x_1) - \varphi(x_2)\| \leq 2\|S_n\|^{-\alpha}\|x_1 - x_2\|^\alpha.$$

The maps S_n, φ_n induce linear operators $\hat{S}_n : \mathcal{F}^{(\alpha)}(B_{\ell_2^n}) \to \mathcal{F}^{(\alpha)}(K)$ and $\Phi_n : \mathcal{F}^{(\alpha)}(K) \to \mathcal{F}(B_{\ell_2^n})$ with $\|\hat{S}_n\| = \|S_n\|^\alpha$ and $\|\Phi_n\| \leq 2\|S_n\|^{-\alpha}$. Clearly $\beta\Phi_n\hat{S}_n = \beta : \mathcal{F}^{(\alpha)}(B_{\ell_2^n}) \to \ell_2^n$. Now if $\mathcal{F}^{(\alpha)}(K)$ is a $\mathcal{L}_1-$space we deduce a uniform bound on $\pi_2(\beta : \mathcal{F}^{(\alpha)}(B_{\ell_2^n}) \to \ell_2^n)$ contradicting Lemma 7.5. □

Remark. To obtain the same result for $\frac{1}{2} < \alpha < 1$ by this method one needs some good information on the existence of extensions of Hölder class α maps into $B_{\ell_2^n}$. We will make this problem explicit in the final section.

Let us now consider the case of ℓ_1.

Proposition 8.6

For $0<\alpha<1$ there is a constant $c=c(\alpha)>0$ so that

$$\gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_1^n}))\geq cn^{\alpha(1-\alpha)}(\log(2n))^{-\alpha/2}$$

when $\frac{1}{2}\leq\alpha<1$ and

$$\gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_1^n}))\geq cn^{\alpha/2}(\log(2n))^{-\alpha/2}$$

when $0<\alpha\leq\frac{1}{2}$. In particular $\mathcal{F}^{(\alpha)}(B_{\ell_1})$ is not a $\mathcal{L}_1-$space.

Proof. Fix $p=1/\alpha$ when $\frac{1}{2}\leq\alpha\leq 1$ and $p=2$ otherwise. Let $\varphi : B_{\ell_p^n}\to B_{\ell_1^n}$ be the Mazur map i.e.

$$\varphi(\xi)_j=\|\xi\|_p^{1-p}|\xi_j|^p\text{sgn }(\xi_j).$$

We recall that

$$\|\varphi(\xi)-\varphi(\eta)\|_1\leq p\|\xi-\eta\|_p,\qquad \xi,\eta\in B_{\ell_p^n}$$

while the inverse satisfies

$$\|\varphi^{-1}(\xi)-\varphi^{-1}(\eta)\|_p\leq C_p\|\xi-\eta\|_1^{1/p}\qquad \xi,\eta\in B_{\ell_1^n}$$

(see [7] p. 198).

For each α there exist constants $c_1, C_1>0$ so that for some $k=k(n)$ with $k\geq c_1\min(n^{2(1-\alpha)},n)$ there are linear maps $S_n:\ell_2^k\to\ell_p^n$ and $T_n:\ell_p^n\to\ell_2^k$ with $T_nS_n=Id_{\ell_2^k}$ and such that $\|S_n\|=1$ and $\|T_n\|\leq C_1$. Let $\psi_n : B_{\ell_p^n}\to B_{\ell_2^k}$ be defined by $\psi_n=r\circ T_n$ where r is the standard retraction of ℓ_2^n onto $B_{\ell_2^n}$.

If we denote by M^α the metric space M with metric d^α we thus have a sequence of Lipschitz maps factoring $Id : B_{\ell_2^k}^\alpha\to B_{\ell_2^k}$:

$$B_{\ell_2^k}^\alpha\xrightarrow{S_n}B_{\ell_p^n}^\alpha\xrightarrow{\varphi}B_{\ell_1^n}^\alpha\xrightarrow{\varphi^{-1}}B_{\ell_p^n}\xrightarrow{\psi_n}B_{\ell_2^k}.$$

The product of the Lipschitz constants is bounded by a constant depending only on α. It follows that $Id:\mathcal{F}^{(\alpha)}(B_{\ell_2^k})\to\mathcal{F}(B_{\ell_2^k})$ factors through $\mathcal{F}^{(\alpha)}(B_{\ell_1^n})$ and hence

$$\begin{aligned}\frac{k^{1/2\alpha}}{4(\log(2k))^{1/2\alpha}}&\leq\pi_2(\beta:\mathcal{F}^{(\alpha)}(B_{\ell_2^k})\to\ell_2^k)\\&\leq C\gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_1^n})).\end{aligned}$$

Taking into account our estimate on k we have

$$\gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_1^n})\geq\begin{cases}cn^{\alpha(1-\alpha)}(\log(2n))^{-\alpha/2} & \frac{1}{2}\leq\alpha<1\\ cn^{\alpha/2}(\log(2n))^{-\alpha/2} & 0<\alpha\leq\frac{1}{2}\end{cases}$$

where $c=c(\alpha)$. $\square$

We now ready to give a general local result.

Theorem 8.7

For fixed $0<\alpha<1$, we have

$$\lim_{n\to\infty}\inf_{\dim X=n}\gamma_1(\mathcal{F}^{(\alpha)}(B_X))=\infty.$$

Proof. If the conclusion fails then there is a constant C_0 and a sequence of finite-dimensional Banach spaces X_n, with $\dim X_n = N_n \to \infty$, and such that

$$\gamma_1(\mathcal{F}^{(\alpha)}(B_{X_n})) \le C_0.$$

For a Banach space X let $b(X,m)$ be the least constant b so that for all $x_1, \cdots, x_m \in X$ we have

$$\Big(\sum_{k=1}^m \|x_k\|^2\Big)^{1/2} \le b\,\Big(\mathbb{E}\Big\|\sum_{k=1}^m \epsilon_k x_k\Big\|^2\Big)^{1/2} \qquad x_1, \cdots, x_m \in X.$$

If $\limsup_{n\to\infty} b(X_n, m) = m^{1/2}$ then there is a subsequence X_{n_k} so that X_{n_k} contains a subspace with Banach-Mazur distance 2 from ℓ_∞^m. Such a subspace is necessarily at most 2-complemented and hence

$$\gamma_1(\mathcal{F}^{(\alpha)}(B_{\ell_\infty^m})) < 4C_0.$$

By Proposition 8.1 we see that for a suitable m_0 we have

$$\limsup_{n\to\infty} b(X_n, m) < m^{1/2} \qquad m > m_0.$$

We similarly argue that Proposition 8.6 implies that X_n cannot contain uniformly complemented ℓ_1^m's of arbitrarily large dimension so that we have for large enough m we also have

$$\limsup_{n\to\infty} b(X_n^*, m) < m^{1/2}.$$

Since $b(X, m_1 m_2) \le b(X, m_1) b(X, m_2)$ this implies the existence of a $\theta > 0$ and a constant C_1 so that

$$b(X_n, m) \le C_1 m^{(1-\theta)/2}, \qquad b(X_n^*, m) \le C_1 m^{(1-\theta)/2}, \qquad m, n \in \mathbb{N}.$$

This implies also the estimate

$$\Big(\sum_{k=1}^m \|x_k\|^2\Big)^{1/2} \le C_1 m^{(1-\theta)/2}\Big(\mathbb{E}\Big\|\sum_{k=1}^m g_k x_k\Big\|^2\Big)^{1/2}, \qquad x_1, \cdots, x_m \in X_n, \quad m, n \in \mathbb{N}$$

and

$$\Big(\sum_{k=1}^m \|x_k^*\|^2\Big)^{1/2} \le C_1 m^{(1-\theta)/2}\Big(\mathbb{E}\Big\|\sum_{k=1}^m g_k x_k^*\Big\|^2\Big)^{1/2} \qquad x_1^*, \cdots, x_m^* \in X_n^*, \quad m, n \in \mathbb{N}$$

where (g_k) is a sequence of independent normalized Gaussians.

We now apply Theorem 2.9 to find $k_n \ge cN_n^\theta$ and operators $S_n : \ell_2^{k_n} \to X_n,\ T_n : X_n \to \ell_2^{k_n}$ with $T_n S_n = Id_{\ell_2^{k_n}}$ and $\|T_n\|\|S_n\| \le C(1+\log N_n)$ for some absolute constant C. Hence by Lemma 8.2 we have

$$\gamma_1(\mathcal{F}^{(\alpha)} B_{\ell_2^{k_n}}) \le C_1(1 + \log N_n)^\alpha \gamma_1(\mathcal{F}^{(\alpha)}(B_{X_n}).$$

Thus

$$\frac{k_n^{\alpha/2}}{(1+\log k_n)^{\alpha/2}} \le C_2(1+\log N_n)^\alpha$$

for some constant C_2. This is a contradiction to the fact that $k_n \ge cN_n^\theta$. $\Box$

The following is now immediate.

Theorem 8.8

Suppose $0 < \alpha < 1$. *Let* K *be a closed bounded convex subset of a separable infinite-dimensional Banach space* X. *Suppose the linear span of* K *is dense in* X. *If there is a sequence of finite-dimensional subspaces* E_n *in* X *which are uniformly complemented then* $\mathcal{F}^{(\alpha)}(K)$ *is not an* $\mathcal{L}_1-$*space and* $Lip^{(\alpha)}(K)$ *is not isomorphic to* ℓ_∞. *If* K *is compact,* $lip^{(\alpha)}(K)$ *is not isomorphic to* c_0.

Remark. In particular if K embeds into some ℓ_p for $1 \le p < \infty$ or c_0 this criterion applies. To summarize our results we see that if K is an infinite-dimensional closed bounded convex subset of a Banach space X such that $\mathcal{F}^{(\alpha)}(K)$ is an $\mathcal{L}_1-$space then we must have:

(1) $0 < \alpha \le \frac{1}{2}$.
(2) X has nontrivial cotype but not nontrivial type.
(3) X does not have finite-dimensional subspaces which are well-complemented.

Obviously the candidate space for a counterexample is Pisier's space [41], [42]; however we doubt such a counterexample exists.

Note also that even in the compact case, some convexity assumption is necessary here. In fact the example of Proposition 6.8 can be embedded in a Banach space (even into c_0 by a result of Aharoni [1]), but not as a convex subset.

If one replaces a power-type gauge by an arbitrary non-trivial gauge one can still give a similar counter-examples:

Proposition 8.9

Suppose ω *is any nontrivial gauge function. Then there is a compact convex subset of* ℓ_2 *so that* $Lip_\omega(K)$ *is not isomorphic to* ℓ_∞.

Proof. Let $a_n \downarrow 0$ and pick $N_n \in \mathbb{N}$ so that if $\omega_n(t) = \omega(a_n t)/\omega(a_n)$ then by Corollary 8.1, $\gamma_1(\mathcal{F}_{\omega_n}(B_{\ell_2^{N_n}})) \to \infty$. Then consider the subsets K of $\ell_2(\ell_2^{N_n})$ of all sequences (ξ_n) such that $\|\xi_n\| \le a_n$. It is not hard to see that $\mathcal{F}_\omega(K)$ contains a complemented copy of $\mathcal{F}_\omega(a_n B_{\ell_2^{N_n}})$ and that this is isometric to $\mathcal{F}_{\omega_n}(B_{\ell_2^{N_n}})$. Thus $\mathrm{Lip}_\omega(K) \approx \ell_\infty$ would imply that $\sup_n \gamma_1(\mathcal{F}_{\omega_n}(B_{\ell_2^{N_n}})) < \infty$. □

9. Approximation properties

Let us say that a metric space (M, d) is a *BL-retract* if M is a Lipschitz retract of some Banach space X. We say (M, d) is a *BU-retract* if it is a uniform retract of some Banach space X. Obviously a BL-retract is also a BU-retract.

Let us make some elementary observations.

Proposition 9.1

(1) *If* K *is a closed convex subset of a Hilbert space then* K *is a BL-retract.*
(2) *For any Banach space* X *then* B_X *is a BL-retract.*

(3) *If K is a closed bounded convex subset of a super-reflexive Banach space X then K is a BU-retract.*
(4) *If K is a compact convex subset of any Banach space then K is a BU-retract.*

Proof. (1) follows from the fact that the nearest point map is contractive in Hilbert spaces. There is a Lipschitz retraction of X onto B_X for all Banach spaces X; this gives (2). For (3) we can assume K contained in B_X and then the nearest point map is uniformly continuous on B_X (see [7] pp. 40–44). In fact K is an absolute uniform retract in these circumstances. For (4) we observe that K is affinely uniformly homeomorphic to a compact convex subset of a Hilbert space. □

Proposition 9.2

(1) *A metric space M is a BL-retract if and only if there is a Lipschitz map $\theta : \mathcal{F}(M) \to M$ such that $\theta \circ \delta = Id_M$.*
(2) *A metric space M is a BU-retract if and only if there is a gauge ω and a uniformly continuous map $\theta : \mathcal{F}_\omega(M) \to M$ such that $\theta \circ \delta = Id_M$.*

Proof. (1) Suppose X is a Banach space and $\phi : M \to X$, $\psi : X \to M$ are Lipschitz maps such that $\psi \circ \phi = Id_M$. Then consider $\hat{\phi} : \mathcal{F}(M) \to \mathcal{F}(X)$ as in Lemma 3.1. Then $\theta = \psi \circ \beta_X \circ \hat{\phi}$ is the required map.

(2) is similar. Let $\phi : M \to X$ and $\psi : X \to M$ be uniformly continuous maps so that $\psi \circ \phi = id_M$. Let ω be a normalized gauge so that $\|\phi(\xi) - \phi(\eta)\| \leq C\omega(\|\xi - \eta\|)$ for some constant C. Consider the map $\hat{\phi} : \mathcal{F}_\omega(M) \to \mathcal{F}(X)$. Then as before let $\theta = \psi \circ \beta_X \circ \hat{\phi}$. □

Theorem 9.3

Let K be a bounded closed convex subset of a separable super-reflexive space X. Then

(1) *There is an equi-uniformly continuous sequence of finite-rank maps $\varphi_n : K \to K$ such that $\lim_{n\to\infty} \varphi_n(x) = x$ for each $x \in X$.*
(2) *Given any gauge ω there exists a gauge $\omega' \geq \omega$ such that the inclusion map $I_{\omega',\omega,K} : \mathcal{F}_{\omega'}(K) \to \mathcal{F}_\omega(K)$ is approximable.*

Proof. (1) We assume K is contained in B_X. Let us note that from Lemma 2.5 of [7] there is gauge ω_1 and constant C_1 so that whenever A is a closed convex subset of B_X there is a uniformly continuous retraction r of X onto A with

$$\|r(x) - r(y)\| \leq \omega_1(\|x - y\|) \qquad x, y \in X.$$

(Assume X is uniformly convex: first retract onto B_X and then use the nearest point map.) Let E_n be an ascending sequence of finite-dimensional subspaces of X whose union is dense and such that $E_1 \cap K$ is non-empty . It follows that there is a retraction φ_n of K onto $K \cap E_n$ so that the sequence (φ_n) is equi-uniformly continuous: indeed

$$\|\varphi_n(x) - \varphi_n(y)\| \leq \omega_1(\|x - y\|) \qquad x, y \in X.$$

Clearly $\lim_{n\to\infty} \varphi_n(x) = x$ for $x \in K$.

(2) For some $0 < \alpha < 1$ let $\omega_2 = \max(\omega, \omega^\alpha)$ and $\omega' = \omega_2 \circ \omega_1$ Since

$$\omega_2(\|\varphi_n(x) - \varphi_n(y)\|) \le \omega'(\|x - y\|) \qquad x, y \in X$$

we can use Lemma 3.1 to induce linear maps $S_n = \hat{\varphi}_n : \mathcal{F}_{\omega'}(K) \to \mathcal{F}_{\omega_2}(K)$ so that $\lim_{n\to\infty} S_n\gamma = I_{\omega',\omega_2}\gamma$ for $\gamma \in \mathcal{F}_{\omega'}(K)$. Note that $S_n(\mathcal{F}_{\omega'}(K)) \subset \mathcal{F}_{\omega_2}(E_n \cap K)$.

Now the space $(E_n \cap K, \omega \circ d)$ (where $d(x, y) = \|x - y\|$) satisfies the doubling condition. Hence $\mathrm{lip}_{\omega^\alpha}(E_n \cap K)$ is isomorphic to c_0 by Theorem 6.5 and thus so is $\mathrm{lip}_{\omega_2}(E_n \cap K)$. It follows that the dual $\mathcal{F}_{\omega_2}(E_n \cap K)$ is isomorphic to ℓ_1 and has (MAP). Thus for each n we can find finite-rank operators $R_{nk} : \mathcal{F}_{\omega_2}(E_n \cap K) \to \mathcal{F}_{\omega_2}(E_n \cap K)$ so that $\|R_{nk}\| \le 1$ and $\lim_{k\to\infty} R_{nk}\gamma = \gamma$ for $\gamma \in \mathcal{F}_{\omega_2}(E_n \cap K)$. Then $I_{\omega',\omega_2,K}$ is in the strong-operator closure of $\{R_{nk}S_n : \ 1 \le n, k < \infty\}$. Hence I_{ω',ω_2} is approximable and hence so is $I_{\omega',\omega}$. $\square$

At this point we will introduce an approximation condition. We will say that a separable BU-retract K has the *uniform compact approximation property* or *(ucap)* if there is an equi-uniformly continuous sequence of maps $\varphi_n : K \to K$ such that $\varphi_n(K)$ is relatively compact for each n and $\lim_{n\to\infty} \varphi_n(x) = x$ for every $x \in K$. It is clear from the above Theorem 9.3 that we have

Corollary 9.4

Every bounded closed convex subset of a separable super-reflexive space has (ucap).

We do not know any example of a separable BU-retract which fails (ucap)! However this condition will be important to us in the next section.

Theorem 9.5

Suppose K is bounded closed convex subset of a Banach space X which is also a BU-retract. Then the following are equivalent:

(1) *K has (ucap)*
(2) *K is a uniform retract of a Banach space with a basis.*
(3) *Given any gauge ω there is a gauge ω' so that the inclusion map $I_{\omega',\omega,K}$ is approximable.*

If $K = B_X$ these conditions are also equivalent to:

(4) *There is a sequence (φ_n) of equi-uniformly continuous finite-rank maps $\varphi_n : K \to K$ such that $\lim_{n\to\infty} \varphi_n(x) = x$ for every $x \in K$.*

Proof. (1) $\implies$ (3). We can assume without loss of generality that ω is a nontrivial gauge. Let $\varphi_n : K \to K$ be equi-uniformly continuous, have relatively compact range and satisfy $\varphi_n(x) \to x$ for $x \in K$. We may assume that $\omega_0 \ge \omega$ is a gauge so that

$$\omega(\|\varphi_n(x) - \varphi_n(y)\|) \le \omega_0(\|x - y\|) \qquad x, y \in K \quad n \in \mathbb{N}.$$

Let K_n be the closed convex hull of $\varphi_n(K)$. We will argue that for each $n \in \mathbb{N}$ we can find a gauge $\omega_n \ge \omega$ so that I_{ω_n,ω,K_n} is approximable. Indeed K_n is a compact convex set and hence is affinely homeomorphic to a compact convex subset K'_n of ℓ_2 which is super-reflexive. Let $h : K_n \to K'_n$ be an affine homeomorphism (which is automatically a uniform homeomorphism since both sets are compact). Suppose ν_n, ν_n' are gauges so that

$$\|h(x) - h(y)\| \le \nu_n(\|x - y\|) \qquad x, y \in K_n$$

and

$$\|h^{-1}(\xi) - h^{-1}(\eta)\| \le \nu_n'(\|\xi - \eta\|), \qquad \xi, \eta \in K_n'.$$

Then h^{-1} induces a norm one operator $L_1 : \mathcal{F}_{\omega\circ\nu_n'}(K_n') \to \mathcal{F}_\omega(K_n)$. By Theorem 9.3 there exists a gauge $\tilde{\nu}_n \ge \omega \circ \nu_n'$ so that $I_{\tilde{\nu}_n,\omega\circ\nu_n',K_n'}$ is approximable. Then h induces a norm one operator $L_2 : \mathcal{F}_{\tilde{\nu}_n\circ\nu_n}(K_n) \to \mathcal{F}_{\tilde{\nu}_n}(K_n')$. If we choose $\omega_n \ge \tilde{\nu}_n \circ \nu_n$ and $\omega_n \ge \omega$ then $I_{\omega_n,\omega,K_n} = L_1L_2$ is approximable.

Now let $\tilde{\omega} = \sum_{n=1}^\infty 2^{-n}\omega_n$. It is clear that $I_{\tilde{\omega},\omega,K_n}$ is approximable for every n. Now we use the facts that $\mathcal{F}_\omega(K_n) = \mathrm{lip}_\omega(K_n)^*$ and $\mathcal{F}_{\tilde{\omega}}(K_n) = \mathrm{lip}_{\tilde{\omega}}(K_n)^*$ (Theorem 6.1) and $I_{\tilde{\omega},\omega}$ is an adjoint map of norm at most one since $\tilde{\omega} \ge \omega$. By Proposition 2.1 there are finite-rank operators $R_{nk} : \mathcal{F}_{\tilde{\omega}}(K_n) \to \mathcal{F}_\omega(K_n)$ with $\|R_{nk}\| \le 1$ and $\lim_{k\to\infty} R_{nk}\gamma = I_{\tilde{\omega},\omega,K_n}\gamma$ for $\gamma \in \mathcal{F}_{\tilde{\omega}}(K_n)$. Next let $\omega' = \tilde{\omega} \circ \omega_0$. Then if $S_n = \hat{\varphi}_n : \mathcal{F}_{\omega'}(K) \to \mathcal{F}_{\tilde{\omega}}(K)$ it is clear that $\|S_n\| \le 1$ and $S_n\gamma \to \gamma$ for all $\gamma \in \mathcal{F}_{\omega'}(K)$. Combining we see that $I_{\omega',\omega,K}$ is in the strong operator closure of $\{R_{nk}S_n : 1 \le k, n < \infty\}$. Thus we have (1) $\implies$ (3).

(3) $\implies$ (2). For this we pick any gauge ω so that there is a uniformly continuous map $\psi : \mathcal{F}_\omega(K) \to K$ with $\psi \circ \delta = Id_K$. Pick $\omega' \ge \omega$ so that $I_{\omega',\omega,K}$ is approximable. Then we can write $I_{\omega',\omega,K} = \sum_{n=1}^\infty A_n$ in the strong operator topology where $A_n : \mathcal{F}_{\omega'}(K) \to \mathcal{F}_\omega(K)$ are finite-rank. We now repeat a trick from [39]. Let $E_n = A_n(\mathcal{F}_{\omega'}(K))$. Define Y to be the space of sequences $(e_n)_{n=1}^\infty$ with $e_n \in E_n$ and such that $\sum_{n=1}^\infty e_n$ converges. This is a Banach space under the norm $\|(e_n)_{n=1}^\infty\|_Y = \sup_n \|\sum_{k=1}^n e_k\|$. Define $S : \mathcal{F}_{\omega'}(K) \to Y$ by $S\gamma = (A_n\gamma)_{n=1}^\infty$ and $T : Y \to \mathcal{F}_\omega(K)$ by $T((e_n)_{n=1}^\infty) = \sum_{n=1}^\infty e_n$. Now $S \circ \delta : K \to Y$ has a left-inverse $\psi \circ T$. Thus K is a uniform retract of Y. Now Y has an (FDD) and so can be embedded as a complemented subspace of a space with a basis ([38], [24]).

(2) $\implies$ (1). This is trivial. If X has basis and $\phi : K \to X$, $\psi : X \to K$ are uniformly continuous with $\psi \circ \phi = Id_K$ then we let $\varphi_n = \psi \circ S_n \circ \phi$ where S_n are the partial sum operators with respect to the basis.

Finally if $K = B_X$ let us show these conditions imply (4). As before there exist gauges $\omega' \ge \omega$ so that $I_{\omega',\omega,K}$ is approximable. Hence $\beta : \mathcal{F}_{\omega'}(K) \to X$ is approximable. Let S_n be finite-rank operators $S_n : \mathcal{F}_{\omega'}(K) \to X$ so that $S_n \to \beta$ in the strong-operator topology. Let r be the natural Lipschitz retraction of X onto B_X. Let $\varphi_n = r \circ S_n \circ \delta$. It is trivial to see that φ_n satisfies the conditions of (4) since r preserves linear subspaces. □

10. Uniform sections of quotient maps revisited

In § 7 we saw that the quotient maps of ℓ_1 onto both ℓ_2 and c_0 fail to have uniformly continuous sections on the ball. On the other hand, if X is a super-reflexive space then any quotient map $X \to X/E$ admits a uniformly continuous section on the ball

([7] Corollary 1.25 p. 28). In this section we will study the problem of the existence of uniformly continuous sections a little more.

We first refine the result on super-reflexive spaces cited above:

Theorem 10.1

Let X be a Banach space and suppose E is a super-reflexive subspace. Then the quotient map $Q : X \to X/E$ admits a uniformly continuous section on the ball $B_{X/E}$.

Proof. Our proof is modelled on the renorming arguments in [6] pp. 272–299 (attributed to Maurey in his proof of the Enflo-Pisier renorming theorems [17], [40]).

Since E is super-reflexive there is exists $p < \infty$ and a constant $0 < c < \frac{1}{2}$ so that whenever $\{f_0, f_1, \cdots, f_n\}$ is a dyadic martingale (with respect to the standard dyadic partition) in $L_p([0,1]; E)$ then

$$\|f_n\|_p^p \geq 2c\left(\|f_0\|_p^p + \sum_{k=1}^{n} \|f_k - f_{k-1}\|_p^p\right).$$

Let us consider the space $L_p([0,1]; X)$ and denote by $\mathbb{E}_n$ the condition expectation onto $L_p(\Sigma_n)$ where Σ_n is the $\sigma-$algebra generated by the atoms $\{[(k-1)2^{-n}, k2^{-n});\ 1 \leq k \leq 2^n\}$. Consider the subspace Z of all functions f such that $f - \mathbb{E}_0 f \in L_p([0,1]; E)$ and define for $x \in X$,

$$\Phi(x) = \inf\left\{\|f\|_p^p - c\left(\sum_{k=1}^{n} \|\mathbb{E}_k f - \mathbb{E}_{k-1} f\|_p^p\right) :\ f \in Z,\ \mathbb{E}_0 f = x\right\}. \tag{10.1}$$

Note that $c\|x\|^p \leq \Phi(x) \leq \|x\|^p$.

We also observe that if $f \in Z$

$$\|f\|_p^p - c\left(\sum_{k=1}^{n} \|\mathbb{E}_k f - \mathbb{E}_{k-1} f\|_p^p\right) \geq \frac{1}{2}\|f\|_p^p.$$

This implies that

$$\Phi(x) = \inf\left\{\|f\|_p^p - c\left(\sum_{k=1}^{n} \|\mathbb{E}_k f - \mathbb{E}_{k-1} f\|_p^p\right) :\ f \in Z,\ \|f\|_p \leq 2\|x\|,\ \mathbb{E}_0 f = x\right\}.$$

Thus for any $x, y \in X$, with $\|y\| \leq \|x\|$,

$$\begin{aligned}\Phi(y) - \Phi(x) &\leq \sup_{\|f\|_p \leq 2\|x\|} \|f + y - x\|_p^p - \|f\|_p^p \\ &\leq (2\|x\| + \|y - x\|)^p - 2^p\|x\|^p \\ &\leq C\|x\|^{p-1}\|y - x\|\end{aligned}$$

for a suitable constant C. It particular

$$|\Phi(x) - \Phi(y)| \leq C(\max(\|x\|, \|y\|))^{p-1}\|x - y\|. \tag{10.2}$$

Note suppose $x, y \in X$ with $x - y \in E$. Then, for $\epsilon > 0$, pick $g, h \in Z$ with $\mathbb{E}_0 g = x,\ \mathbb{E}_0 h = y$ and

$$\|g\|^p - c^p\left(\sum_{k=1}^{\infty} \|\mathbb{E}_k g - \mathbb{E}_{k-1} g\|_p^p\right) < \Phi(x) + \tfrac{1}{2}\epsilon$$

and

$$\|h\|^p - c^p \Big(\sum_{k=1}^{\infty} \|\mathbb{E}_k h - \mathbb{E}_{k-1} h\|_p^p \Big) < \Phi(y) + \tfrac{1}{2}\epsilon.$$

Let $f(t) = g(2t)$ for $t \leq \frac{1}{2}$ and $f(t) = h(2t-1)$ for $t > \frac{1}{2}$. Then

$$\begin{aligned} \Phi\Big(\tfrac{1}{2}(x+y)\Big) &\leq \|f\|^p - c^p \Big(\sum_{k=1}^{\infty} \|\mathbb{E}_k f - \mathbb{E}_{k-1} f\|^p \Big) \\ &\leq \tfrac{1}{2}(\Phi(x) + \Phi(y)) - 2^{-p} c \|x-y\|^p + \epsilon. \end{aligned}$$

It follows that

$$\Phi\Big(\tfrac{1}{2}(x+y)\Big) \leq \tfrac{1}{2}(\Phi(x) + \Phi(y)) - a\|x-y\|^p, \qquad x, y \in X,\ x - y \in E \tag{10.3}$$

where $a = 2^{-p} c$.

Now if $\xi \in X/E$ we define $\theta(\xi) = \inf\{\Phi(x) :\ Qx = \xi\}$. Then $c\|\xi\|^p \leq \theta(xi) \leq \|\xi\|^p$. Notice that if $Qx, Qy = \xi$ then $\Phi(\frac{1}{2}(x+y)) \geq \theta(\xi)$. Hence

$$\frac{1}{2}(\Phi(x) + \Phi(y)) - \theta(\xi) \geq a\|x-y\|^p. \tag{10.4}$$

It follows that if (x_n) is any sequence with $\lim_{n\to\infty} \Phi(x_n) = \theta(\xi)$ and $Qx_n = \xi$ then (x_n) is convergent. Thus there is a unique $\varphi(\xi)$ such that $Q\varphi(\xi) = \xi$ and $\Phi(\varphi(\xi)) = \theta(\xi)$. We will show that φ is a Hölder continuous function on $B_{X/E}$. First notice that $c\|\varphi(x)\|^p \leq \theta(\xi) \leq \|\xi\|^p$ so that $\|\varphi(\xi)\| \leq c^{-1/p}$ for $\xi \in B_{X/E}$.

Indeed suppose $\xi, \eta \in B_{X/E}$. We may pick $u \in X$ with $Qu = \xi - \eta$ and $\|u\| = \|\xi - \eta\|$. Then $\Phi(\varphi(\eta) + u) \geq \theta(\xi)$. Now $\|\varphi(\eta) + u\| \leq (1 + c^{-1/p})$ and so for some suitable constant B, using (10.2), we have

$$\Phi(\varphi(\eta) + u) \leq \theta(\eta) + B\|\xi - \eta\|$$

and

$$\theta(\eta) \geq \theta(\xi) - B\|\xi - \eta\|.$$

Hence

$$\Phi(\varphi(\xi) - u) \leq \theta(\xi) + B\|\xi - \eta\| \leq \theta(\eta) + 2B\|\xi - \eta\|.$$

This implies by (10.4) that

$$a\|\varphi(\xi) - u - \varphi(\eta)\|^p \leq B\|\xi - \eta\|$$

and so

$$\|\varphi(\xi) - \varphi(\eta)\| \leq (Ba^{-1})^{1/p} \|\xi - \eta\|^{1/p} + \|\xi - \eta\|.$$

This completes the proof. □

Notice that we can localize this argument. In particular if $E_n \subset \ell_1$ is a sequence of uniformly Euclidean subspaces of ℓ_1 with $\dim E_n \to \infty$ then the quotient map $\ell_1(\ell_1) \to \ell_1(\ell_1/E_n)$ admits a uniformly continuous selection on the ball. This shows that there exist some spaces X other than ℓ_1 such that every quotient map onto X admits a uniformly continuous selection on the ball. Let us remark that a Lipschitz selection on the ball would a global Lipschitz selection and that is enough to imply X is isomorphic to ℓ_1 (see [19]).

Corollary 10.2

Let X be a reflexive subspace of L_1. Then $B_{L_1/X}$ is uniformly homeomorphic to B_{ℓ_2}.

Proof. X is super-reflexive [46] and by Theorem 10.1 there is a uniformly continuous selection $\varphi : B_{L_1/X} \to L_1$ of the quotient map Q. By Proposition 7.2, we conclude that B_{L_1} is uniformly homeomorphic to $B_X \times B_{L_1/X}$. Now B_{L_1} is uniformly homeomorphic to B_{ℓ_2} and also B_X is uniformly homeomorphic to B_{ℓ_2} (Corollary 9.11 of [7]). Thus $B_{L_1/X} \times B_{\ell_2}$ is uniformly homeomorphic to B_{ℓ_2}. Now L_1/X is isomorphic to $\ell_1 \oplus L_1/X$ and so we also have a uniform homeomorphism between $B_{L_1/X} \times B_{\ell_2}$ and $B_{L_1/X}$. □

We will next study conditions where the existence of local sections for a quotient map is already sufficient for the existence of a global section. Let us say that a quotient map $Q : Y \to Y/E$ has a *local uniformly continuous section on the ball* if there exists a gauge ω such that for every finite-dimensional subspace F of X/E there is a uniformly continuous map $\varphi_F : B_F \to X$ with

$$\|\varphi(f) - \varphi(g)\| \leq \omega(\|f - g\|) \qquad f, g \in B_F$$

and $Q \circ \varphi_F = Id_{B_F}$. As before, we can assume that each φ_F is a homogeneous function.

A typical example where local sections exist is the case when Q locally splits (see Lemma 2.2).

We next introduce a rather technical condition. Let X be a Banach space and ω a gauge. A sequence of functions $a_n : \partial B_X \to [0, 1]$ will be called an $\omega-$*partition* if

$$\sum_{k=1}^{\infty} a_k(x) = 1 \qquad x \in \partial B_X \tag{10.5}$$

$$\sum_{k=1}^{\infty} |a_k(x) - a_k(y)| \leq \omega(\|x - y\|) \qquad x, y \in \partial B_X \tag{10.6}$$

and given $\epsilon > 0$ there exists $\nu > 0$ and a sequence of compact sets K_n so that

$$\sum_{k=1}^{n} a_k(x) \geq 1 - \nu \Rightarrow d(x, K_n) \leq \epsilon. \tag{10.7}$$

Let us note that (10.7) implies that given $\epsilon > 0$ there exists $\nu' > 0$ and a sequence of finite sets F_n so that

$$\sum_{k=1}^{n} a_k(x) \geq 1 - \nu' \Rightarrow d(x, F_n) \leq \epsilon. \tag{10.8}$$

Indeed take $\nu' = \nu(\frac{1}{2}\epsilon)$ and then cover K_n with finitely many balls of radius $\frac{1}{2}\epsilon$.

We will say that X has a *good partition* if it has an $\omega-$partition for some gauge ω.

Lemma 10.3

Suppose X is a separable Banach space with a good partition. Suppose also B_X has (ucap). Then, whenever $Q : Y \to X$ is a quotient map admitting a local uniformly continuous section, there is a uniformly continuous section of Q.

Proof. We assume ω_1 is a gauge and $\varphi_n : B_X \to B_X$ are uniformly continuous homogeneous functions with finite-dimensional range E_n such that $\varphi_n(x) \to x$ for all x and $\|\varphi_n(x)-\varphi_n(y)\| \le \omega_1(\|x-y\|)$ for $x, y \in B_X$. We assume further that for some gauge ω_2, and every finite-dimensional subspace E of X we can find a uniformly continuous homogeneous map $\psi_E : X \to Y$ such that $Q\psi_E = Id_E$ and $\|\psi_E(x) - \psi_E(y)\| \le \omega_2(\|x - y\|)$. Finally we assume (a_n) is an ω_3-partition.

Pick $0 < \epsilon < 1/10$ so that $\omega_1(\epsilon) < 1/10$. We can then choose $\nu > 0$ and a sequence K_n of compact sets so that

$$\sum_{k=1}^{n} a_k(x) \ge 1 - \nu \Rightarrow d(x, K_n) \le \epsilon. \tag{10.9}$$

Next we define a function $h : [0, 1] \to [0, 1]$ to be affine on $[0, \frac{1}{10}]$ and $[\frac{1}{10}, 1]$ and such that $h(0) = 0,\ h(\frac{1}{10}) = 1 - \nu$, and $h(1) = 1$. Let

$$b_k(x) = h\Big(\sum_{j=1}^{k} a_j(x)\Big) - h\Big(\sum_{j=1}^{k-1} a_j(x)\Big).$$

We first show that (b_n) is also a good partition. Clearly (10.5) and (10.7) hold. We consider (10.6). Let $\kappa_0 = 10(1 - \nu)$ and $\kappa_1 = 10\nu/9$ and $\kappa = \max(\kappa_0, \kappa_1)$. Suppose $x, y \in \partial B_X$. Let n_1 be the first integer so that $\sum_{k=1}^{n_1} a_k(x) \ge 1 - \nu$ and n_2 the first integer such that $\sum_{k=1}^{n_2} a_k(x) \ge 1 - \nu$. For $k \ne n_1, n_2$ it is clear that $|b_k(x) - b_k(y)| \le \kappa|a_k(x) - a_k(y)|$. If $k = n_1$ or $k = n_2$ we have an estimate

$$|b_k(x) - b_k(y)| \le \kappa\Big|\sum_{j=1}^{k-1} a_j(x) - \sum_{j=1}^{k-1} a_j(y)\Big| + \kappa\Big|\sum_{j=1}^{k} a_j(x) - \sum_{j=1}^{k} a_j(y)\Big| \le 2\kappa\omega_3(\|x - y\|).$$

Since there are at most two such k we obtain

$$\sum_{k=1}^{\infty} |b_k(x) - b_k(y)| \le 5\kappa\omega_3(\|x - y\|). \tag{10.10}$$

By passing to a subsequence of the (φ_n) we can assume that

$$\|\varphi_n(x) - x\| \le \tfrac{1}{10} \qquad x \in K_n. \tag{10.11}$$

If $\sum_{k=1}^{n} a_k(x) \ge 1 - \nu$ then we may find $y \in K_n$ so that $\|x - y\| \le \epsilon$. Hence

$$\|\varphi_n(x) - \varphi_n(y)\| \le \tfrac{1}{10}.$$

Hence

$$\sum_{k=1}^{n} a_k(x) \ge 1 - \nu \Rightarrow \|\varphi_n(x) - x\| \le \tfrac{3}{10}. \tag{10.12}$$

Now let $\psi_n = \psi_{E_n}$ and define

$$\rho(x) = \sum_{k=1}^{\infty} b_k(x)\psi_k(\varphi_k(x)).$$

First we consider $Q \circ \rho$.

$$Q\rho(x) = \sum_{k=1}^{\infty} b_k(x)\varphi_k(x).$$

Suppose $\|x\| = 1$ and let r be the least integer such that $\sum_{k=1}^{r} a_k(x) > 1 - \nu$. Then by (10.12) we have $\|x - \varphi_k(x)\| \le \frac{3}{10}$ for $k \ge r$. Thus

$$\|Q\rho(x) - x\| \le \sum_{k=1}^{r-1} b_k(x)\|x - \varphi_k(x)\| + \tfrac{3}{10}.$$

The first sum is estimated by $2\sum_{j=1}^{r-1} b_k(x) \le \frac{1}{5}$. Hence

$$\|Q\rho(x) - x\| \le \tfrac{1}{2}. \tag{10.13}$$

Next we show that ρ is uniformly continuous. Indeed,

$$\Big\| \sum_{k=1}^{\infty} b_k(x)(\psi_k(\varphi_k(x) - \psi_k(\varphi_k(y))\Big\| \le \omega_2 \circ \omega_1(\|x - y\|)$$

while by (10.10)

$$\Big\| \sum_{k=1}^{\infty}(b_k(x) - b_k(y))\psi_k(\varphi_k(y)\Big\| \le 5\omega_2(1)\kappa\omega_3(\|x - y\|).$$

Hence ρ is uniformly continuous with modulus of continuity bounded by $C(\omega_2 \circ \omega_1 + \omega_3)$. We can now extend ρ to B_X by insisting that it is positively homogeneous and ρ will have the same modulus of continuity up to constant. By (10.13) we have

$$\|Q\rho(x) - x\| \le \tfrac{1}{2}\|x\| \qquad x \in B_X.$$

We now obtain our section by an iteration procedure. Let $g(x) = x - Q\rho(x)$. Then g is uniformly continuous with modulus of continuity $\omega_4 \le C(\omega_2 \circ \omega_1 + \omega_3)$. Define

$$\psi(x) = \sum_{n=0}^{\infty} \rho(g^n(x)).$$

Since ρ is positive homogeneous it satisfies an estimate $\|\rho(x)\| \le M\|x\|$ and we have $\|g^n(x)\| \le 2^{-n}\|x\|$ so this is well-defined. Note that $Q\rho(g^n(x)) = g^n(x) - g^{n+1}(x)$ so that $Q\psi(x) = x$. Finally we note that by uniform convergence ψ is uniformly continuous. □

Remark. In general the modulus of continuity of ψ seems rather poor, but in specific cases it can be improved. In the special case when Q is the quotient map of ℓ_1 onto L_1 we may take $\omega_j(t) \approx t$ for $j = 1, 2, 3$. In this case ρ is Lipschitz and the section ψ is Hölder-continuous. Unfortunately it does not seem clear exactly what type of argument Hölder-continuity is optimal for this situation.

We now recall from § 2 the Johnson-Zippin space C_1 [25]. This is the ℓ_1−sum of a sequence of finite-dimensional spaces G_n dense in all finite-dimensional spaces for

Banach-Mazur distance. Clearly any ℓ_1-sum of finite-dimensional spaces is embedded complementably in C_1.

Lemma 10.4

Let X be a separable Banach space.

1. *If B_X embeds uniformly in a space Y with a good partition then X has a good partition.*
2. *If B_X embeds uniformly in C_1 then X has a good partition.*
3. *If X is super-reflexive and B_X embeds uniformly in a super-reflexive space with (UFDD) then B_X has a good partition.*

Proof. (1) Let $\psi : B_X \to Y$ be a homogeneous uniform embedding with $\psi(0) = 0$. Then $x \to \|\psi(x)\|^{-1}\psi(x)$ uniformly embeds ∂B_X into ∂B_Y. Therefore we assume $\psi : \partial B_X \to \partial B_Y$ is a uniform embedding. Let (a_n) be a good partition for Y. Then $b_n = a_n \circ \psi$ is a good partition for X. Only the verification of (10.7) requires some explanation. If $\epsilon > 0$ then there exists $0 < \epsilon_1 < \epsilon$ so that

$$\|\psi(x_1) - \psi(x_2)\| \leq \epsilon_1 \implies \|x_1 - x_2\| \leq \tfrac{1}{2}\epsilon \qquad x_1, x_2 \in B_X.$$

Using (10.8) we may find $\nu > 0$ and finite-subsets F_n of ∂B_Y so that

$$\sum_{k=1}^{n} a_k(y) \geq 1 - \nu \implies y \in F_n + \epsilon_1 B_Y.$$

Let G_n be the finite subset of ∂B_X obtained by taking one point in each non-empty $\psi^{-1}(y + \epsilon_1 B_Y)$ where $y \in F_n$. Then

$$\psi^{-1}(F_n + \epsilon_1 B_Y) \subset G_n + \epsilon B_X$$

and so (10.7) holds with $K_n = G_n$.

(2) If $x = (x_n)_{n=1}^{\infty} \in \mathrm{C}_1$ define $a_n(x) = \|x_n\|$; then (a_n) is a good partition of C_1. We then appeal to (1).

(3) It is only necessary to show that any super-reflexive space with a UFDD has a good partition. Let (E_n) be a UFDD for X and let $R_n : X \to E_n$ be associated projections. We can assume that X is uniformly convex and uniformly smooth and that the UFDD is 1-unconditional. (Just equip X with any uniformly convex, uniformly smooth norm $\|\cdot\|_0$ and then renorm it by $\|x\| = (\mathbb{E}\|\sum_{n=1}^{\infty} \epsilon_n R_n x\|^2)^{\frac{1}{2}}$.) Now the duality map $D : \partial B_X \to \partial B_{X^*}$ defined so that $\langle x, Dx\rangle = 1$ is uniformly continuous. We define

$$a_n(x) = |\langle R_n x, Dx\rangle|.$$

Then for $x, y \in \partial B_X$ we have

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n(x) - a_n(y)| &\leq \sum_{n=1}^{\infty} |\langle R_n(x-y), Dx\rangle| + \sum_{n=1}^{\infty} |\langle R_n y, Dy - Dx\rangle| \\ &\leq \|x - y\| + \|Dx - Dy\|.\end{aligned}$$

If $\|x\| = 1$ then

$$\Big\| x - \sum_{k=1}^{n} R_k x \Big\| \leq \frac{1}{2}\sigma_X^{-1}\Big(1 - \Big\|\sum_{k=1}^{n} R_k x\Big\|\Big)$$

where σ_X is the modulus of uniform convexity i.e.

$$\sigma_X(\epsilon) = \inf \left\{1 - \|\tfrac{1}{2}(x+y)\| : \ \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon \right\}.$$

Hence if

$$\sum_{k=1}^{n} a_k(x) \geq 1-\nu \Rightarrow \Big\| \sum_{k=1}^{n} R_k(x) \Big\| \geq 1-\nu \Rightarrow \Big\| x - \sum_{k=1}^{N} R_k(x) \Big\| \leq \frac{1}{2}\sigma_X^{-1}(\nu).$$

This shows that (a_n) is a good partition. □

We now can state our main theorem on local sections:

Theorem 10.5

Let X be a separable Banach space. The following conditions on X are equivalent:

(1) *Whenever $Q : Y \to X$ is a quotient map with locally complemented kernel then there a uniformly continuous section $\psi : B_X \to Y$.*
(2) *Whenever $Q : Y \to X$ is a quotient map with a locally uniformly continuous section on the ball then there is a uniformly continuous section on the ball.*
(3) *B_X has (ucap) and a good partition.*
(4) *B_X has (ucap) and B_X is uniformly homeomorphic to a subset of C_1.*
(5) *B_X is a uniform retract of B_{C_1}.*
(6) *B_X is a uniform retract of $\ell_1(E_n)$ where E_n is some sequence of finite-dimensional subspaces of X.*

If, additionally, X is super-reflexive, these conditions are equivalent to:

(7) *B_X is uniformly homeomorphic to a subset of C_1.*
(8) *B_X is uniformly homeomorphic to a subset of a super-reflexive space with a UFDD.*

Proof. (6) $\implies$ (5) $\implies$ (4) is trivial. (4) $\Rightarrow$ (3) follows from Lemma 10.4 (2). (3) $\Rightarrow$ (2) is Lemma 10.3. (2) $\Rightarrow$ (1) is trivial.

It remains to show (1) $\implies$ (6). Let $(E_n)_{n=1}$ be any increasing sequence of finite-dimensional subspaces such that $\cup E_n$ is dense. Define the quotient map $Q : Y = \ell_1(E_n) \to X$ by $Q((e_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} e_n$. By (1) there is a uniformly continuous lifting $\psi : B_X \to \ell_1(E_n)$. We can assume $\psi(B_X) \subset cB_Y$ where $c > 1$. Define $\rho : cB_Y \to B_X$ by $\rho(y) = Qy/\max(\|Qy\|, 1)$. Then ρ is a uniformly continuous map and $\rho \circ \psi = Id_{B_X}$. Hence B_X is a uniform retract of B_Y.

If X is super-reflexive then B_X is absolute uniform retract and must have (ucap). Thus we can simplify the conditions to give (7) and (8) using Lemma 10.4 (3). □

Remarks. The space C_1 has a stable norm. Hence c_0 and any non-reflexive space X with nontrivial type are examples of spaces for which the theorem fails ([44] and [7] p. 214). For example there is no uniformly continuous lifting on the ball of a quotient map of $\ell_1(\ell_\infty^n)$ onto c_0.

The space $X = \ell_1(\ell_{p_n}^n)$ where $p_n \to \infty$ satisfies the conclusions of the Theorem but its ball is not an absolute uniform retract [7] p. 30.

Suppose X is a separable dual and is not a Schur space. If ω is nontrivial and strictly normalized, the quotient map $\beta : \mathcal{F}_\omega(B_X) \to X$ has a uniformly continuous lifting on the ball, but cannot have a linear lifting (since X does not embed into $\mathcal{F}_\omega(X)$

by Theorem 4.6). Since β is easily seen to be the adjoint of the canonical embedding of the predual of X into $\mathrm{lip}_{\omega,*}(B_X)$, it also follows that the kernel of β cannot be locally complemented (i.e. there is no local linear lifting).

Odell and Schlumprecht [37] showed that any Banach with cotype and an unconditional basis has B_X uniformly homeomorphic to B_{ℓ_2} or, equivalently B_{ℓ_1}. The same result for Banach lattices is due to Chaatit [12]. For extensions see [7] and [13]. Thus we have.

Corollary 10.6

If X is a subspace of a separable Banach lattice with cotype then X satisfies the conditions of Theorem 10.5 if and only if B_X has (ucap). In particular if X is a super-reflexive subspace of a Banach lattice then the conditions of Theorem 10.5 hold.

Corollary 10.7

If X is a separable $\mathcal{L}_1$-space then any quotient map $Q : Y \to X$ admits a uniformly continuous lifting on the ball.

Remark. In particular the quotient map $Q : \ell_1 \to L_1$ admits a uniformly continuous lifting on the ball.

Proof. X has a basis [24] and hence B_X has (ucap). It embeds into L_1 and so by Corollary it verifies the conditions of Theorem 10.5. Any quotient map onto X has a locally complemented kernel. □

Theorem 10.8

Let X be the quotient of a separable $\mathcal{L}_1-$space by a reflexive subspace. Then any quotient map onto X admits a section which is uniformly continuous on the ball.

Proof. Let Y be a separable $\mathcal{L}_1$-space and suppose $Q : Y \to X$ is a quotient map. Let $Q_Y : \ell_1 \to Y$ be any quotient map. Then combining Corollary and Theorem there is a uniformly continuous section $\psi : B_X \to \ell_1$ of QQ_Y. Now let $q : Z \to X$ be any quotient map. Then by Theorem it follows that if $q_Z : \ell_1 \to Z$ is any quotient map then qq_Z admits a uniformly continuous section. Thus so does q. □

It has been conjectured that every separable super-reflexive space X has B_X uniformly homeomorphic to B_{ℓ_2}. This conjecture would imply that every super-reflexive space satisfies Theorem 10.5. Partial results on this problem can be found in [7] pp. 199–204; in particular if X is a subspace of a super-reflexive Banach lattice then B_X is uniformly homeomorphic to B_{ℓ_2}. If we relax the assumption of super-reflexivity we know of no example of a closed subspace X of a Banach lattice with cotype so that B_X is not uniformly homeomorphic to B_{ℓ_2}. Let us consider this problem for subspaces of L_1. Our next theorem shows that the problem is in a certain sense local.

Theorem 10.9

Let X be a closed infinite-dimensional subspace of L_1. Suppose (E_n) is an increasing sequence of finite-dimensional subspaces of X whose union is dense. Then:

(1) *If for each n there is a homeomorphism $\varphi_n : B_{E_n} \to B_{\ell_2^{\dim E_n}}$ with uniform bounds on the modulus of continuity of φ_n and φ_n^{-1}, then B_X is uniformly homeomorphic to B_{ℓ_2}.*

(2) *If the sets $(B_{E_n})_{n=1}^\infty$ are uniformly absolute uniform retracts then B_X is an absolute uniform retract.*

Proof. In both cases we observe that the sets $(B_{E_n})_{n=1}^\infty$ are uniformly absolute uniform retracts and hence B_X has (ucap). Since X embeds into L_1, Lemma guarantees that X has a good partition. Consider the natural quotient map $Q : Y = \ell_1(E_n) \to X$ defined by $Q(e_n) = \sum_{n=1}^\infty e_n$. By Theorem 10.5 we can find a uniformly continuous section $\psi : B_X \to Y$. By Proposition 7.2 this implies that B_Y is uniformly homeomorphic to $B_X \times B_{\ker Q}$.

In case (1) we argue first that B_Y and $B_{\ell_1(Y)}$ are uniformly homeomorphic to B_{ℓ_2}. We now apply the above argument to $\ell_1(X)$ and deduce the existence of a space Z so that B_{ℓ_2} is uniformly homeomorphic to $B_{\ell_1(X)} \times B_Z$ and hence also to $B_X \times B_{\ell_1(X)} \times B_Z$ and thus to $B_X \times B_{\ell_2}$. On the other hand if X is super-reflexive we are done by Corollary 9.11 of [7]. If X is not super-reflexive then X is isomorphic to $X \oplus \ell_1$ and hence $B_X \times B_{\ell_2}$ is uniformly homeomorphic to B_X and the proof is complete.

In case (2) we argue B_Y is an absolute uniform retract, and this will suffice. Indeed suppose M is a metric space containing B_Y. Let $h : B_Y \to B_{\ell_1}$ be defined by $h((e_n)_{n=1}) = \|e_n\|$. Then h can be extended to a uniformly continuous map $\tilde{h} : M \to B_{\ell_1}$ since B_{ℓ_1} is an absolute uniform retract; let $\tilde{h}(x) = (\tilde{h}_n(x))_{n=1}^\infty$. Next for each n let $g_n((e_k)_{k=1}^\infty) = e_n$ and consider the sequence of maps $g_n : B_Y \to B_{E_n}$. The maps (g_n) can be extended to equi-uniformly continuous maps $\tilde{g}_n : M \to B_{E_n}$. Finally define $F : M \to B_Y$ by

$$F(x) = \frac{\sum_{n=1}^\infty \tilde{h}_n(x)\tilde{g}_n(x)}{\max(1, \|\tilde{h}(x)\|)}.$$

Then F is uniformly continuous. □

11. Conclusion

We close with a few remarks and unsolved problems. Several of our results can probably be proved by an alternate approach if we understood more about the Banach space structure of $\mathcal{F}(M)$.

Let us start with Proposition 4.4. It is natural to ask:

Problem 11.1

For which uniformly discrete metric spaces M is it true that $\mathcal{F}(M)$ embeds into C_1 (i.e. into an ℓ_1-sum of finite-dimensional spaces)? In particular what if M is a net in arbitrary Banach space X (e.g. $X = \ell_2$)?

We saw in Example 5.6 that when M is a net associated to c_0 then $\mathcal{F}(M)$ does not embed into C_1. A similar problem is the following:

Problem 11.2

If $0 < \alpha < 1$, does $\mathcal{F}^{(\alpha)}(B_{\ell_2})$ (or $\mathcal{F}^{(\alpha)}(\ell_2)$) embed into L_1? Does it have cotype two?

A negative answer to this problem would give an alternative route to the results of § 8. In fact, one could then complete Theorem 8.5 for the case $\frac{1}{2} < \alpha < 1$. Theorem 8.5 could also be completed if the following problem has the right answer:

Problem 11.3

If $n \in \mathbb{N}$ and $\frac{1}{2} < \alpha < 1$ what is the least constant c_α so that whenever M is a metric space, E a subset of M and $\varphi : E \to \ell_2^n$ is a map satisfying:

$$\|\varphi(x) - \varphi(y)\|^\alpha \leq d(x,y) \qquad x, y \in E$$

then there is an extension $\psi : M \to \ell_2^n$ with

$$\|\psi(x) - \psi(y)\|^\alpha \leq c_\alpha d(x,y) \qquad x, y \in M.$$

Minty's theorem [35] shows that $c_\alpha = 1$ if $0 < \alpha \leq \frac{1}{2}$ while it is trivial that $c_1 = n^{\frac{1}{2}}$. It is a natural conjecture that $c_\alpha \leq C n^{\alpha - \frac{1}{2}}$ if $\frac{1}{2} < \alpha < 1$. If this conjecture is correct it would allow us to extend Theorem 8.5 to $0 < \alpha < 1$. The work of Ball [5] and Naor [36] may well be useful here.

Problem 11.4

Is there any example of a Banach space X so that B_X fails (ucap)? What if X embeds into L_1?

Of course such a space would have to fail (BAP) and fail to be super-reflexive. One way to approach this problem is to find a subspace of C_1 whose unit ball fails to be a uniform retract of B_{C_1} (see Theorem 10.5).

We have remarked before that it has been conjectured that if X is a separable super-reflexive space then B_X is uniformly homeomorphic to B_{ℓ_2}. We raise the following problems:

Problem 11.5

If X is a subspace of L_1 is B_X uniformly homeomorphic to B_{ℓ_2}?

Problem 11.6

If X is a subspace of L_1 is B_X an absolute uniform retract (or equivalently, is there a uniform retraction from B_{L_1} onto B_X)?

Theorem 10.9 suggests that these problems are local in character. In fact let $(E_n)_{n=1}^\infty$ be a sequence of finite-dimensional subspaces of L_1 which is dense in the Banach-Mazur sense in the collection of all such subspaces. If the answer to Problem 11.6 is positive then we can use $X = \ell_1(E_n)$ and hence deduce that the sets B_E are uniformly absolute uniform retracts as E runs through all finite-dimensional subspaces of L_1. Conversely if the sets B_E are uniformly absolute uniform retracts then Theorem 10.6 already implies that for every subspace X of L_1, we have that B_X is an absolute uniform retract.

We remark that we also do not know if there is a subspace of L_1 so that B_X is an absolute uniform retract but is not uniformly homeomorphic to B_{ℓ_2}.

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