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Collect. Math. **54**, 3 (2003), 255–267

On some properties of partial intersection schemes

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Received September 6, 2002

Abstract

Partial intersection subschemes of $\mathbb{P}^r$ of codimension c were used to furnish various graded Betti numbers which agree with a fixed Hilbert function. Here we study some further properties of such schemes; in particular, we show that they are not in general licci and we give a large class of them which are licci. Moreover, we show that all partial intersections are glicci. We also show that for partial intersections the first and the last Betti numbers, say m and p respectively, give bounds each other; in particular, in the codimension 3 case we see that $\left\lceil \frac{p+5}{2} \right\rceil \leq m \leq 2p+1$ and each m and p satisfying the above inequality can be realized.

Introduction

The aim of this paper is to produce some further applications of the partial intersection schemes which were introduced in [6]. These schemes were used essentially to try to understand which are all possible graded Betti numbers of aCM schemes with an assigned Hilbert function. Here we use these schemes, which seem very related to the class of artinian monomial ideals of the polynomial ring (see for instance [4]), to study their behavior with respect to the property licci and glicci (i.e. to be ci-linked or g-linked to a complete intersection). Moreover, for general aCM schemes it is known that the Hilbert function gives bounds on the graded Betti numbers (hence on the Betti numbers) but no restriction on the Betti numbers can be done in terms of one of them; in this paper we will show that for partial intersections schemes the number of last minimal syzygies gives a bound on the number of minimal generators (see Theorem 3.3)

Keywords: Hilbert function, Betti numbers, liaison, arithmetically Cohen-Macaulay schemes.
MSC2000: 13D40, 13H10.

and, consequently, conversely, the number of minimal generators limits the number of minimal last syzygies.

Here is a very short sketch of the paper. The first section is devoted to recall definitions, properties and facts on partial intersection schemes as in [6]. In Section 2, after observing that not all partial intersections can be licci, using a very special type of liaison, "ad hoc" for partial intersections, we give a large class of such schemes which are licci. Then, using g-liaison and suitable Gorenstein schemes, we show that all partial intersections are glicci. In the last section, we deal with the problem of giving relationships between the number of minimal generators and the last Betti number; precisely, we will first show that in any codimension a partial intersection X with p last syzygies has a number of minimal generators bounded by $f(p)$, for some function of p. The codimension 3 case is studied more in details giving as a result that $\lceil \frac{p+5}{2} \rceil \leq \nu(I_X) \leq 2p+1$ (Corollary 3.8) and each pair $\nu(I_X)$ and p satisfying the above inequalities can be reached by some partial intersection.

1. Partial intersections: definitions, properties and facts

Throughout this paper k will denote an algebraically closed field, $\mathbb{P}^r$ the r-dimensional projective space over k,

$$R = k[x_0, x_1, \dots, x_r] = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{O}_{\mathbb{P}^r}(n)).$$

If $V \subset \mathbb{P}^r$ is a subscheme, I_V will denote its defining ideal and $H_V(n) = \dim_k R_n - \dim_k (I_V)_n$ its Hilbert function. Moreover, if $V \subset \mathbb{P}^r$ is a c-codimensional aCM scheme with minimal free resolution

$$0 \to \oplus R(-j)^{\alpha_{cj}} \to \cdots \to \oplus R(-j)^{\alpha_{2j}} \to \oplus R(-j)^{\alpha_{1j}} \to I_V \to 0$$

then the integers $\{\alpha_{ij}\}_j$ will denote the i-th graded Betti numbers.

In this section we recall the construction of the c-codimensional partial intersection schemes made in [6] and we collect from there the main facts that will be used in this paper.

Let $(\mathcal{P}, \leq)$ be a poset. We denote, for every $H \in \mathcal{P}$,

$$\mathcal{S}_H = \{K \in \mathcal{P} \mid K < H\}, \quad \overline{\mathcal{S}}_H = \{K \in \mathcal{P} \mid K \leq H\}.$$

DEFINITION 1.1. A subset $\mathcal{A}$ of the poset $\mathcal{P}$ is said to be a *left segment* if for every $H \in \mathcal{A}$, $\mathcal{S}_H \subseteq \mathcal{A}$. In particular, when $\mathcal{P} = \mathbb{N}^c$ with the order induced by the natural order on $\mathbb{N}$, a finite left segment will be mentioned as a *c-left segment*.

Note that every c-left segment $\mathcal{A}$ has sets of generators but there is a unique minimal set of generators consisting of the maximal elements of $\mathcal{A}$; we will denote it by $G(\mathcal{A})$.

If $\pi_i : \mathbb{N}^c \to \mathbb{N}$ will denote the projection to the i-th component, and $\mathcal{A}$ is a c-left segment, we set $a_i = \max\{\pi_i(H) \mid H \in \mathcal{A}\}$, for $1 \leq i \leq c$. The c-tuple $T = T(\mathcal{A}) = (a_1, \dots, a_c)$ will be called the *size* of $\mathcal{A}$. Moreover we denote by $v(H) = \sum_{i=1}^{c} \pi_i(H)$.

A c-left segment is said to be *degenerate* if $a_i = 1$ for some i.

If $\mathcal{A}$ is a c-left segment, $F(\mathcal{A})$ will denote the set of minimal elements of $\mathbb{N}^c \setminus \mathcal{A}$, i.e.

$$F(\mathcal{A}) = \{H \in \mathbb{N}^c \setminus \mathcal{A} \mid \mathcal{S}_H \subseteq \mathcal{A}\}.$$

Note that, if $H = (m_1, \dots, m_c) \in F(\mathcal{A})$ and $m_i > 1$, then $H_i = (m_1, \dots, m_i - 1, \dots, m_c) \in \mathcal{A}$. Moreover, the elements

$$T_1 = (a_1 + 1, 1, \dots, 1), \dots, T_c = (1, 1, \dots, a_c + 1)$$

always belong to $F(\mathcal{A})$, and we will call them *canonical* c-tuples.

In the sequel we denote the c-tuple $(1, \dots, 1)$ by I and, for every subset Z of $\overline{\mathcal{S}}_T$, we denote

$$C_T(Z) = \{T + I - H \mid H \in Z\}.$$

Finally, for every c-left segment $\mathcal{A}$ we define

$$\mathcal{A}^* = C_T(\overline{\mathcal{S}}_T \setminus \mathcal{A}).$$

Observe that $\mathcal{A}^*$ is a c-left segment.

Proposition 1.2

If $\mathcal{A}$ is a c-left segment, then

1. $F(\mathcal{A}) = C_T(G(\mathcal{A}^*)) \cup \{T_1, \dots, T_c\}$,
2. $F(\mathcal{A}^*) = C_T(G(\mathcal{A})) \cup \{T_1^*, \dots, T_c^*\}$.
3. *If $T_i^* \neq T_i$, for some i, then $T_i^* \in C_T(G(\mathcal{A}))$.*

Proof. See Proposition 1.3 in [6]. □

Fix a c-left segment $\mathcal{A}$ and consider c families of hyperplanes of $\mathbb{P}^r$, $c \leq r$,

$$\{A_{1j}\}_{1 \leq j \leq a_1}, \quad \{A_{2j}\}_{1 \leq j \leq a_2}, \quad \dots, \quad \{A_{cj}\}_{1 \leq j \leq a_c}$$

sufficiently generic, in the sense that $A_{1j_1} \cap \ldots \cap A_{cj_c}$ are $\prod_{i=1}^{c} a_i$ pairwise distinct linear varieties of codimension c.

For every $H = (j_1, \dots, j_c) \in \mathcal{A}$, we denote by

$$L_H = \bigcap_{h=1}^{c} A_{hj_h}.$$

With this notation we have the following.

Definition 1.3. The subscheme of $\mathbb{P}^r$

$$V = \bigcup_{H \in \mathcal{A}} L_H$$

will be called a *c-partial intersection* with respect to the hyperplanes $\{A_{ij}\}$ and support on the c-left segment $\mathcal{A}$.

Theorem 1.4

Every c-partial intersection X of $\mathbb{P}^r$ is a reduced aCM subscheme consisting of a union of c-codimensional linear varieties.

Proof. See Theorem 1.9 in [6]. □

Here are the main results on c-codimensional partial intersections.

Theorem 1.5

If $V \subset \mathbb{P}^r$ is a partial intersection of codimension c with support on $\mathcal{A}$, then the $(r-c+1)$-th difference of its Hilbert function is

$$\Delta^{r-c+1} H_V(n) = \Big| \{H \in \mathcal{A} \mid v(H) = n + c\} \Big| .$$

Proof. See Theorem 2.1 in [6]. □

Now, if X is a c-codimensional partial intersection with support on $\mathcal{A}$ and with respect to the families of hyperplanes A_{ij} whose defining forms are f_{ij}, to every $H = (m_1, \dots, m_c) \in \mathcal{A}$ we associate the following form

$$P_H = \prod_{i=1}^{c} \prod_{j=1}^{m_i - 1} f_{ij}.$$

Theorem 1.6

Let $V \subset \mathbb{P}^r$ be a partial intersection of codimension c with support $\mathcal{A}$. Then a minimal set of generators for I_V is

$$\{P_H \mid H \in F(\mathcal{A})\}.$$

Proof. See Theorem 3.1 in [6]. □

Remark 1.7. The previous theorem shows that partial intersection schemes can be regarded as pseudo-liftings of some monomial ideals (according to the terminology used in [4]). Nevertheless this combinatorial approach essentially looks at the last syzygies of the defining ideal and it permits us to have an easier control both of Hilbert function and of first and last graded Betti numbers.

Corollary 1.8

Let V be as above then its first graded Betti numbers depend only on $\mathcal{A}$ and they are the following integers

$$d_H = v(H) - c \quad \forall\ H \in F(\mathcal{A})$$

i.e. the number of minimal generators of degree j is equal to the number of $H \in F(\mathcal{A})$ such that $v(H) = j + c$.

And finally:

Theorem 1.9

Let $V \subset \mathbb{P}^r$ be a partial intersection of codimension c with support $\mathcal{A}$. Then the last graded Betti numbers of V are

$$s_H = v(H) \quad \forall\ H \in G(\mathcal{A})$$

i.e. the number of minimal generators of the last syzygy module of degree j is equal to the number of $H \in G(\mathcal{A})$ such that $v(H) = j$.

Proof. See Theorem 3.4 in [6]. □

2. Licci and glicci property for partial intersections

Many authors in the last few years studied subschemes of $\mathbb{P}^n$ which are linked (by a complete intersection) to a complete intersection (briefly *licci schemes*). It is well known that not all aCM subschemes are licci. The easiest example is given by 4 general points of $\mathbb{P}^3$, or more generally by $\binom{d+3}{3}$ general points of $\mathbb{P}^3$. Indeed, if $X \subset \mathbb{P}^n$ is a c-codimensional aCM scheme, by a result of Huneke and Ulrich (see [2] Theorem 5.8), if we denote by α the minimum degree of a generator for I_X and by θ the maximum degree of the last graded Betti numbers, if $\theta \leq (c-1)\alpha$ then X cannot be licci. Since α and θ are determined by the Hilbert function of X, we deduce that if H is an admissible Hilbert function for a c-codimensional aCM subscheme of $\mathbb{P}^n$ such that $\theta(H) \leq (c-1)\alpha(H)$ then every aCM scheme with such an Hilbert function cannot be licci.

In codimension 2 is well known that every aCM scheme is licci; but in codimension $c \geq 3$ we do not know which are the Hilbert functions for which there exist licci schemes with such Hilbert function. Therefore it seems interesting to produce classes of aCM schemes of codimension $c \geq 3$ which are licci.

Of course, not all partial intersection schemes (defined in the previous section) are licci. Here we present some class of partial intersections which are licci.

For partial intersection schemes we can perform liaison using complete intersections which are partial intersections, i.e. with support $\mathcal{A} = \langle H \rangle$.

Definition 2.1. We say that two c-partial intersections, $X, Y \subset \mathbb{P}^n$ are directly $*$-linked if $X \cup Y = Z$ where Z is a partial intersection complete intersection (and $X \cap Y = \emptyset$). More generally, X and Y are said to be $*$-linked if there is a sequence of partial intersections $X = Y_1, Y_2, \ldots, Y_t = Y$ such that each Y_i is directly $*$-linked to Y_{i+1}.

Following the previous definition a partial intersection X is said to be $*$-*licci* if it is $*$-linked to a complete intersection.

Next result gives a condition in order to have partial intersections which are $*$-linked.

Let $\mathcal{A}$ be a c-left segment with $G(\mathcal{A}) = \{H_1, \ldots, H_p\}$ and denote $T(\mathcal{A})$ the smallest c-tuple greater or equal to each $H_i \in G(\mathcal{A})$. Now define, inductively, the following sets

$$G_1 = \begin{cases} \emptyset & \text{if each } \ T(\mathcal{A}) - H_i \\ & \text{has at least two components different from zero;} \\ & \text{in this case we set } \ G(\mathcal{A}_1) = G(\mathcal{A}). \\ \{H_{i_1}\} & \text{where } \ T(\mathcal{A}) - H_{i_1} \\ & \text{has only one component different from zero;} \\ & \text{in this case we set } \ G(\mathcal{A}_1) = G(\mathcal{A}) \setminus \{H_{i_1}\}. \end{cases}$$

Now, once G_h was built, $h < p$, we define

$$G_{h+1} = \begin{cases} G_h & \text{if each } \ T(\mathcal{A}_h) - H_i, \ H_i \in G(\mathcal{A}_h) \\ & \text{has at least two components different from zero;} \\ & \text{in this case we set } \ G(\mathcal{A}_{h+1}) = G(\mathcal{A}_h). \\ G_h \cup \{H_{i_{h+1}}\} & \text{where } \ T(\mathcal{A}_h) - H_{i_{h+1}} \\ & \text{has only one component different from zero;} \\ & \text{in this case we set } \ G(\mathcal{A}_{h+1}) = G(\mathcal{A}_h) \setminus \{H_{i_{h+1}}\}. \end{cases}$$

Of course, we have $G_1 \subseteq G_2 \subseteq \ldots \subseteq G_p$. Notice that, despite the fact we made choices during the previous construction, G_p does not depend on these choices.

Theorem 2.2

Let $X \subset \mathbb{P}^n$ be a c-codimensional partial intersection with support on the c-left segment $\mathcal{A}$ and $G(\mathcal{A}) = \{H_1, \ldots, H_p\}$. If $G_p = G(\mathcal{A})$ then X is $*$-licci; in particular X is licci.

Proof. We use induction on $p = |G(\mathcal{A})|$. The case $p = 1$ is trivial since X is a complete intersection. Let us assume the result true for partial intersections whose left segment support $\mathcal{A}'$ is generated by $i < p$ elements and $G_i(\mathcal{A}') = G(\mathcal{A}')$. Since $G_p = G(\mathcal{A})$ there are elements $H_j \in G(\mathcal{A})$ such that $T(\mathcal{A}) - H_j$ have only one component different from zero. Without loss of generality we can assume that $H_1, \ldots, H_t$ have this property (of course, $t \leq p$) and moreover (just reordering the components) we can assume that the only non zero component of $T(\mathcal{A}) - H_i$, $i \leq t$, is a_{ii} at the i-th position.

Claim. $G(\mathcal{A}^{**}) = \{H_i - (T(\mathcal{A}) - T(\mathcal{A}^*)) \mid 1 \leq i \leq p\} \cap \mathbb{N}^c$.

The claim will be proved in the next lemma. Now, it is easy to verify that $Z = T(\mathcal{A}) - T(\mathcal{A}^*) = (a_{11}, \ldots, a_{tt}, 0, \ldots, 0)$, therefore $H_i - Z \notin \mathbb{N}^c$ for $i = 1, \ldots, t$. Hence we get $G(\mathcal{A}^{**}) = \{H_i - Z \mid t+1 \leq i \leq p\}$. So, $|G(\mathcal{A}^{**})| = p - t < p$. Let us denote $U = \{H_1, \ldots, H_t\}$ and observe that $|G_i(\mathcal{A}^{**})| = |G(\mathcal{A}_{i+t}) \setminus U|$ for $1 \leq i \leq p-t$. Indeed, we see that

$$H \in G_i(\mathcal{A}^{**}) \iff H + Z \in G_{i+t}(\mathcal{A}) \setminus U;$$

now, since Z is constant we get the required equality. In particular, $|G_{p-t}(\mathcal{A}^{**})| = p-t$, hence $G_{p-t}(\mathcal{A}^{**}) = G(\mathcal{A}^{**})$. Now the induction concludes the proof. $\square$

Lemma 2.3

Let $\mathcal{A}$ be a c-left segment with $G(\mathcal{A}) = \{H_1, \dots, H_p\}$. Then

$$G(\mathcal{A}^{**}) = \{H_i - (T(\mathcal{A}) - T(\mathcal{A}^*))\} \cap \mathbb{N}^c.$$

Proof. By part 2) of Proposition 1.2 in [6] we have $F(\mathcal{A}^*) = C_T(G(\mathcal{A})) \cup \{T_1^*, \dots, T_c^*\}$; now applying to $\mathcal{A}^*$ part 1) of the same proposition we get

$$F(\mathcal{A}^*) = C_{T^*}(G(\mathcal{A}^{**})) \cup \{T_1^*, \dots, T_c^*\}.$$

Hence,

$$C_T(G(\mathcal{A})) \cup \{T_1^*, \dots, T_c^*\} = C_{T^*}(G(\mathcal{A}^{**})) \cup \{T_1^*, \dots, T_c^*\}.$$

Apply the operator $C_{T^*}(-)$ to both sides of the previous equality to get

$$\begin{aligned} &C_{T^*}\big(C_T(G(\mathcal{A}))\big) \cup \{T^* + I - T_1^*, \dots, T^* + I - T_c^*\} \\ &\quad = G(\mathcal{A}^{**}) \cup \{T^* + I - T_1^*, \dots, T^* + I - T_c^*\}. \end{aligned}$$

Therefore

$$\begin{aligned} &\big[C_{T^*}\big(C_T(G(\mathcal{A}))\big) \cup \{T^* + I - T_1^*, \dots, T^* + I - T_c^*\}\big] \cap \mathbb{N}^c \\ &\quad = \big[G(\mathcal{A}^{**}) \cup \{T^* + I - T_1^*, \dots, T^* + I - T_c^*\}\big] \cap \mathbb{N}^c. \end{aligned}$$

Finally, since $T^* + I - T_i^* \notin \mathbb{N}^c$ for all i and $G(\mathcal{A}^{**}) \subset \mathbb{N}^c$ we get the conclusion. □

Despite of the fact that not every partial intersection is licci if we generalize complete intersection liaison to Gorenstein liaison we are able to prove that every partial intersection is G-linked to a complete intersection, i.e. is *glicci*. It is still an open question to understand if every aCM scheme is glicci. Affirmative answers are, for instance, in [1] by Hartshorne for generic points on quadrics and cubics and in [3] where the authors show that every standard determinantal projective scheme is glicci. Migliore and Nagel proved in [5], Theorem 3.1, that schemes which are lifting of artinian monomial ideals are glicci. Our result is in some sense in this direction.

Theorem 2.4

Every partial intersection is glicci.

Proof. We work by induction on the codimension c. The case $c = 1$ is trivially true. So we can assume that every $(c-1)$-codimensional partial intersection is glicci. Let $X \subset \mathbb{P}^n$ be a c-partial intersection with support on $\mathcal{A}$. Then we can write $X = \bigcup_{j=1}^s V_j \cap A_{cj}$, where V_j are $(c-1)$-partial intersections and A_{cj} hyperplanes. Now we proceed by induction on s. If $s = 1$ $X = V_1 \cap A_{c1}$ is trivially glicci since V_1 is glicci. Now, let $Y = \bigcup_{j=2}^s V_j \cap A_{cj}$, (glicci by inductive hypothesis) we show that $X = Y \cup (V_1 \cap A_{c1})$ is G-linked to Y. Let $\langle T \rangle$ be the smallest rectangle containing the $(c-1)$-left segment $\mathcal{A}_1 = \{H \in \mathbb{N}^{c-1} \mid (H, 1) \in \mathcal{A}\}$ and denote $V_1' = \bigcup_{H \in \langle T \rangle \setminus \mathcal{A}_1} L_H$. Since $V_1 \cup V_1'$ is a complete intersection $G = V_1 \cap V_1'$ is a c-codimensional Gorenstein scheme.

Claim 1. $G_1 = V_1 \cap \left(\bigcup_{j=2}^{s} A_{cj}\right) \cup G$ is a Gorenstein scheme.

Thus Y is G-linked by G_1 to $G \cup Y'$ where $Y' = V_1 \cap \left(\bigcup_{j=2}^{s} A_{cj}\right) \setminus Y$.

Claim 2. $G_1 \cup (V_1 \cap A_{c1})$ is a Gorenstein scheme.

Then $G \cup Y'$ is G-linked by G_2 to $Y \cup (V_1 \cap A_{c1}) = X$. The proofs of the claims work in the same line. One observes that the module $[I_{V_1'} + I_{V_1}]/I_{V_1}$ has a minimal free resolution of Cohen-Macaulay type 1. Now using Lemma 4.8 in [3] we get that $I_{V_1} + [I_{V_1'} + I_{V_1}] \prod_{j=2}^{s} f_{cj}$ and $I_{V_1} + [I_{V_1'} + I_{V_1}] \prod_{j=1}^{s} f_{cj}$ are Gorenstein (here f_{cj} means the linear form defining A_{cj}) and the two claims are done. □

3. Bounds between first and last Betti numbers for partial intersections

It is well known that for aCM schemes (of codimension ≥ 3) there is no "a priori" bound for the last graded Betti numbers of the defining ideal in terms of the minimal number of generators and conversely the number of minimal last syzygies does not limit the number of minimal generators. It is enough for this to remark that (at least in codimension 3) there are arithmetically Gorenstein schemes with any (odd) numbers of minimal generators. Here we want to show that for partial intersection schemes the minimal number of generators gives a bound to the last Betti number and conversely the number of last syzygies furnishes a bound for the minimal number of generators.

If $\mathcal{A}$ is a c-left segment with $G(\mathcal{A}) = \{H_1, \ldots, H_p\}$ we set $a_i = \pi_1(Hi)$, for $1 \leq i \leq p$ and we reorder the elements in $G(\mathcal{A})$ in such a way $a_1 \leq \ldots \leq a_p$. Hence we can write $H_i = (a_i, K_i)$ with $K_i \in \mathbb{N}^{c-1}$. Finally, we set for all i $\mathcal{A}_i = \{K \in \mathbb{N}^{c-1} \mid (i, K) \in \mathcal{A}\}$.

Lemma 3.1

Let $i, j \in \mathbb{N}$ be such that $a_h + 1 \leq i \leq j \leq a_{h+1}$. Then $\mathcal{A}_i = \mathcal{A}_j$.

Proof. Since $\mathcal{A}_1 \supseteq \ldots \supseteq \mathcal{A}_i \supseteq \ldots \supseteq \mathcal{A}_j \supseteq \ldots$ we need to prove $\mathcal{A}_i \subseteq \mathcal{A}_j$. Let $K \in \mathcal{A}_i$, i.e. $(i, K) \in \mathcal{A}$; there exists $H_r \in G(\mathcal{A})$ such that $(i, K) \leq H_r$. Now, $H_r = (a_r, K_r)$ hence from $a_r \geq i > a_h$ we get $p \geq r \geq h + 1$. Moreover, form $K \leq K_r$ we have $(j, K) \leq (a_{h+1}, K_r) \leq (a_r, K_r) = H_r$, which means $(j, K) \in \mathcal{A}$, i.e. $K \in \mathcal{A}_j$. □

We start proving that the number of minimal generators of the defining ideal of a partial intersection X is bounded by some integer depending on the last Betti number. Before we prove the following.

Lemma 3.2

Let $\mathcal{A}$ be a left segment minimally generated by p elements. Then there exists a left segment $\mathcal{B}$, minimally generated by $p + 1$ elements such that $|F(\mathcal{B})| \geq |F(\mathcal{A})|$.

Proof. Let $G(\mathcal{A}) = \{H_1, H_2, \dots, H_p\}$. We set $2\mathcal{A} := \langle 2H_1, 2H_2, \dots, 2H_p\rangle$. Of course $|G(2\mathcal{A})| = |G(\mathcal{A})|$. More precisely we will show that Moreover $F(2\mathcal{A}) = \{2K - I \mid K \in F(\mathcal{A})\}$. In fact if $2K - I \leq 2H_i$, with $K \in F(\mathcal{A})$ for some i, then $2K \leq 2H_i + I$ i.e. $K \leq H_i$, since the components of K and H are integers, that is a contradiction. So $2K - I \notin \mathcal{A}$, for every $K \in F(\mathcal{A})$. If $L < 2K - I$, with $K \in F(\mathcal{A})$, then there is L' having all components even, such that $L \leq L' < 2K$; so $1/2L' < K$, i.e. there is an integer i such that $1/2L' \leq H_i$, therefore $L \leq L' \leq 2H_i$, so each $2K - I$ is minimal for $\mathbb{N}^c \setminus 2\mathcal{A}$; finally if $L \notin 2\mathcal{A}$, denote by L' the c-tuple such that $\pi_i(L') = 2\left\lceil \frac{\pi_i(L)}{2} \right\rceil$. Then $L' \geq L$ and L' has all components even. So $L' \notin 2\mathcal{A}$ and $1/2L' \notin \mathcal{A}$, hence there is $K \in F(\mathcal{A})$ such that $1/2L' \geq K$, i.e. $L' \geq 2K$, that implies $L \geq 2K - I$. In particular we have $|F(2\mathcal{A})| = |F(\mathcal{A})|$.

Let m be the maximum among the first components of the elements of $2\mathcal{A}$; it is obvious that $T_1 = (m+1, 1, \dots, 1) \in F(2\mathcal{A})$ and the left segment $\mathcal{B} := \langle 2H_1, 2H_2, \dots, 2H_p, T_1\rangle$ is minimally generated by these $p+1$ elements. Moreover $F(2\mathcal{A}) \setminus \{T_1\} \subseteq F(\mathcal{B})$; indeed if $K \in F(2\mathcal{A}) \setminus \{T_1\}$, $K \notin \mathcal{B}$, and if $L < K$ then $L \in 2\mathcal{A} \subset \mathcal{B}$, so $K \in F(\mathcal{B})$. But in $F(\mathcal{B})$ there is also $(m+2, 1, \dots, 1)$, therefore $|F(\mathcal{B})| \geq |F(2\mathcal{A})| = |F(\mathcal{A})|$. □

Theorem 3.3

Let c and p be two integers. Then there is an integer $f_c(p)$ such that $\nu(I_X) \leq f_c(p)$, for every partial intersection X of codimension c and with last Betti number p.

Proof. Let X be a c-partial intersection with last Betti number p whose support is the c-left segment $\mathcal{A}$ with $G(\mathcal{A}) = \{H_1, \dots, H_p)\}$. We need to show that there is an integer $f_c(p)$, depending only on c and p, such that $|F(\mathcal{A})| \leq f(p)$. Following the terminology of the previous lemma we set $H_i := (a_i, K_i)$ and $a_1 \leq \dots \leq a_p$. We proceed by induction on the codimension c. The case $c = 1$ is trivial since $|G(\mathcal{A})| = |F(\mathcal{A})| = 1$ (indeed also the case $c = 2$ is trivial as $|F(\mathcal{A})| = p + 1$). Therefore suppose the result true for each $(c-1)$-left segment. Observe first that $\mathcal{A}_i = \langle \{K_j \mid a_j \geq i\}\rangle$; namely, if $K \in \mathcal{A}_i$ then $(i, K) \in \mathcal{A}$, therefore $(i, K) \leq (a_r, K_r) = H_r$, so $K \leq K_r$ with $a_r \geq i$. This means $|G(\mathcal{A}_i)| \leq p$ and by the inductive hypothesis and by Lemma 3.2 $|F(\mathcal{A}_i)| \leq f_{c-1}(p)$.

Now denote $\alpha_0 = 1, \alpha_1 = a_1 + 1, \dots, \alpha_p = a_p + 1$ and consider the following set

$$U = \{(\alpha_i, L) \mid 0 \leq i < p,\ L \in F(\mathcal{A}_{\alpha_i})\} \cup \{(a_p + 1, 1, \dots, 1)\}$$

Of course, $U \cap \mathcal{A} = \emptyset$. We show that $F(\mathcal{A}) \subseteq U$. Let $H \in F(\mathcal{A})$ and set $H = (x, K)$ (of course, $H \in \mathbb{N}^c \setminus \mathcal{A}$). If $x > a_p$ then $(a_p + 1, 1, \dots, 1) \leq H$ and since H is minimal in $\mathbb{N}^c \setminus \mathcal{A}$ we get $H = (a_p + 1, 1, \dots, 1)$. So we can assume $1 \leq x \leq a_p$, hence $a_h + 1 \leq x \leq a_{h+1}$; notice that $K \notin \mathcal{A}_x$ therefore there exists an element $V \in F(\mathcal{A}_x)$ with $V \leq K$. Now, by previous lemma $\mathcal{A}_x = \mathcal{A}_{a_h+1}$ hence $V \in F(\mathcal{A}_{a_h+1}$ and this means $(a_h + 1, V) \in U$ and $(a_h + 1, V) \leq (x, K) = H$; by the minimality of H in $\mathbb{N}^c \setminus \mathcal{A}$ we get $H = (a_h + 1, V) \in U$. In conclusion,

$$|F(\mathcal{A})| \leq |U| \leq \sum_{i=0}^{p} |F(\mathcal{A}_{\alpha_i})| + 1 \leq (p+1) f_{c-1}(p) + 1 := f_c(p). \ \square$$

As consequence of the previous theorem, it makes sense to define the following function

$$\Phi_c(p) := \max\{|F(\mathcal{A})| : \mathcal{A} \text{ is a } c\text{-left segment and } |G(\mathcal{A})| = p\}.$$

It is known that $\Phi_1(p) = 1$ and $\Phi_2(p) = p + 1$.

Problem 3.4 Compute explicitly Φ_c for any c.

We will solve this problem for $c = 3$.

Corollary 3.5

Let $X \subset \mathbb{P}^r$ be a c-codimensional partial intersection. If I_X has m minimal generators and p minimal last syzygies then

$$p \leq \Phi_c(m - c).$$

Proof. If $p = 1$ it is trivial. If $p > 1$, let $\mathcal{A}$ be the c-left segment, support of X. Then $\mathcal{A}^* \neq \emptyset$ and set $m^* = |F(\mathcal{A}^*)|$, $p^* = |G(\mathcal{A}^*)|$; we have that $m^* \leq \Phi_c(p^*)$. On the other hand, by liaison, $p \leq m^* \leq p + c$ and $p^* = m - c$, so we have $p \leq m^* \leq \Phi_c(m - c)$. □

Now we compute explicitly Φ_3.

Theorem 3.6

Let $X \subset \mathbb{P}^r$ be a 3-codimensional partial intersection with support $\mathcal{A}$. If $|G(\mathcal{A})| = p$, i.e. if I_X has last Betti number p, then $\nu(I_X) \leq 2p + 1$.

Proof. Denote $\mathcal{A} = \langle H_1, \dots, H_p \rangle$, i.e. $G(\mathcal{A}) = \{H_1, \dots, H_p\}$ and set $H_i = (a_i, b_i, c_i)$, $a = \max\{a_i\}$, $b = \max\{b_i\}$, $c = \max\{c_i\}$. We first show that if $a_i \neq a_j$, $b_i \neq b_j$, $c_i \neq c_j$ for all $i \neq j$ then $|F(\mathcal{A})| = 2p + 1$, that for [6] Theorem XX will mean $\nu(I_X) = 2p + 1$. Of course, we can suppose $a_1 < \ldots < a_p$. Now denote $BC_i = \{(b_h, c_h) \mid h \geq i\}$ and observe that, for every $j > h$, cannot happen $(b_h, c_h) \leq (b_j, c_j)$. Moreover, $BC_{i-1} \supseteq BC_i$, $|BC_i| = p + 1 - i$ and $BC_{i-1} \setminus BC_i = \{(b_{i-1}, c_{i-1})\}$. If $\max BC_i$ is the set of maximal elements in the poset BC_i, by the previous observations we see that $|\max BC_{i-1} \setminus \max BC_i| = 1$. We are interested on the elements in $\max BC_i$ which are not on any $\max BC_r$ with $r < i$; but one notes that if $K \in \max BC_i \cap \max BC_r$, $r < i$ then $K \in \max BC_{i-1}$. Therefore we can restrict ourselves to $\max BC_i \setminus \max BC_{i-1}$. Finally, set $n_i = |\max BC_i|$ and $m_i = |\max BC_i \setminus \max BC_{i-1}|$.

Claim 1: $\sum\limits_{1=1}^{p} m_i = p$.

Indeed, $U_i = \max BC_i \setminus \max BC_{i-1}$ are disjoint sets and $\bigcup\limits_{i=0}^{p} U_i = BC_1$.

Now denote $\max BC_i = \{(b_{i_1}, c_{i_1}), \dots, (b_{i_{n_i}}, c_{i_{n_i}})\}$; without lost of generality we can assume $b_{i_1} < \ldots < b_{i_{n_i}}$; $c_{i_1} > \ldots > c_{i_{n_i}}$. Consider the element (b_{i-1}, c_{i-1}) and define the integers h and k as follows

$$b_{i_h} < b_{i-1} < b_{i_{h+1}}; \quad c_{i_k} > c_{i-1} > c_{i_{k+1}}.$$

If $k \geq h+1$ then $(b_{i-1}, c_{i-1}) < (b_{i_{h+1}}, c_{i_{h+1}})$ and since $i_{h+1} \geq i$, we have $(a_{i-1}, b_{i-1}, c_{i-1}) < (a_{i_{h+1}}, b_{i_{h+1}}, c_{i_{h+1}})$ a contradiction with the minimality of the elements in $G(\mathcal{A})$. So, $k \leq h$. Then

$$\max BC_i \setminus \max BC_{i-1} = \{(b_{i_{k+1}}, c_{i_{k+1}}), \dots, (b_{i_h}, c_{i_h})\}.$$

Finally, set

$$G_i = \{(a_{i-1}+1, b_{i_k}+1, c_{i_{k+1}}+1), \dots, (a_{i-1}+1, b_{i_h}+1, c_{i_{h+1}}+1)\}$$

for $i = 1, \dots, p+1$ (here we set $b_{i_0} = c_{i_{n_i+1}} = 0$). Note that $G_{p+1} = \{(a_p+1, 1, 1)\}$). Notice that, even in the case $\max BC_i \setminus BC_{i-1} = \emptyset$, (i.e. $h=k$) G_i contains 1 element, precisely $(a_{i-1}+1, b_{i_h}+1, c_{i_{h+1}}+1)$. And finally, note that $|G_i| = m_i + 1$ and G_i are pairwise disjoint sets.

Claim 2: $F(\mathcal{A}) = \bigcup_{i=1}^{p+1} G_i$.

Once we have Claim 2 we get

$$|F(\mathcal{A})| = \sum_{1=1}^{p} (m_i + 1) = \sum_{1=1}^{p} m_i + p + 1 = 2p+1.$$

Proof of Claim 2. Take $H = (a_{i-1}+1, b_{i_r}+1, c_{i_{r+1}}+1) \in G_i$.

– $H \notin \mathcal{A}$: since (b_{i_r}, c_{i_r}), $(b_{i_{r+1}}, c_{i_{r+1}})$ are maximal elements in BC_i there is no maximal element (x, y) in BC_i with $x > b_{i_r}$ and $y > c_{i_{r+1}}$; if $H \in \mathcal{A}$ then $H \leq (a_j, b_j, c_j)$ for some j; now $a_j \geq a_{i-1}+1$ hence $j \geq i$ therefore $(b_j, c_j) \in BC_i$; now $b_j > b_{i_r}$ and $c_j > c_{i_{r+1}}$ gives a contradiction.

– $H \in F(\mathcal{A})$: we need to verify that H is minimal in $\mathbb{N}^c \setminus \mathcal{A}$.

i) note that $(a_{i-1}, b_{i_r}+1, c_{i_{r+1}}+1) \leq (a_{i-1}, b_{(i-1)_s}, c_{(i-1)_s})$ for some $(b_{(i-1)_s}, c_{(i-1)_s}) \in \max BC_{i-1}$: indeed, if this is not the case (b_{i_r}, c_{i_r}) and/or (b_{i_r+1}, c_{i_r+1}) should be maximal also in BC_{i-1}. Thus $(a_{i-1}, b_{i_r}+1, c_{i_{r+1}}+1) \in \mathcal{A}$.

ii) $(a_{i-1}+1, b_{i_r}, c_{i_{r+1}}+1) \leq (a_{i_r}, b_{i_r}, c_{i_r}) \in \mathcal{A}$.

iii) $(a_{i-1}+1, b_{i_r}+1, c_{i_{r+1}}) \leq (a_{i_{r+1}}, b_{i_{r+1}}, c_{i_{r+1}}) \in \mathcal{A}$.

To complete this part of the proof we need to show that every element in $F(\mathcal{A})$ is in $\bigcup_{i=1}^{p+1} G_i$ or, equivalently, every element which is not in $\mathcal{A}$ is greater or equal to some element in some G_i. So, take $H = (x, y, z) \notin \mathcal{A}$; we have $a_{i-1} < x \leq a_i$ for some $i = 1, \dots, p+1$ (here we use $a_0 := 0$, $a_{p+1} := \infty$). Then (y, z) cannot be $\leq$ to the elements $(b_{i_1}, c_{i_1}), \dots, (b_{i_{n_i}}, c_{i_{n_i}})$. Moreover, $b_{i_h} < y \leq b_{i_{h+1}}$ and $c_{i_k} \geq z > c_{i_{k+1}} \geq c_{i_{h+1}}$ (note that $h \leq k$), hence $(b_{i_h}+1, c_{i_{h+1}}) \leq (y, z)$. Now $(b_{i_{k+1}}, c_{i_{k+1}}), \dots, (b_{i_h}, c_{i_h})$ are maximal elements in some BC_j but not in BC_{j-1} for some $j \leq i$, therefore $(a_{j-1}+1, b_{i_h}+1, c_{i_{h+1}}+1) \in G_j$ and $(a_{j-1}+1, b_{i_h}+1, c_{i_{h+1}}+1) \leq (x, y, z)$.

To finish the proof we observe that when the initial condition $a_i \neq a_j$, $b_i \neq b_j$, $c_i \neq c_j$ for all $i \neq j$ is not satisfied, repeating the argument we can just show that $F(\mathcal{A}) \subseteq \bigcup_{i=1}^{p+1} G_i$ and, moreover, G_i are not necessarily disjoint sets; hence $|F(\mathcal{A})| \leq |\bigcup_{i=1}^{p+1} G_i| \leq 2p+1$. $\square$

Corollary 3.7

$\Phi_3(p) = 2p+1$.

Proof. From the proof of Theorem 3.6 we can deduce that for every p there exists a 3-left segment $\mathcal{A}$ with $|G(\mathcal{A})| = p$ and $|F(\mathcal{A})| = 2p + 1$. $\square$

Corollary 3.8

Let $X \subset \mathbb{P}^r$ be a 3-codimensional partial intersection with support $\mathcal{A}$. If I_X has p minimal last syzygies then

$$\left\lceil \frac{p+5}{2} \right\rceil \leq \nu(I_X) \leq 2p + 1.$$

Proof. We have to proof only the first inequality. By Corollaries 3.5 and 3.7, $p \leq \Phi_3(m-3) = 2(m-3) + 1 = 2m - 5$. $\square$

Corollary 3.9

Let $X \subset \mathbb{P}^r$ be a 3-codimensional partial intersection with support $\mathcal{A}$. If I_X has m minimal generators then

$$\left\lceil \frac{m-1}{2} \right\rceil \leq s(I_X) \leq 2m - 5,$$

where $s(I_X)$ is the number of minimal last syzygies of I_X.

Proof. This is an immediate consequence of the previous corollary. $\square$

Now we show that every possibilities between $\lceil \frac{p+5}{2} \rceil$ and $2p + 1$ can occur.

Theorem 3.10

Let $p \geq 1$ and $\lceil \frac{p+5}{2} \rceil \leq m \leq 2p + 1$, integers. Then there exists a 3-left segment $\mathcal{A}$ such that $|G(\mathcal{A})| = p$ and $|F(\mathcal{A})| = m$.

Proof. If $p+2 \leq m \leq 2p+1$, we set $h := m - p - 1$ and let us consider the left segment

$$\mathcal{L}_{p,m} = \langle \{(i, i, p+1-i) \mid 1 \leq i \leq h\} \cup \{(i, h, p+1-i) \mid h+1 \leq i \leq p\} \rangle.$$

Since these generators are minimal $|G(\mathcal{L}_{p,m})| = p$. Moreover

$$F(\mathcal{L}_{p,m}) = S := \{T_1, T_2, T_3\} \cup \{(1, j, p+2-j) \mid 2 \leq j \leq h\}$$

$$\cup \{(j, 1, p+2-j) \mid 2 \leq j \leq p\},$$

where $T_1 = (p+1, 1, 1), T_2 = (1, h+1, 1), T_3 = (1, 1, p+1)$ are the three canonical 3-tuples. It is trivial that $S \subseteq F(\mathcal{L}_{p,m})$; to verify that $F(\mathcal{L}_{p,m}) \subseteq S$ take $N := (x, y, z) \in \mathbb{N}^3 \setminus \mathcal{L}_{p,m}$; of course we can suppose $x \leq p$, $y \leq h$ and $z \leq p$, otherwise $N \geq T_i$ for some i. If $z \geq p + 2 - x$ (that implies $x \geq 2$) then $N \geq (x, 1, p+2-x) \in S$. If $z \leq p+1-x$ let us consider

$$H := (x, x, p+1-x), H' := (y, y, p+1-y) \in \mathcal{L}_{p,m}$$

and $K := (1, y, p+2-y) \in S$; since $N \notin \mathcal{L}_{p,m}$, $N \not\leq H$ so $y > x$ and $N \not\leq H'$ so $z > p+1-y$; therefore $N \geq K$.

Then

$$|F(\mathcal{L}_{p,m})| = 3 + h - 1 + p - 1 = 3 + m - p - 1 - 1 + p - 1 = m.$$

If $\lceil \frac{p+5}{2} \rceil \leq m \leq p+1$, we set $p' := m-3$ and $m' := p$. Since $m \leq p+1$, $m'+1 \geq p'+3$ i.e. $m' \geq p'+2$. So we can build the left segment $\mathcal{L}_{p',m'} = \mathcal{L}_{m-3,p}$. Now we set $U := (m-2, m-2, m-2)$ and let $\mathcal{A} := C_U(\langle U \rangle \setminus \mathcal{L}_{m-3,p})$. By liaison we obtain that $|G(\mathcal{A})| = m' = p$ and $|F(\mathcal{A})| = p'+3 = m$. $\square$

Corollary 3.8 gives us a further restriction to the schemes whose graded Betti numbers can be realized using partial intersections; so the following question arises in a natural way.

Question 3.11 Let X a 3-codimensional aCM scheme, whose first Betti number is m and whose last Betti number is p, such that

$$\left\lceil \frac{p+5}{2} \right\rceil \leq m \leq 2p+1.$$

Is there a partial intersection Y having the same graded Betti numbers of X?

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