Collect. Math. **54**, 3 (2003), 217–254

Frames associated with expansive matrix dilations

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Received June 24, 2002. Revised January 15, 2003

Abstract

We construct wavelet-type frames associated with expansive matrix dilation on the Anisotropic Triebel-Lizorkin spaces, $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$. We also show the a.e. convergence of the frame expansion which includes multi-wavelet expansion as a special case.

1. Introduction

The basic idea of a frame was used by many authors; in particular, we cite Paley and Wiener [14], and Duffin and Schaeffer [3]. Paley and Wiener were interested in the question of which collections $\{e^{ix_n\xi} : x_n \in \mathbb{R}, n \in \mathbb{Z}\}$ form a Riesz basis for $B_l = \{\hat{f} : f \in L^2(-l,l)\}$, $l > 0$. Recall that a Riesz basis of a Hilbert space is the image of an orthonormal basis under an invertible linear operator. Duffin and Schaeffer, in fact, considered the "dual" setting of the above problem: what is the sufficient conditions for $\{x_n : n \in \mathbb{Z}\}$ to be a *sample set* of B_l. That is, there exist constants $0 < C \leq B$ such that, for any $f \in B_l$,

$$C\|f\|_{L^2}^2 \leq \sum_{n\in\mathbb{Z}} |f(x_n)|^2 \leq B\|f\|_{L^2}^2.$$

Keywords: Anisotropic Triebel-Lizorkin spaces, molecules, frames, almost diagonal matrices, convergence of frame expansions.

MSC2000: Primary 42B25, 42B35, 42C40; Secondary 46E35, 47B38.

The results of Duffin and Schaeffer stimulated several directions of research in numerical analysis, sampling theory, and nonharmonic analysis. Their research had an impact that led to the study of "wavelet frames". The wavelet frames in $\mathbb{R}^n$ is a collection of functions $\{\varphi_{j,k}(x) = 2^{nj/2}\varphi(2^j x - k)\}$ with the following property: there exist constants $B \geq C > 0$ such that for any $f \in L^2(\mathbb{R}^n)$, we have

$$C\|f\|_{L^2}^2 \leq \sum_{j,k} |\langle f, \varphi_{j,k}\rangle|^2 \leq B\|f\|_{L^2}^2.$$

Furthermore, if $\{\varphi_{j,k}(x) = 2^{nj/2}\varphi(2^j x - k)\}$ is orthonormal, then it is a "wavelet basis" of $L^2(\mathbb{R}^n)$. We refer to [9] for a complete discussion about wavelets and frames.

There are three main results in this article. The first one, Theorem 4.1, is the existence of frames on the Anisotropic Triebel-Lizorkin spaces that introduced in [1] and [11]. The second one, Theorem 4.2, is the study of the smoothness of the dual frames, and the last one, Theorem 4.3, is the a.e. and L^p convergence of the truncated frame expansions. Theorem 4.1 and Theorem 4.2 are the anisotropic version (That is, the dilation is an "expansive matrix" defined below) of the results in [6] and [8]. Theorem 4.3 is the anisotropic version of the results obtained in the paper [13].

This article is organized as follow. Section 2 contains some background materials about anisotropic function spaces. We introduce the notion of strong molecules in Section 3. The main theorems are presented in Section 4. Section 5 to Section 7 are the proofs for the main theorems. This is part of the author's Ph.D. Dissertation and I would like to thank my Ph.D. supervisor, Guido Weiss, for his patience and teaching.

2. Background materials

This article is based on [1], therefore, we start by some background materials about anisotropic function spaces. Moreover, with respect to the results of [1], we only interest in the unweighted version of $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \omega dx)$, that is, we take $\omega \equiv 1$.

A real $n \times n$ matrix A is an *expansive matrix*, if $\min_{\lambda\in\sigma(A)} |\lambda|>1$, where $\sigma(A)$ is the set of all eigenvalues (the spectrum) of A.

A basic notion in our study is a quasi-norm ρ_A associated with A, which induces a quasi-distance making $\mathbb{R}^n$ a space of homogeneous type, see Coifman and Weiss [2].

DEFINITION 2.1. A quasi-norm associated with an expansive matrix A is defined by

$$\rho_A(x) = \sum_{k=-\infty}^{\infty} |\det A|^k \chi_{O_k}(x) \tag{2.1}$$

where $O_k = A^k(B(0,1)) \setminus \bigcup_{i=-\infty}^{k-1} A^i(B(0,1))$, and $B = B(0,1) = \{\xi : |\xi| \leq 1\}$ is the unit ball.

Here are some basic properties of ρ_A, the proof for these properties can be found in [1], [11]:

$$\begin{aligned} &\rho_A(x) > 0, && \text{for } x \neq 0,\\ &\rho_A(Ax) = |\det A|\rho_A(x) && \text{for } x \in \mathbb{R}^n,\\ &\rho_A(x+y) \leq H(\rho_A(x) + \rho_A(y)) && \text{for } x, y \in \mathbb{R}^n, \end{aligned} \tag{2.2}$$

where $H \geq 1$ is a constant.

It is clear that ρ_A given by (2.1) satisfies (2.2) with the constant $H = |\det A|^{j_0}$, where j_0 is the smallest integer such that $\bigcup_{j\le 0} A^j(B(0,2)) \subset A^{j_0}(B(0,1))$.

Proposition 2.1

For any expansive matrix A, we have

1. *there is a constant $C > 0$ such that $C^{-1}|\det A|^k \le |O_k| \le C|\det A|^k$ for any $k \in \mathbb{Z}$,*
2. $\int_{B(0,1)} \rho_A(x)^{\epsilon-1}dx < \infty$ *and* $\int_{\mathbb{R}^n\setminus B(0,1)} \rho_A(x)^{-1-\epsilon}dx < \infty$ *for any* $\epsilon > 0$.

Lemma 2.2

Suppose A is expansive matrix, and λ_- and λ_+ are any positive real numbers such that $\lambda_- < \min_{\lambda\in\sigma(A)} |\lambda|$ and $\lambda_+ > \max_{\lambda\in\sigma(A)} |\lambda|$. Let $\tau = \frac{\ln\lambda_+}{\ln|\det A|}$, $\zeta = \frac{\ln\lambda_-}{\ln|\det A|}$. Then for any quasi-norm ρ_A there exists a constant C such that,

$$C^{-1}\rho_A(x)^{\zeta} \le |x| \le C\rho_A(x)^{\tau} \quad if \quad \rho_A(x) \ge 1 \tag{2.3}$$

and

$$C^{-1}\rho_A(x)^{\tau} \le |x| \le C\rho_A(x)^{\zeta} \quad if \quad \rho_A(x) \le 1. \tag{2.4}$$

Furthermore, if A is diagonalizable over $\mathbb{C}$, we may take $\lambda_- = \min_{\lambda\in\sigma(A)} |\lambda|$ and $\lambda_+ = \max_{\lambda\in\sigma(A)} |\lambda|$.

2.1 Definition of $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$

For any $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{j,k} = A^{-j}([0,1]^n + k)$ be the dilated cube, and $x_{Q_{j,k}} = A^{-j}(k)$ be its “lower-left corner”. Let

$$\mathcal{Q} = \{Q_{j,k} | j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$$

be the collection of all dilated cubes. Define

$$\varphi_j(x) = |\det A|^j \varphi(A^j x) \qquad \text{for } j \in \mathbb{Z},$$

$$\varphi_{j,k}(x) = \varphi_Q(x) = |\det A|^{j/2}\varphi(A^j x - k) = |Q|^{1/2}\varphi_j(x - x_Q) \qquad \text{for } Q = Q_{j,k} \in \mathcal{Q}.$$

The definition for the anisotropic Triebel-Lizorkin spaces is based on the “Littlewood-Paley function” as used in the definition of classical Triebel-Lizorkin spaces, see [6].

Definition 2.2. For $\alpha \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$, the *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$ is the collection of all $f \in \mathcal{S}'/\mathcal{P}$ ($\mathcal{P}$ is the class of polynomials) such that,

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,dx)} = \left\| \Big(\sum_{j\in\mathbb{Z}} \big(|\det A|^{j\alpha} |f * \varphi_j| \big)^q \Big)^{1/q} \right\|_{L^p(dx)} < \infty,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.5) and (2.6),

$$\operatorname{supp}\hat{\varphi} \subset [-\pi,\pi]^{\mathrm{n}} \setminus \{0\}, \tag{2.5}$$

$$\sup_{j\in\mathbb{Z}} |\hat{\varphi}((A^*)^j\xi)| > 0 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.6}$$

In [1], we show that this definition is independent of φ.

The sequence space, $\dot{\mathbf{f}}_p^{\alpha,q}(A,dx)$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}(A,dx)} = \left\| \Big(\sum_{Q\in\mathcal{Q}} (|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q)^q \Big)^{1/q} \right\|_{L^p(dx)} < \infty,$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of the cube Q.

2.2 $\phi - \psi$ transform

The $\phi - \psi$ transform is a basic tool in [6]. They use it to develop the atomic and molecular decompositions of the classical Triebel-Lizorkin spaces. In [1], we follow that idea and obtain the corresponding results for the anisotropic Triebel-Lizorkin spaces. Their basic definition and result is given below.

Suppose that φ, ψ are test functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\hat{\varphi}, \operatorname{supp}\hat{\psi} \subset [-\pi,\pi]^{\mathrm{n}} \setminus \{0\} \tag{2.7}$$

$$\sum_{j\in\mathbb{Z}} \overline{\hat{\varphi}\left((A^*)^j\xi\right)}\hat{\psi}\left((A^*)^j\xi\right) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.8}$$

where A^* is the adjoint (transpose) of A, and the Fourier transform of f is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi\rangle}dx.$$

Definition 2.3. The φ-transform S_φ is the map taking each $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ to the sequence $S_\varphi f = \{(S_\varphi f)_Q\}_{Q\in\mathcal{Q}}$ defined by $(S_\varphi f)_Q = \langle f, \varphi_Q\rangle$. (This is well-defined, since $\int x^\gamma \varphi_Q(x)dx = 0$ for any multi-index γ.) The inverse φ-transform, T_ψ, is the map taking the sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$ to $T_\psi s = \sum_{Q\in\mathcal{Q}} s_Q\psi_Q$.

Theorem 2.3

Suppose $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ are such that $\operatorname{supp}\hat{\varphi}$, $\operatorname{supp}\hat{\psi}$ are compact and bounded away from the origin. Then the operators $S_\varphi : \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx) \to \dot{\mathbf{f}}_p^{\alpha,q}(A,dx)$ and $T_\psi : \dot{\mathbf{f}}_p^{\alpha,q}(A,dx) \to \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$ are bounded. In addition, if φ, ψ satisfy (2.7), (2.8) then $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$ and

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle\psi_Q, \qquad \text{for any } f \in \mathcal{S}'/\mathcal{P}, \tag{2.9}$$

where the convergence of the above series, as well as the equality, is in $\mathcal{S}'/\mathcal{P}$.

3. Strong molecules

3.1 Some basis facts about the quasi-norm ρ_A

Let A be an expansive matrix, $\{\lambda_i\}_{i=1}^n$ (allowing multiplicities) be the set of eigenvalues of A which order as $|\lambda_j| \le |\lambda_i|$, if $j \le i$ and $e_i = (e_{1i}, e_{2i}, \dots, e_{ni})$ be the generalized eigenvectors associated with λ_i. That is, the matrix representation of A in term of the basis $\{e_i\}_{i=1}^n$ is its Jordan canonical form; in this case, for some $r \in \mathbb{N}$,

$$\mathbf{A} = \begin{pmatrix} \mathbf{J}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{J}_2 & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{J}_r \end{pmatrix}$$

where, if the eigenvalue λ_i is real, $\mathbf{J}_i$ is a $k_i \times k_i$ matrix and each $\mathbf{J}_i$, $1 \le i \le r$, is either a $k_i \times k_i$ diagonal matrix,

$$\mathbf{J}_i = \begin{pmatrix} \lambda_i & 0 & \cdots & 0 \\ 0 & \lambda_i & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_i \end{pmatrix}$$

or a $k_i \times k_i$ Jordan block,

$$\mathbf{J}_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}.$$

If λ_i is complex with $\mathrm{R}e(\lambda_i) = c_i$ and $\mathrm{I}m(\lambda_i) = d_i \ne 0$, then $\mathbf{J}_i$, $1 \le i \le r$, is either

$$\mathbf{J}_i = \begin{pmatrix} \mathbf{D}_i & 0 & \cdots & 0 \\ 0 & \mathbf{D}_i & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{D}_i \end{pmatrix} \quad \text{where} \quad \mathbf{D}_i = \begin{pmatrix} c_i & d_i \\ -d_i & c_i \end{pmatrix} \tag{3.1}$$

or

$$\mathbf{J}_i = \begin{pmatrix} \mathbf{D}_i & \mathbf{I}_{2\times 2} & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{D}_i & \mathbf{I}_{2\times 2} & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{D}_i & \cdots & 0 & 0 \\ \multicolumn{6}{c}{\dotfill} \\ 0 & 0 & 0 & \cdots & \mathbf{D}_i & \mathbf{I}_{2\times 2} \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{D}_i \end{pmatrix}.$$

For a reference of this decomposition, see [10, p. 129]. (In [10], the matrix is represented in lower triangular form, it is easy to obtain the upper triangular representation by taking the transpose.)

For $m \in \mathbb{Z}^n$, we may define $\mathbf{A}^m$ as,

$$\mathbf{A}^m = \begin{pmatrix} \mathbf{J}_1^m & 0 & \cdots & 0 \\ 0 & \mathbf{J}_2^m & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{J}_r^m \end{pmatrix}$$

where $\mathbf{J}_i^m$ is

$$\mathbf{J}_i{}^m = \begin{pmatrix} \lambda_i^m & 0 & \cdots & 0 \\ 0 & \lambda_i^m & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_i^m \end{pmatrix} \quad \text{or} \quad \mathbf{J}_i{}^m = \begin{pmatrix} \mathbf{D}_i^m & 0 & \cdots & 0 \\ 0 & \mathbf{D}_i^m & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathbf{D}_i^m \end{pmatrix}$$

if $\mathbf{J}_i$ is a diagonal matrix or matrices of the form (3.1). If $\mathbf{J}_i$ is a $k_i \times k_i$ Jordan block corresponding to a real eigenvalue λ_i, we have

$$\mathbf{J}_i{}^m = \begin{pmatrix} \frac{\lambda_i^m}{0!} & \frac{m\lambda_i^{m-1}}{1!} & \frac{m(m-1)\lambda_i^{m-2}}{2!} & \cdots & \frac{m(m-1)\ldots(m-k_i+2)\lambda_i^{m-k_i+1}}{(k_i-1)!} \\ 0 & \frac{\lambda_i^m}{0!} & \frac{m\lambda_i^{m-1}}{1!} & \cdots & \frac{m(m-1)\ldots(m-k_i+3)\lambda_i^{m-k_i+2}}{(k_i-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{m\lambda_i^{m-1}}{0!} \\ 0 & 0 & 0 & \cdots & \frac{\lambda_i^m}{0!} \end{pmatrix} \tag{3.2}$$

or, in the complex case,

$$\mathbf{J}_i{}^m = \begin{pmatrix} \frac{\mathbf{D}_i^m}{0!} & \frac{m\mathbf{D}_i^{m-1}}{1!} & \frac{m(m-1)\mathbf{D}_i^{m-2}}{2!} & \cdots & \frac{m(m-1)\ldots(m-k_i+2)\mathbf{D}_i^{m-k_i+1}}{(k_i-1)!} \\ 0 & \frac{\mathbf{D}_i^m}{0!} & \frac{m\mathbf{D}_i^{m-1}}{1!} & \cdots & \frac{m(m-1)\ldots(m-k_i+3)\mathbf{D}_i^{m-k_i+2}}{(k_i-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{m\mathbf{D}_i^{m-1}}{0!} \\ 0 & 0 & 0 & \cdots & \frac{\mathbf{D}_i^m}{0!} \end{pmatrix}. \tag{3.3}$$

Let $a_i = \frac{\ln|\lambda_i|}{\ln|\det A|}$, $a = (a_1, \ldots, a_n)$, notice that $\sum_{i=1}^n a_i = 1$. Let the differential operators ∂_i, $1 \le i \le n$, be defined by

$$\partial_j f(x) = \sum_{i=1}^n e_{ij} \frac{\partial f}{\partial x_i}(x)$$

and $\partial^\gamma = \prod_{i=1}^n \partial_i^{\gamma_i}$ where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$. For any $f \in C^1$ and $m \in \mathbb{Z}$, let $f_{A^m}(x) = f(A^m x)$. We have

$$\begin{aligned}
\partial_j(f_{A^m}(x)) &= \sum_{i=1}^n e_{ij} \frac{\partial f_{A^m}(x)}{\partial x_i} = \sum_{i=1}^n e_{ij} \sum_{k=1}^n a_{ki}^m \frac{\partial f}{\partial x_k}(A^m x) \\
&= \sum_{k=1}^n \left(\sum_{i=1}^n a_{ki}^m e_{ij} \right) \frac{\partial f}{\partial x_k}(A^m x)
\end{aligned}$$

where $\{a_{ki}^m\}_{1\le k,i\le n}$ is the matrix representation of A^m with respect to the basis $\{e_i\}_{i=1}^n$. Therefore, for any $\triangle > 0$, there exists a $C_\triangle > 0$ such that

$$\begin{aligned}
|\partial_j(f_{A^m})(x)| &\le C(|\lambda_j^m| + |m\lambda_j^{m-1}| + \cdots + |m(m-1)\cdots(m-s)\lambda^{s+1}|)\|\nabla f\|_{L^\infty} \quad (3.4) \\
&\le C_\triangle |\lambda_j|^{(m+|m|\triangle)}
\end{aligned}$$

for some $s > 0$. If λ_j is real, the estimates (3.4) is an easy consequence of the representation (3.2). If λ_j is a complex number, the estimates follows from representation (3.3) and

$$\frac{1}{2}|z| \le \max(\mathrm{Re}(z), \mathrm{Im}(z)) \le |z| \quad \text{for any complex number } z \in \mathbb{C}.$$

Furthermore, for any $\gamma \in \mathbb{N}^n$ and any $\triangle > 0$, there exists a $C_{\triangle,\gamma}$ such that

$$|\partial^\gamma(f_{A^m}(x))| \le C_{\triangle,\gamma} \prod_{i=1}^n |\det A|^{a_i\gamma_i(m+|m|\triangle)} = C_{\triangle,\gamma} |\det A|^{\langle a,\gamma\rangle(m+|m|\triangle)}. \qquad (3.5)$$

From now on, we represent any $x \in \mathbb{R}^n$ by the basis $\{e_i\}_{1<i<n}$. That is, if $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$ we write $x = (x_1, x_2, \ldots, x_n)$ and $x^\beta = \prod_{i=1}^n x_i^{\beta_i}$ for any $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n$.

We have $\rho_A(x) = |\det A|^k$ for some $k \in \mathbb{Z}$, that is, there exist x_o with $A^k x_o = x$ and $|x_o| \le 1$. (k is determined by the condition $x \in O_k$). Let $E \subset \mathbb{R}^n$ be the reduced subspace corresponding to the Jordan block with eigenvalue λ. For any $x \in \mathbb{R}$, let $x_E \in E$ be the orthogonal projection of x onto E. We are going to show that, for any $\triangle > 0$, there exists a $C_\triangle$, such that

$$|x_E| \le C_\triangle |\lambda|^{k+|k|\triangle} \quad \text{for any} \quad x \in \mathbb{R}^n.$$

By Lemma 2.2, we have the following inequality for any matrices,

$$|(A^k x_o)_E| = |A_E^k (x_o)_E| \le C_\triangle |\lambda|^{k+|k|\triangle} |(x_o)_E| \le C_\triangle |\lambda|^{k+|k|\triangle} |x_o|$$

where A_E is the restriction of A to E. We have,

$$|(A^k x_o)_E| \le C_\triangle |\lambda|^{k+|k|\triangle}$$

because E is a reduced subspace of A and $|x_o|$ is bounded by one. Therefore,

$$|x_E| \le C_\triangle |\lambda|^{k+|k|\triangle}.$$

Furthermore, we obtain

$$|x_i| \le C_\triangle |\lambda_i|^{k+|k|\triangle} = C_\triangle |\det A|^{ka_i+|k|a_i\triangle} = \begin{cases} C_\triangle \rho_A(x)^{a_i+a_i\triangle} & \rho_A(x) \ge 1 \\ C_\triangle \rho_A(x)^{a_i-a_i\triangle} & \rho_A(x) < 1 \end{cases} \qquad (3.6)$$

for any $1 < i < n$.

The extra $\triangle$ in (3.6) and (3.5) shows the difficulty in the analysis involving non-diagonalizable expansive matrix dilations on $\mathbb{R}^n$.

3.2 Definition of strong molecules and strongly almost diagonal matrix

The basic notion of "molecules" we shall use is given in the following definition. That definition is motivated by the type I and type II molecules used in [6] for the isotropic case and [1] in the anisotropic case. Notice that if K and M are large enough, the "strong molecules" defined below is an anisotropic type I and type II molecules. Therefore, all results about type I and type II molecules in [1] are still valid for strong molecules.

Definition 3.1. Let A be an expansive matrix. Let $K, M > 0$ with $M > 1 + K$. For each $Q = Q_{j,k} \in \mathcal{B}$, we say that $\{h_Q\}_{Q=Q_{j,k}}$ is a *strong molecule of order* K, M, if it satisfies

$$\int x^\gamma h_Q(x)dx = 0 \quad \text{if} \quad \langle a, \gamma\rangle \leq K, \tag{3.7}$$

and, for $\langle a, \gamma\rangle \leq K$,

$$\left|\partial^\gamma (h_Q(A^{-j}x))\right| \leq C|\det A|^{j/2} \frac{1}{(1 + \rho_A(x - A^j x_Q))^M} \tag{3.8}$$

where C is a positive constant.

The class of these molecules is denoted by $\mathcal{M}_{K,M}$. Let $\|\cdot\|_{\mathcal{M}_{K,M}}$ be the infimum of the C's in (3.8). It is easy to see that $\mathcal{M}_{K,M}$ with the norm $\|\cdot\|_{\mathcal{M}_{K,M}}$ is a Banach space.

The "almost diagonal" operators associated with strong molecules satisfy a stronger inequality which is crucial for further estimations.

Definition 3.2. Let $K, M > 0$, we say that the matrix $\{a_{QP}\}_{QP}$ is a *strongly almost diagonal matrix, or strongly almost diagonal operator*, of order K, M, if

$$\sup_{QP} |a_{QP}| / s\kappa_{QP}(K, M) < C \tag{3.9}$$

for some constant C, where

$$s\kappa_{QP}(K, M) = \min\left(\left(\frac{|Q|}{|P|}\right)^{K+1/2}, \left(\frac{|P|}{|Q|}\right)^{K+1/2}\right)\left(1 + \frac{\rho_A(x_Q - x_P)}{\max(|P|, |Q|)}\right)^{-M}.$$

The class of these operators is denoted by $s\kappa(K, M)$. Let the norm $\|\cdot\|_{s\kappa(K,M)}$ be the infimum of the constants C in (3.9).

Remark 3.1. Notice that P, Q in Definition 3.2 are symmetric. The definition of the strongly almost diagonal operators is motivated by the "symmetrization" of the almost diagonal operators in [1], [6]. The matrix $\{a_{PQ}\}$ can be thought of as an operator acting a sequence space $\dot{\mathbf{f}}_p^{\alpha,q}(A, dx)$, see [6] for the isotropic case and [1] for the anisotropic case. It is easy to see that if

$$K \geq \max\left(\alpha + \epsilon, \frac{1}{\min(1, p, q)} - 1 - \alpha + \epsilon\right) \quad \text{and} \quad M > \frac{1}{\min(1, p, q)} + \epsilon$$

for some $\epsilon > 0$, then a strongly almost diagonal operator is an almost diagonal operator. Therefore, it is bounded on $\dot{\mathbf{f}}_p^{\alpha,q}(A, dx)$.

Theorem 3.1

Let $K, M > 0$. If $\{\Phi_Q\}$ and $\{\Psi_Q\}$ are type III strong molecules of order K', M' with $K' > K+a_n$ and $M' > \max(M, K'+a_n+1)$, then the matrix $\{a_{QP}\}_{QP} = \langle \Phi_Q, \Psi_P\rangle$ is strongly almost diagonal of order K, M with

$$\|\langle \Phi_Q, \Psi_P\rangle_{QP}\|_{s\kappa(K,M)} \le C\|\{\Phi_Q\}_Q\|_{\mathcal{M}_{K',M'}}\|\{\Psi_Q\}_Q\|_{\mathcal{M}_{K',M'}}$$

where C is an constant depending only on the matrix A.

Proof. Without loss of generality, we may assume

$$\|\{\Phi_Q\}_Q\|_{\mathcal{M}_{K',M'}} = \|\{\Psi_Q\}_Q\|_{\mathcal{M}_{K',M'}} = 1.$$

Let $\delta > 0$ satisfy $\delta < \min(K' - K - a_n, M' - 1 - K' - a_n)$ and $\triangle < \delta$. If $|Q| = |\det A|^{-\beta} \le |\det A|^{-\lambda} = |P|$, by Lemma 8.2 in Appendix with $R = M'$, $i = \beta$, $j = \lambda$, $x_0 = x_Q$,

$$g(x) = \overline{\Phi_P(x_P - x)} \quad \text{and} \quad h(x) = \Psi_Q(x),$$

we have,

$$R = M' > K' + \delta + 1 + a_n, \quad K' > K + a_n + \delta.$$

Hence,

$$\begin{aligned}|\langle \Phi_P, \Psi_Q\rangle| &= |\Psi * \Phi(A^\lambda x_P)|\\ &\le C|\det A|^{-(\beta-\lambda)(K+\delta-\triangle+1/2)}\big(1 + |\det A|^\lambda \rho_A(x_P - x_Q)\big)^{-M'}\\ &\le C|\det A|^{-(\beta-\lambda)(K+1/2)}\big(1 + |\det A|^\lambda \rho_A(x_P - x_Q)\big)^{-M}.\end{aligned}$$

We interchange the role of h_P and k_Q, if $|Q| = |\det A|^{-\beta} \ge |\det A|^{-\lambda} = |P|$, and we have

$$|\langle \Phi_P, \Psi_Q\rangle| \le C|\det A|^{-(\lambda-\beta)(K+1/2)}\big(1 + |\det A|^\beta \rho_A(x_P - x_Q)\big)^{-M}.$$

Combining these two inequalities, we have

$$|\langle \Phi_P, \Psi_Q\rangle| \le Cs\kappa_{QP}(K, M). \ \Box$$

3.3 Molecular decomposition of $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$

Let $J = \frac{1}{\min(1,p,q)}$ and $N = \max(J - 1 - \alpha, -1)$.

Theorem 3.2 (Smooth molecular decomposition).

Suppose A is an expansive matrix and $\delta > 0$. There exists a constant $C > 0$, such that, if $f = \sum_{Q\in\mathcal{Q}} s_Q\Phi_Q$, where $\{\Phi_Q\}_Q$ is a family of strong molecules of order K', M' with $K' > \max(\alpha + a_n + \delta, N + \delta)$ and $M' > \max(J + \delta, \alpha + a_n + \delta + 1)$, then

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,dx)} \le C\,\|\{s_Q\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha,q}(A,dx)} \qquad \text{for all } \{s_Q\}_Q \in \dot{\mathbf{f}}_p^{\alpha,q}(A,dx).$$

Proof. In [6] (that is, the isotropic case), the proof is made into two steps. The first part is an estimate on the inner product of molecules, [6] Corollary B.3, and the second step is a simple application of step 1, [6] Theorem 3.5. We already obtained the estimate on the inner product of strong molecules on Theorem 3.1.

By Theorem 2.3, we may write

$$\Phi_P = \sum_Q \langle \Phi_P, \varphi_Q \rangle \psi_Q.$$

If $\mathcal{A}$ is the operator on $\dot{\mathbf{f}}_p^{\alpha,q}(A, dx)$ with matrix $\{a_{QP}\}_{Q,P} = \{\langle \Phi_P, \varphi_Q \rangle\}_{Q,P}$, then $\{a_{QP}\}_{Q,P}$ is a strongly almost diagonal matrix of order $\tilde{K}, \tilde{M}$ with $\tilde{K} = \max(\alpha + \delta, N + \delta)$ and $\tilde{M} = J + \delta$.

Returning to the estimate, we have

$$\mathrm{T}_\psi \mathcal{A} s = \sum_Q \sum_P a_{QP} s_P \psi_Q = \sum_P s_P \sum_Q \langle \Phi_P, \varphi_Q \rangle \psi_Q = \sum_P s_P \Phi_P = f$$

and

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)} = \|\mathrm{T}_\psi \mathcal{A} s\|_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)} \leq C \|\mathcal{A} s\|_{\dot{\mathbf{f}}_p^{\alpha,q}(A, dx)} \leq C \|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}(A, dx)}.$$

Since T_ψ is bounded and $\mathcal{A}$ is a strongly almost diagonal and, hence, bounded by Remark 3.1. □

4. Main theorems

In what follows, we are going to study frames associated with expansive matrices and obtain some results that generalize the work of Frazier and Jawerth [6] and Glibert, Han, Hogan, Lakey, Weiland and Weiss [8] of wavelet-type frames associated with diagonal matrices having identical eigenvalues. In [8], they constructed the wavelet-type frame by showing that $I - \mathcal{F}$ ($\mathcal{F}$ is the frame operator that will be defined right the way) is a Calderón-Zygmund operator having operator norm less than 1. Moreover, they showed that a class of "molecules" is invariant under the mapping by $I - \mathcal{F}$. Hence, it makes sense to study the inverse of $I - \mathcal{F}$ by using the Neumann series. Instead of using Calderón-Zygmund operator, we obtain the Anisotropic wavelet-type frame by a discrete method. Let us first introduce the basic operator of this section, the frame operator:

DEFINITION 4.1. We say that $\{\phi_\gamma\}_{\gamma \in \Gamma} \in \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx) \bigcap (\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx))^*$ is a *frame for the Anisotropic Triebel-Lizorkin Spaces*, $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$, if the frame operator

$$\mathcal{F} : f \to \sum_{\gamma \in \Gamma} \langle f, \phi_\gamma \rangle \phi_\gamma$$

is bounded and invertible on $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$.

For a given φ satisfying the modified Calderón reproducing formula, there are several ways for constructing a frame associated with it. In [8] one can find a method for such construction which relies heavily on the Calderón-Zygmund operators. This is not easily applied in the anisotropic case (see the estimates in Section 2 of [11]). There is another idea, however, arising from the φ, ψ transform theory that can be found on p. 63–69 of Frazier and Jawerth's paper [6] which does not rely on the Calderón-Zygmund operators.

Theorem 4.1

Let A be an expansive matrix, $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and $B \in M_{n\times n}(\mathbb{R})$ with $|\det B| \neq 0$. There exist $K, M > 0$ such that for any function φ satisfies the modified Calderón reproducing formula,

$$\sum_{j=-\infty}^{\infty} \hat{\varphi}\left((A^*)^j \xi\right) \hat{\tilde{\varphi}}\left((A^*)^j \xi\right) = 1 \quad \textit{for} \quad \xi \neq 0;$$

$$\int x^{\gamma} \varphi(x) dx = 0 \tag{4.1}$$

if $\langle a, \gamma\rangle \leq K$, and

$$|\partial^{\gamma} \varphi(x)| \leq \frac{1}{(1 + \rho_A(x))^M} \tag{4.2}$$

if $\langle a, \gamma\rangle \leq K$, there exists an η_0 such that, for any fixed $\eta < \eta_0$,

$$\left\{\varphi_{j,k}(x) = |\det A|^{j/2} |\det A|^{\eta} \varphi\left(A^{\eta} A^j x - A^{\eta} B k\right)\right\}_{j\in\mathbb{Z}, k\in\mathbb{Z}^n}$$

is a frame on $\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, dx)$.

Remark 4.1. In fact, we may take $M > \max(J, J - \alpha + a_n)$ and $K > \max(\alpha + a_n, J - 1 - \alpha + a_n)$. That estimates of K and M can be obtained by a detailed investigation of Theorem 3.2. We state the theorem in this way in order to avoid checking the indices.

Theorem 4.1 shows the invertibility of the frame operator, $\mathcal{F}$. In order to begin the study of the analysis of convergence of the frame expansion, we are going to study the regularity of the frame operator. By this we mean that the strong molecules are invariant under the mapping of the frame operators and their inverses. The regularity of the frame operator in the isotropic case can be found in [8].

The following theorem addresses the problem of regularity of the inverse of the frame operators.

Theorem 4.2

Suppose A is an expansive matrix and $B \in M_{n\times n}$ with $|\det B| \neq 0$. Let $K, M, \delta > 0$ satisfy $M > K + 2a_n + 1$ (This condition comes from Theorem 3.1.). Let $K' =$

$K+\delta+a_n$ and $M' = M+\delta$. Suppose that φ satisfies the modified Calderón reproducing formula,

$$\sum_{j=-\infty}^{\infty} \hat{\varphi}\left((A^*)^j\xi\right)\hat{\tilde{\varphi}}\left((A^*)^j\xi\right) = 1 \quad \text{for} \quad \xi \neq 0;$$

$$\int x^\gamma \varphi(x)dx = 0 \tag{4.3}$$

if $\langle a, \gamma\rangle \leq K'$; and

$$|\partial^\gamma \varphi(x)| \leq \frac{1}{(1+\rho_A(x))^{M'}}, \tag{4.4}$$

if $\langle a, \gamma\rangle \leq K' + a_n$.

Then, for any $L, \tilde{M}$ satisfying $1 + K > \tilde{M}$, $K > L$, $M > \tilde{M}$ (These conditions arise from Theorem 6.4) and $2K + a_n + 1 > M$ (This condition comes from Corollary 6.3), $I - |\det B|\mathcal{F}_\eta$ maps a strong molecule of order K', M' into a strong molecule of order $L, \tilde{M}$. Furthermore, there is an $\eta_0 \in \mathbb{R}$ such that the dual frame of $\{\varphi_Q\}$, $\{\mathcal{F}_\eta^{-1}\varphi_Q\}$, consist of a family of strong molecules of order $L, \tilde{M}$.

The last result is the almost everywhere convergence of the wavelet-type frame. Notice that it also includes an important case, the convergence of multiwavelet expansions. The main idea comes from the paper [13] by Kelly, Kon and Raphael. Let φ be a function satisfying the condition stated in Theorem 4.2. We have the frame expansion

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle \phi_{j,k} \tag{4.5}$$

where $\phi_{j,k} = \mathcal{F}^{-1}(\varphi_{j,k})$, and $\mathcal{F}$ is the frame operator. Notice that the $\phi_{j,k}$'s are not necessarily translations and dilations of a single function. On the other hand, we have

$$\phi_{j,k}(x) = |\det A|^{j/2}\phi_{0,k}(A^j x).$$

This follows from the fact that the frame operator

$$\mathcal{F}f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle \varphi_{j,k}(x),$$

commutes with dilations,

$$\mathcal{D}_A f(x) = |\det A| f(Ax)$$

because,

$$\begin{aligned}
\mathcal{D}_A\mathcal{F}f(x) &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle |\det A|^{j/2+1}\varphi(A^j(Ax) - Bk) \\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle |\det A|^{(j+2)/2}\varphi(A^{j+1}x - Bk) \\
&= \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{l-1,k}\rangle |\det A|^{(l+1)/2}\varphi(A^l x - Bk)
\end{aligned}$$

by the change of index $l = j + 1$.

Since

$$\begin{aligned}\langle f, \varphi_{l-1,k} \rangle &= \int_{\mathbb{R}^n} f(y)\overline{|\det A|^{(l-1)/2}\varphi(A^{l-1}y-k)}dy \\ &= \int_{\mathbb{R}^n} f(Ay)\overline{|\det A|^{(l+1)/2}\varphi(A^{l}y-k)}dy\end{aligned}$$

by the change of variable $y \to Ay$, we have,

$$\mathcal{D}_A \mathcal{F} f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \langle \mathcal{D}_A f, \varphi_{j,k}\rangle \varphi_{j,k}(x) = \mathcal{F}\mathcal{D}_A f(x).$$

Therefore,

$$\begin{aligned}\phi_{j,k}(x) &= \mathcal{F}^{-1}(\varphi_{j,k})(x) = \mathcal{F}^{-1}(|\det A|^{-j/2}\mathcal{D}_{A^j}\varphi_{0,k})(x) \\ &= |\det A|^{-j/2}\mathcal{F}^{-1}(\mathcal{D}_{A^j}\varphi_{0,k})(x) \\ &= |\det A|^{-j/2}\mathcal{D}_{A^j}(\mathcal{F}^{-1}\varphi_{0,k})(x) = |\det A|^{-j/2}\mathcal{D}_{A^j}(\phi_{0,k})(x) \\ &= |\det A|^{-j/2}|\det A|^{j}(\phi_{0,k})(A^j x) = |\det A|^{j/2}(\phi_{0,k})(A^j x).\end{aligned}$$

Consider the operator:

$$\mathcal{T}_N f = \sum_{j\le N}\sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle \phi_{j,k}.$$

We then have what we consider to be the main result of this paper:

Theorem 4.3

Suppose that $\{\varphi_{j,k}\}$ is a frame and $\{\phi_{j,k}\}$ is its dual-frame. Then, for any $f \in L^p$, $1 < p < \infty$, $\mathcal{T}_N f(x)$ converges to $f(x)$ for every x in the Lebesgue set of f, in the sense of the space $\mathbb{R}^n$ endowed with ρ_A and Lebesgue measure.

5. Frames on $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$

First of all, we claim that there exists a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ that satisfies the Calderón reproducing formula associated with the expansive matrix A.

Theorem 5.1

For any expansive matrix, A, there is a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\sum_{k=-\infty}^{\infty} \left|\hat{\psi}\left((A^*)^k\xi\right)\right|^2 = 1 \quad \textit{for} \quad \xi \neq 0.$$

Proof. See [1] or [11]. □

Corollary 5.2

For any expansive matrix, A, there is a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\sum_{k=-\infty}^{\infty} \hat{\varphi}\left((A^*)^k\xi\right)\hat{\tilde{\varphi}}\left((A^*)^k\xi\right) = 1 \quad \textit{for} \quad \xi \neq 0 \tag{5.1}$$

where $\tilde{\varphi}(x) = \overline{\varphi(-x)}$.

Proof. See [1] or [11]. □

Proof of Theorem 4.1 Let $B_{j,k} = A^{-j}(B([0,1]^n) + Bk)$, $\mathcal{B} = \{B_{j,k}\}_{j,k}$ and

$$\varphi_{j,k}(x) = \varphi_{B_{j,k}}(x) = |\det A|^{j/2}\varphi(A^j x - Bk) \qquad \varphi_\eta(x) = |\det A|^\eta \varphi(A^\eta x).$$

Let the operator $\mathcal{F}_\eta$ on $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$ be the frame operator,

$$\begin{aligned}\mathcal{F}_\eta f(x) &= \sum_{B_{j,k}} \langle f, (\varphi_\eta)_{j,k}\rangle (\varphi_\eta)_{j,k} \\ &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\det A|^{-j} \left[\int_{\mathbb{R}^n} f(x) |\det A|^{j+\eta} \varphi(A^{j+\eta}x - A^\eta Bk) dx \right] \\ &\quad \times |\det A|^{j+\eta}\varphi(A^{j+\eta}x - A^\eta Bk).\end{aligned}$$

Boundedness of the frame operator:

First of all, we are going to prove the boundedness of $\mathcal{F}_\eta$. Let

$$\mathrm{S}_{\varphi_{\eta,B}} : \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx) \to \dot{\mathbf{f}}_p^{\alpha,q}(A, dx) \quad \text{and} \quad \mathrm{T}_{\varphi_{\eta,B}} : \dot{\mathbf{f}}_p^{\alpha,q}(A, dx) \to \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, dx)$$

be defined by

$$(S_{\varphi_{\eta,B}} f)_{B_{j,k}} = \langle f, (\varphi_\eta)_{j,k}\rangle \quad \text{and} \quad T_{\varphi_{\eta,B}} s = \sum_{j,k} s_{B_{j,k}} (\varphi_\eta)_{j,k}.$$

By an easy modification of Theorem 2.3, these operators $\mathrm{S}_{\varphi,B}$, $\mathrm{T}_{\varphi,B}$ are bounded. Since

$$\mathcal{F}_\eta f(x) = \mathrm{T}_{\varphi_{\eta,B}} \circ \mathrm{S}_{\varphi_{\eta,B}} f(x),$$

we have the boundedness of the frame operator $\mathcal{F}_\eta$.

A representation of $I - |\det B|\mathcal{F}_\eta$:

Here is the first step toward the invertibility of the frame operator. We are going to represent the operator $I - |\det B|\mathcal{F}_\eta$ by a series of strong molecules. Let

$$\mathbb{Z}_k^n(j,\eta) = \{l \in \mathbb{Z}^n : B_{j-\eta,l} \cap B_{j,k} \neq \emptyset\}.$$

By the modified Calderón reproducing formula (5.1),

$$f(x) = \sum_{j\in\mathbb{Z}} \varphi_j * \tilde{\varphi}_j * f(x)$$

and a change of variable $j \to j+\eta$, we have, for an η to be determined later,

$$\begin{aligned}
f(x) &= \sum_{j\in\mathbb{Z}} \varphi_j * \tilde{\varphi}_j * f(x) = \sum_{j\in\mathbb{Z}} \varphi_{j+\eta} * \tilde{\varphi}_{j+\eta} * f(x) \\
&= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \int_{B_{j+\eta,k}} (\tilde{\varphi}_{j+\eta} * f(y))\varphi_{j+\eta}(x-y)dy \\
&= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(j+\eta,\eta)} \int_{B_{j,l}\cap B_{j+\eta,k}} (\tilde{\varphi}_{j+\eta} * f(y))[\varphi_{j+\eta}(x-y) - \varphi_{j+\eta}(x-A^{-j}Bl)]dy \\
&\quad + \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(j+\eta,\eta)} \int_{B_{j,l}\cap B_{j+\eta,k}} (\tilde{\varphi}_{j+\eta} * f(y))\varphi_{j+\eta}(x-A^{-j}Bl)dy
\end{aligned}$$

and, hence,

$$\begin{aligned}
f(x) &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(j+\eta,\eta)} \int_{B_{j,l}\cap B_{j+\eta,k}} (\tilde{\varphi}_{j+\eta} * f(y))[\varphi_{j+\eta}(x-y) - \varphi_{j+\eta}(x-A^{-j}Bl)]dy \\
&\quad + \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(j+\eta,\eta)} \int_{B_{j,l}\cap B_{j+\eta,k}} [\tilde{\varphi}_{j+\eta} * f(y) - \tilde{\varphi}_{j+\eta} * f(A^{-j}Bl)]\varphi_{j+\eta}(x-A^{-j}Bl)dy \\
&\quad + \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(j+\eta,\eta)} \int_{B_{j,l}\cap B_{j+\eta,k}} (\tilde{\varphi}_{j+\eta} * f(A^{-j}Bl))\varphi_{j+\eta}(x-A^{-j}Bl)dy \\
&= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(j+\eta,\eta)} \int_{B_{j,l}\cap B_{j+\eta,k}} (\tilde{\varphi}_{j+\eta} * f(y))[\varphi_{j+\eta}(x-y) - \varphi_{j+\eta}(x-A^{-j}Bl)]dy \\
&\quad + \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(j+\eta,\eta)} \int_{B_{j,l}\cap B_{j+\eta,k}} [\tilde{\varphi}_{j+\eta} * f(y) - \tilde{\varphi}_{j+\eta} * f(A^{-j}Bl)]\varphi_{j+\eta}(x-A^{-j}Bl)dy \\
&\quad + |\det B|\mathcal{F}_\eta f(x).
\end{aligned}$$

Replacing the index $j+\eta$ by β and rearranging the last term, we have

$$\begin{aligned}
&(I - |\det B|\mathcal{F}_\eta)f(x) \\
&= \sum_{\beta\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} (\tilde{\varphi}_\beta * f(y))[\varphi_\beta(x-y) - \varphi_\beta(x-A^{-\beta+\eta}Bl)]dy \\
&\quad + \sum_{\beta\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}_k^n(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} [\tilde{\varphi}_\beta * f(y) - \tilde{\varphi}_\beta * f(A^{-\beta+\eta}Bl)]\varphi_\beta(x-A^{-\beta+\eta}Bl)dy \\
&= I + II.
\end{aligned}$$

Estimate on I:

We consider the first term. Let $\varepsilon < \frac{\ln|\lambda_-|}{\ln|\det A|}$,

$$s_{B_{\beta,k}} = |\det A|^{\beta/2}|\det B| \int_{B_{\beta,k}} |\tilde{\varphi}_\beta * f(y)|dy$$

and

$$m_{B_{\beta,k}}(x) = |\det A|^{-\eta\varepsilon} s_{B_{\beta,k}}^{-1} |\det B|^{-\varepsilon} \\ \times \sum_{l\in\mathbb{Z}_k^n(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} \tilde{\varphi}_\beta * f(y)[\varphi_\beta(x-y)-\varphi_\beta(x-A^{-\beta+\eta}Bl)]dy.$$

Therefore,

$$I = \sum_{\beta\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\det B|^{\varepsilon} |\det A|^{\eta\varepsilon} s_{B_{\beta,k}} m_{B_{\beta,k}}(x).$$

We are going to show that each $m_{B_{\beta,k}}(x)$ is a strong molecule.

The vanishing moment conditions of $m_{B_{\beta,k}}(x)$ is inherited from $\varphi(x)$. We only need to check the size conditions. Notice that if $y \in B_{\beta-\eta,l}$ and $\langle a,\gamma\rangle \le K' = K - a_n$, the size conditions of φ and inequalities (2.3) and (2.4) yield,

$$\begin{aligned}
&\big|\partial_x^\gamma\big(\varphi_\beta(A^{-\beta}x - y) - \varphi_\beta(A^{-\beta}x - A^{-\beta+\eta}Bl)\big)\big| \\
&\le C\sum_{i=1}^n |\det A|^\beta \big|\big((A^\beta y)_i - (A^\eta Bk)_i\big)\big| \sup_{z\in B_{\beta-\eta,l}} \big|\big(\partial_i\partial_x^\gamma\varphi\big)\big(A^\beta(A^{-\beta}x - z)\big)\big| \\
&\le C\sum_{i=1}^n |\det A|^{\beta(\varepsilon+1)} \rho_A\big(y_i - (A^{-\beta+\eta}Bk)_i\big)^\varepsilon \sup_{z\in B_{\beta,k}} \big(1+\rho_A(x - A^\beta z)\big)^{-M} \\
&\le C\sum_{i=1}^n |\det A|^{\beta(\varepsilon+1)} \big(|\det A|^{-\beta+\eta}|\det B|\big)^\varepsilon \big(1+\rho_A(x - Bk)\big)^{-M} \\
&\le C|\det A|^\beta \big(|\det A|^\eta|\det B|\big)^\varepsilon \big(1+\rho_A(x - A^\beta(A^{-\beta}Bk))\big)^{-M},
\end{aligned}$$

for an universal constant $C > 0$. Hence, for $\langle a,\gamma\rangle < K'$, we have

$$\begin{aligned}
&\big|\partial^\gamma m_{B_{\beta,k}}(A^{-\beta}x)\big| \\
&\le C\frac{1}{s_{B_{\beta,k}}} |\det A|^\beta \big(1+\rho_A(x - A^\beta x_{B_{\beta,k}})\big)^{-M} \sum_{l\in\mathbb{Z}_k^n(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} |\tilde{\varphi}_\beta * f(y)|dy \\
&\le C|\det A|^{\beta/2} \big(1+\rho_A(x - A^\beta x_{B_{\beta,k}})\big)^{-M}. \qquad (5.2)
\end{aligned}$$

Thus, $\{m_{B_{\beta,k}}\}$ is a family of strong molecules of order K', M.

Estimate on II:

For the second term, let

$$t_{B_{\beta,k}} = |\det A|^{\beta/2} \sum_{l\in\mathbb{Z}_k^n(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} |\tilde{\varphi}_\beta * f(y) - \tilde{\varphi}_\beta * f(A^{-\beta+\eta}Bl)|dy$$

and, if $t_{B_{\beta,k}} \neq 0$,

$$n_{B_{\beta,k}}(x) = t_{B_{\beta,k}}^{-1} \sum_{l\in\mathbb{Z}_k^n(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} [\tilde{\varphi}_\beta * f(y) - \tilde{\varphi}_\beta * f(A^{-\beta+\eta}Bl)]\varphi_\beta(x - A^{-\beta+\eta}Bl)dy.$$

Therefore,

$$II = \sum_{\beta \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} t_{B_{\beta,k}} n_{B_{\beta,k}}(x).$$

For any $l \in \mathbb{Z}^n_k(\beta,\eta), y \in B_{\beta-\eta,l}$ and $\langle a,\gamma\rangle \leq K'$, we have

$$\begin{aligned}
&\big|\partial_x^\gamma\big(\tilde{\varphi}_\beta(A^{-\beta}x - y) - \tilde{\varphi}_\beta(A^{-\beta}x - A^{-\beta+\eta}Bl)\big)\big| \qquad (5.3)\\
&\quad \leq C|\det A|^\beta \rho_A\big(A^\beta(y - A^{-\beta+\eta}Bl)\big)^\varepsilon \big(1+\rho_A(x - A^\beta x_{B_{\beta,k}})\big)^{-M}\\
&\quad \leq C|\det A|^\beta |\det A|^{\eta\varepsilon}\big(1+\rho_A(x - A^\beta x_{B_{\beta,k}})\big)^{-M},
\end{aligned}$$

where the constant C is independent of $y \in B_{\beta-\eta,l}$.

Therefore, if we write

$$h_{B_{\beta,k}}(x) = |\det A|^{-\beta/2}\big(\tilde{\varphi}_\beta(y-x) - \tilde{\varphi}_\beta(A^{-\beta+\eta}Bl - x)\big),$$

then $h_{B_{\beta,k}}(x)$ is a strong molecule with the same order as $\varphi_{B_{\beta,k}}$. Furthermore, by the φ-ψ transform reproducing property, we write $f = \sum_P r_P\psi_P$ where $r = \{r_P\}_P$ satisfies $\|r\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,dx)} \leq C\|f\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)}$. If we set

$$b_{PQ} = |\det A|^{\beta/2} \sum_{l \in \mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap Q} |\tilde{\varphi}_\beta * \psi_P(y) - \tilde{\varphi}_\beta * \psi_P(A^{-\beta+\eta}Bl)|dy,$$

where $Q = B_{\beta,k}$, then $|t_Q| \leq \sum_P |b_{PQ}||r_P|$. By the estimation in Theorem 3.1, we have,

$$|\langle\psi_P, h_Q\rangle| \leq C|\det A|^{\eta\varepsilon}\kappa_{PQ}(K',M).$$

Hence, for $Q = B_{\beta,k}$,

$$\begin{aligned}
|b_{PQ}| &\leq |\det A|^{\beta/2} \sum_{l \in \mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} |\tilde{\varphi}_\beta * \psi_P(y) - \tilde{\varphi}_\beta * \psi_P(A^{-\beta+\eta}Bl)|dy\\
&\leq C|\det A|^\beta \int_{B_{\beta,k}} |\langle\psi_P, h_Q\rangle|dy\\
&\leq C|\det A|^\beta \int_{B_{\beta,k}} |\det A|^{\eta\varepsilon}\kappa_{PQ}(K',M)dy \leq C|\det B||\det A|^{\eta\varepsilon}\kappa_{PQ}(K',M),
\end{aligned}$$

because

$$\int_{B_{\beta,k}} dy = |\det A|^{-\beta}|\det B|.$$

Therefore, the operator $\mathcal{B} = \{b_{PQ}\}$ is bounded on $\dot{\mathbf{f}}^{\alpha,q}_p(A,dx)$ if K', M are large enough. We now consider $n_{B_{\beta,k}}$. The vanishing moment conditions of $n_{B_{\beta,k}}$ is inherited by φ. Furthermore, for any $\langle a,\gamma\rangle \leq K'$, we see that

$$\begin{aligned}
&\big|\partial^\gamma(n_{B_{\beta,k}}(A^{-\beta}x))\big| \qquad (5.4)\\
&\leq t^{-1}_{B_{\beta,k}} \sum_{l \in \mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}}
\end{aligned}$$

$$
\begin{aligned}
&\times \big|\tilde\varphi_\beta * f(y) - \tilde\varphi_\beta * f(A^{-\beta+\eta}Bl)\big|\big|\partial^\gamma\big(\varphi_\beta(A^{-\beta}x - A^{-\beta+\eta}Bl)\big)\big|dy\\
&\le |\det A|^{\beta/2}\big(1+\rho_A(x - A^\beta(A^{-\beta+\eta}Bl))\big)^{-M}\\
&\times t^{-1}_{B_{\beta,k}} \sum_{l\in\mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} |\tilde\varphi_\beta * f(y) - \tilde\varphi_\beta * f(A^{-\beta+\eta}Bl)|dy\\
&\le C|\det A|^{\beta/2}\big(1+\rho_A(x - A^\beta x_{B_{\beta,k}})\big)^{-M},
\end{aligned}
$$

because $A^{-\beta+\eta}Bl \in B_{\beta-\eta,l}$ and

$$
\sup_{z\in B_{\beta,k}} \big(1+\rho_A(x-A^\beta z)\big)^{-M} \sim \inf_{z\in B_{\beta,k}} \big(1+\rho_A(x-A^\beta z)\big)^{-M}.
$$

Therefore $\{n_{B_{\beta,k}}\}_{\beta,k}$ is a family of strong molecules of order K', M.

Estimate of $(I - |\det B|\mathcal{F}_\eta)$**:**

We may write

$$
\big|(I - |\det B|\mathcal{F}_\eta)f(x)\big| \le C|\det A|^{\eta\varepsilon}\bigg|\sum_{\beta,k} s_{B_{\beta,k}} m_{B_{\beta,k}}\bigg| + C\bigg|\sum_{\beta,k} t_{B_{\beta,k}} n_{B_{\beta,k}}\bigg|.
$$

Taking the $\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, dx)$ norm on both sides, we have

$$
\begin{aligned}
&\|(I - |\det B|\mathcal{F}_\eta)f(x)\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)}\\
&\le C|\det A|^{\eta\varepsilon}\bigg\|\sum_{\beta,k} s_{B_{\beta,k}} m_{B_{\beta,k}}\bigg\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)} + C\bigg\|\sum_{\beta,k} t_{B_{\beta,k}} n_{B_{\beta,k}}\bigg\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)}.
\end{aligned}
$$

For the first term, if K', M are large enough, by Theorem 3.2 and

$$
s_{B_{\beta,k}} = |\det A|^{\beta/2}|\det B|\int_{B_{\beta,k}} |\tilde\varphi_\beta * f(y)|dy \le C|\det A|^{-\beta/2}\sup_{z\in B_{\beta,k}} |\tilde\varphi_\beta * f(y)|,
$$

we have

$$
\bigg\|\sum_{\beta,k} s_{B_{\beta,k}} m_{B_{\beta,k}}\bigg\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)} \le C\|\{s_{B_{\beta,k}}\}\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,dx)} \le C\|f\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)},
$$

because $\{m_{B_{\beta,k}}\}_{\beta,k}$ is a family of strong molecules. For the second term, if K', M are large enough, we have

$$
\begin{aligned}
\bigg\|\sum_{\beta,k} t_{B_{\beta,k}} n_{B_{\beta,k}}\bigg\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)} &\le C\|\{t_{B_{\beta,k}}\}\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,dx)} \le C|\det A|^{\eta\varepsilon}\|\{r_{B_{\beta,k}}\}\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,dx)}\\
&\le C|\det A|^{\eta\varepsilon}\|f\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)}
\end{aligned}
$$

because $t_Q = \sum_P b_{PQ} r_P$, $\|r\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,dx)} \le C\|f\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,dx)}$ and

$$
|b_{PQ}| \le C|\det A|^{\eta\varepsilon}\kappa_{PQ}(K', M).
$$

Therefore,

$$\|(I - |\det B|\mathcal{F}_\eta)f(x)\|_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,dx)} \le C|\det A|^{\eta\varepsilon}\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,dx)}.$$

Choose an $\eta_0 < 0$ so that $C|\det A|^{\eta\varepsilon} < 1$, for any $\eta < \eta_0$; we then have

$$\|I - |\det B|\mathcal{F}_\eta\|_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,dx)\to\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,dx)} < 1$$

and by the Neumann series $(I-T)^{-1} = I + T + T^2 + \cdots = \sum_{i\ge 0} T^i$,

$$|\det B|\mathcal{F}_\eta = I - (I - |\det B|\mathcal{F}_\eta)$$

is invertible, and, hence, $\mathcal{F}_\eta$ is invertible because $|\det B| \ne 0$. □

6. Regularity of the frame operator

We begin with some lemmas. The following is an anisotropic version of Lemma B.2 of [6].

Lemma 6.1

Let $|R| = |\det A|^{-k} \ge |\det A|^{-i} = |P|$, $i,j,k \in \mathbb{Z}$, $M > 1$, $x \in \mathbb{R}^n$, we have

$$\sum_{|Q|=|\det A|^{-j}} \left(1 + \frac{\rho_A(x_Q - x_R)}{\max(|Q|,|R|)}\right)^{-M} \left(1 + \frac{\rho_A(x - x_Q)}{\max(|P|,|Q|)}\right)^{-M}$$
$$\le C\left(1 + \frac{\rho_A(x - x_R)}{\max(|Q|,|R|)}\right)^{-M} \max\left(1, \frac{|P|}{|Q|}\right)$$

for a constant $C > 0$ depending only on M.

Proof. Without loss of generality, we may assume $k = 0$. First, we deal with the case $|R| = 1 \ge |P| \ge |Q|$. For any $y \in Q$, we have

$$C^{-1}\big(1 + \rho_A(x_R - y)\big)^{-M} \le \big(1 + \rho_A(x_Q - x_R)\big)^{-M} \le C\big(1 + \rho_A(x_R - y)\big)^{-M}$$

because $|R| = 1 \ge |\det A|^{-j} = |Q|$; moreover,

$$C^{-1}\left(1 + \frac{\rho_A(x-y)}{|P|}\right)^{-M} \le \left(1 + \frac{\rho_A(x - x_Q)}{|P|}\right)^{-M} \le C\left(1 + \frac{\rho_A(x-y)}{|P|}\right)^{-M}$$

because $|P| \ge |Q|$. Notice that C depends continuously on M. Hence,

$$\sum_{|Q|=|\det A|^{-j}} \frac{1}{(1+\rho_A(x_Q - x_R))^M} \frac{1}{(1 + \frac{\rho_A(x-x_Q)}{|P|})^M}$$
$$\le C \sum_{|Q|=|\det A|^{-j}} |\det A|^j \int_Q \frac{1}{(1+\rho_A(x_R - y))^M} \frac{1}{(1+\frac{\rho_A(x-y)}{|P|})^M} dy$$
$$\le C\frac{|\det A|^j}{|\det A|^i}|\det A|^i \int_{\mathbb{R}^n} \frac{1}{(1+\rho_A(x_R - y))^M} \frac{1}{(1+\frac{\rho_A(x-y)}{|P|})^M} dy$$
$$\le C\frac{|P|}{|Q|}\frac{1}{(1+\rho_A(x - x_R))^M}$$

by Lemma 8.1 in Appendix.

For the case $|R| = 1 \geq |Q| > |P|$, we have

$$\sum_{|Q|=|\det A|^{-j}} \left(1 + \frac{\rho_A(x_Q - x_R)}{\max(|Q|, |R|)}\right)^{-M} \left(1 + \frac{\rho_A(x - x_Q)}{\max(|P|, |Q|)}\right)^{-M}$$
$$= \sum_{|Q|=|\det A|^{-j}} (1 + \rho_A(x_Q - x_R))^{-M} \left(1 + \frac{\rho_A(x - x_Q)}{|Q|}\right)^{-M}$$
$$\leq C \sum_{|Q|=|\det A|^{-j}} |\det A|^j \int_Q \frac{1}{(1 + \rho_A(x_R - y))^M} \frac{1}{(1 + \frac{\rho_A(x-y)}{|Q|})^M} dy$$
$$\leq C|\det A|^j \int_{\mathbb{R}^n} \frac{1}{(1 + \rho_A(x_R - y))^M} \frac{1}{(1 + \frac{\rho_A(x-y)}{|Q|})^M} dy \leq C \frac{1}{(1 + \rho_A(x - x_R))^M}.$$

Similarly, for the last case $|Q| > |R| \geq |P|$, we have

$$\sum_{|Q|=|\det A|^{-j}} \left(1 + \frac{\rho_A(x_Q - x_R)}{\max(|Q|, |R|)}\right)^{-M} \left(1 + \frac{\rho_A(x - x_R)}{\max(|P|, |Q|)}\right)^{-M}$$
$$\leq C \sum_{|Q|=|\det A|^{-j}} |\det A|^j \int_Q \frac{1}{(1 + \frac{\rho_A(x_R-y)}{|Q|})^M} \frac{1}{(1 + \frac{\rho_A(x-y)}{|Q|})^M} dy$$
$$\leq C|\det A|^j \int_{\mathbb{R}^n} \frac{1}{(1 + \frac{\rho_A(x_R-y)}{|Q|})^M} \frac{1}{(1 + \frac{\rho_A(x-y)}{|Q|})^M} dy \leq C \frac{1}{(1 + \frac{\rho_A(x-x_R)}{|Q|})^M}. \square$$

With these lemmas, we can show the following theorem which guarantees that the composition of two strongly almost diagonal matrices is a strongly almost diagonal matrix.

Theorem 6.2

Let $K, L, M, \tilde{M} > 0$ satisfy $K \neq L$ and $K + L + 1 > \min(M, \tilde{M})$. Suppose $\mathcal{A} = \{a_{PQ}\}_{PQ} \in s\kappa(K, M)$ and $\mathcal{B} = \{b_{QR}\}_{QR} \in s\kappa(L, \tilde{M})$ be strongly almost diagonal matrices. Then the matrix $\mathcal{A} \circ \mathcal{B} = \mathcal{C} = \{c_{PR}\}_{PR}$ where

$$c_{PR} = \sum_Q a_{PQ} b_{QP}$$

is a strongly almost diagonal matrix and $\mathcal{C} = \{c_{PR}\}_{PR} \in s\kappa(\min(K, L), \min(M, \tilde{M}))$. Moreover, we have a constant $C > 0$, depending continuously on $M, \tilde{M}$ only, such that

$$\|\{c_{PR}\}_{PR}\|_{s\kappa(\min(K,L),\min(M,\tilde{M}))} \leq C\|\{a_{PQ}\}_{PQ}\|_{s\kappa(K,M)}\|\{b_{QR}\}_{QR}\|_{s\kappa(L,\tilde{M})}.$$

Proof. Let $|P| = 2^{-i}$, $|Q| = 2^{-j}$ and $|R| = 2^{-k}$. Without loss of generality, we may assume $|P| \leq |R|$, $K > L$, $M \geq \tilde{M}$, $\|\{a_{PQ}\}_{PQ}\|_{s\kappa(K,M)} = 1$ and $\|\{b_{QR}\}_{QR}\|_{s\kappa(L,\tilde{M})} = 1$. We split c_{PR} into three terms

$$|c_{PR}| \leq \sum_{|P|\leq|R|\leq|Q|} |a_{PQ}||b_{QR}| + \sum_{|P|\leq|Q|\leq|R|} |a_{PQ}||b_{QR}| + \sum_{|Q|\leq|P|\leq|R|} |a_{PQ}||b_{QR}|$$
$$= I + II + III.$$

For I, we have

$$I \le \sum_{|P|\le|R|\le|Q|} \left(\frac{|P|}{|Q|}\right)^{K+1/2} \left(1+\frac{\rho_A(x_Q-x_P)}{|Q|}\right)^{-M} \left(\frac{|R|}{|Q|}\right)^{L+1/2} \times \left(1+\frac{\rho_A(x_Q-x_R)}{|Q|}\right)^{-\tilde{M}}.$$

Because $M>\tilde{M}$,

$$\begin{aligned} I &\le |P|^{K+1/2}|R|^{L+1/2} \sum_{|P|\le|R|\le|Q|} |Q|^{-K-L-1} \left(1+\frac{\rho_A(x_Q-x_P)}{|Q|}\right)^{-\tilde{M}}\left(1+\frac{\rho_A(x_Q-x_R)}{|Q|}\right)^{-\tilde{M}} \\ &\le |P|^{K+1/2}|R|^{L+1/2} \sum_{j=-\infty}^{k} 2^{j(K+L+1)} \sum_{|Q|=2^{-j}} \left(1+\frac{\rho_A(x_Q-x_P)}{|Q|}\right)^{-\tilde{M}}\left(1+\frac{\rho_A(x_Q-x_R)}{|Q|}\right)^{-\tilde{M}}. \end{aligned}$$

Therefore, by Lemma 6.1,

$$I \le C|P|^{K+1/2}|R|^{L+1/2} \sum_{j=-\infty}^{k} 2^{j(K+L+1)} \left(1+\frac{\rho_A(x_P-x_R)}{|Q|}\right)^{-\tilde{M}}.$$

Because $|Q|\ge|R|$, we have

$$1+\frac{\rho_A(x_P-x_R)}{|R|} \le 1+\frac{\rho_A(x_P-x_R)}{|Q|}\frac{|Q|}{|R|} \le \frac{|Q|}{|R|}\left(1+\frac{\rho_A(x_P-x_R)}{|Q|}\right).$$

Hence, we have

$$\begin{aligned} I &\le C|P|^{K+1/2}|R|^{L+1/2} \sum_{j=-\infty}^{k} 2^{j(K+L+1)} \frac{|Q|}{|R|}^{\tilde{M}} \left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}} \\ &\le C|P|^{K+1/2}|R|^{L-\tilde{M}+1/2} \sum_{j=-\infty}^{k} 2^{j(K+L+1-\tilde{M})} \left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}} \\ &\le C|P|^{K+1/2}|R|^{L-\tilde{M}+1/2}|R|^{-K-L-1+\tilde{M}} \left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}} \\ &\le C\frac{|P|^{K+1/2}}{|R|^{K+1/2}} \left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}} \le C\frac{|P|^{L+1/2}}{|R|^{L+1/2}} \left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}}. \end{aligned}$$

Since $K+L+1>\tilde{M}$, $K>L$ and $|P|\le|R|$.

For II, we have

$$\begin{aligned} II &\le \sum_{|P|\le|Q|\le|R|} \left(\frac{|P|}{|Q|}\right)^{K+1/2}\left(1+\frac{\rho_A(x_Q-x_P)}{|Q|}\right)^{-M}\left(\frac{|Q|}{|R|}\right)^{L+1/2}\left(1+\frac{\rho_A(x_Q-x_R)}{|R|}\right)^{-\tilde{M}} \\ &\le \frac{|P|^{K+1/2}}{|R|^{L+1/2}} \sum_{|P|\le|Q|\le|R|} |Q|^{-K+L} \left(1+\frac{\rho_A(x_Q-x_P)}{|Q|}\right)^{-\tilde{M}} \left(1+\frac{\rho_A(x_Q-x_R)}{|R|}\right)^{-\tilde{M}} \\ &\le \frac{|P|^{K+1/2}}{|R|^{L+1/2}} \sum_{j=k}^{i} 2^{j(K-L)} \sum_{|Q|=2^{-j}} \left(1+\frac{\rho_A(x_Q-x_P)}{|Q|}\right)^{-\tilde{M}} \left(1+\frac{\rho_A(x_Q-x_R)}{|R|}\right)^{-\tilde{M}}. \end{aligned}$$

Because $K > L$,

$$II \le C\frac{|P|^{K+1/2}}{|R|^{L+1/2}}|P|^{-K+L}\left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}}$$
$$\le C\frac{|P|^{L+1/2}}{|R|^{L+1/2}}\left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}}.$$

For the last term, we have

$$III \le \sum_{|Q|\le|P|\le|R|}\left(\frac{|Q|}{|P|}\right)^{K+1/2}\left(1+\frac{\rho_A(x_Q-x_P)}{|P|}\right)^{-M}\left(\frac{|Q|}{|R|}\right)^{L+1/2}$$
$$\times\left(1+\frac{\rho_A(x_Q-x_R)}{|R|}\right)^{-\tilde{M}}$$
$$\le \frac{1}{|P|^{K+1/2}|R|^{L+1/2}}\sum_{|Q|\le|P|\le|R|}|Q|^{K+L+1}\left(1+\frac{\rho_A(x_Q-x_P)}{|P|}\right)^{-\tilde{M}}$$
$$\times\left(1+\frac{\rho_A(x_Q-x_R)}{|R|}\right)^{-\tilde{M}}.$$

By Lemma 6.1,

$$III \le C|P|^{-K-1/2}|R|^{-L-1/2}\sum_{j=i}^{\infty}\left(2^{-j(K+L+1)}\left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}}\frac{|P|}{|Q|}\right)$$
$$\le C|P|^{-K+1/2}|R|^{-L-1/2}\sum_{j=i}^{\infty}\left(2^{-j(K+L)}\left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}}\right)$$
$$\le C|P|^{-K+1/2}|R|^{-L-1/2}|P|^{K+L}\left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}}$$
$$\le C\frac{|P|^{K+1/2}}{|R|^{L+1/2}}\left(1+\frac{\rho_A(x_P-x_R)}{|R|}\right)^{-\tilde{M}}.$$

Combining these inequalities, we show that $\{c_{PR}\}_{PR}$ satisfies condition (3.9) and

$$\|\{c_{PR}\}_{PR}\|_{s\kappa(L,\tilde{M})} \le C\|\{a_{PQ}\}_{PQ}\|_{s\kappa(K,M)}\|\{b_{QR}\}_{QR}\|_{s\kappa(L,\tilde{M})}. \ \Box$$

By iterating the estimates, we obtain the following corollary.

Corollary 6.3

Let $K', M' > 0$. Suppose A_i, $1 \le i \le m$, are strongly almost diagonal matrices with order K', M', then for any $K, M > 0$ satisfying $K' > K$, $M' > M$, and $K + K' + 1 > M$, the composition of A_i, $A_1 \circ A_2 \cdots \circ A_m$ is a strongly almost diagonal matrix with order K, M and

$$\|A_1 \circ A_2 \cdots \circ A_m\|_{s\kappa(K,M)} \le C^{m-1}\|A_1\|_{s\kappa(K',M')}\|A_2\|_{s\kappa(K',M')}\cdots\|A_m\|_{s\kappa(K',M')}$$

for a constant $C > 0$ depending only on M, M'.

We are now going to prove the main result of this section that asserts that the class of strong molecule is invariant under the mapping by a strongly almost diagonal operator. Moreover, it also generalizes a result of Frazier and Jawerth [6], Lemma 9.14.

Theorem 6.4

Let $K > L > 0$, $M \geq \tilde{M}$ and $1 + K > \tilde{M}$. Let $\{k_Q\}_{Q_{j,k}}$ be a family of strong molecules with order $L, \tilde{M}$, and $\{a_{PQ}\}$ be a strongly almost diagonal matrix with order K, M. Let

$$h_Q(x) = \sum_P a_{PQ} k_P(x).$$

We have

$$\int x^\gamma h_Q(x)dx = 0 \quad \text{if} \quad \langle a, \gamma \rangle \leq L; \tag{6.1}$$

and, for a constant C depending only on the matrix A and $\tilde{M}$,

$$\left|\partial^\gamma (h_Q(A^{-j}x))\right| \leq C|\det A|^{j/2} \frac{1}{(1+\rho_A(x - A^j x_Q))^{\tilde{M}}} \tag{6.2}$$

if $\langle a, \gamma \rangle \leq L$.

Moreover, we have

$$\|\{h_Q\}_Q\|_{\mathcal{M}_{L,\tilde{M}}} \leq C \|\{a_{PQ}\}_{PQ}\|_{s\kappa(K,M)} \|\{k_Q\}_Q\|_{\mathcal{M}_{L,\tilde{M}}}$$

for a constant $C > 0$.

Proof. Without loss of generality, we assume $\|\{a_{PQ}\}_{PQ}\|_{s\kappa(K,M)} = \|\{k_Q\}_Q\|_{\mathcal{M}_{L,\tilde{M}}} = 1$. To estimate $|\partial^\gamma h_Q(x)|$ for $|\langle a, \gamma \rangle| \leq L$, with $|Q| = |\det A|^{-i}$, we split the sum into two terms.

$$\begin{aligned}\left|\partial^\gamma (h_Q(A^{-i}x))\right| \leq & \sum_{|P|\leq|Q|} \left|a_{PQ}\right|\left|\partial^\gamma (k_P(A^{-i}x))\right| \\ & + \sum_{|P|>|Q|} \left|a_{PQ}\right|\left|\partial^\gamma (k_P(A^{-i}x))\right| = I + II.\end{aligned}$$

By the definition of strong molecules and strongly almost diagonal matrices, we have,

$$\begin{aligned} I \leq & \sum_{j=i}^{\infty} \sum_{|P|=|\det A|^{-j}} \left(1 + \frac{\rho_A(x_Q - x_P)}{|Q|}\right)^{-M} \left(\frac{|P|}{|Q|}\right)^{K+1/2} \\ & \times |\det A|^{j/2 + \langle a,\gamma\rangle(j-i) + |j-i|\triangle} \left(1 + \frac{\rho_A(A^{-i}x - x_P)}{|P|}\right)^{-\tilde{M}} \\ \leq & |\det A|^{i/2} \sum_{j=i}^{\infty} \sum_{|P|=|\det A|^{-j}} \left(1 + \frac{\rho_A(x_Q - x_P)}{|Q|}\right)^{-\tilde{M}} |\det A|^{(-j+i)K} \\ & \times |\det A|^{(-j+i)/2 + (j-i)/2 + \langle a,\gamma\rangle(j-i) + |j-i|\triangle} \left(1 + \frac{\rho_A(A^{-i}x - x_P)}{|P|}\right)^{-\tilde{M}} \end{aligned}$$

because $M \geq \tilde{M}$. Therefore,

$$I \leq \left(|\det A|^{i/2} \sum_{j=i}^{\infty} |\det A|^{-(j-i)(K-\langle a,\gamma\rangle)+(j-i)\triangle} \right)$$
$$\times \sum_{|P|=|\det A|^{-j}} \left(1 + \frac{\rho_A(x_Q - x_P)}{|Q|}\right)^{-\tilde{M}} \left(1 + \frac{\rho_A(A^{-i}x - x_P)}{|P|}\right)^{-\tilde{M}}$$

because $j \geq i$ and $\langle a,\gamma\rangle \leq L < K - \triangle$ if we take $\triangle < K - L$. Furthermore, by Lemma 6.1, we have,

$$I \leq C|\det A|^{i/2} \left(1 + \frac{\rho_A(A^{-i}x - x_Q)}{|Q|}\right)^{-\tilde{M}} \leq C|\det A|^{i/2} \left(1 + \rho_A(x - A^i x_Q)\right)^{-\tilde{M}}.$$

For II, we have

$$II \leq \sum_{j=-\infty}^{i-1} \sum_{|P|=|\det A|^{-j}} \left(1 + \frac{\rho_A(x_Q - x_P)}{|P|}\right)^{-M} \left(\frac{|Q|}{|P|}\right)^{K+1/2}$$
$$\times |\det A|^{j/2+\langle a,\gamma\rangle(j-i)+|j-i|\triangle} \left(1 + \frac{\rho_A(A^{-i}x - x_P)}{|P|}\right)^{-\tilde{M}}$$
$$\leq |\det A|^{i/2} \sum_{j=-\infty}^{i-1} \sum_{|P|=|\det A|^{-j}} \left(1 + \frac{\rho_A(x_Q - x_P)}{|P|}\right)^{-\tilde{M}} |\det A|^{(-i+j)(1/2+K)}$$
$$\times |\det A|^{(j-i)/2+\langle a,\gamma\rangle(j-i)+|j-i|\triangle} \left(1 + \frac{\rho_A(A^{-i}x - x_P)}{|P|}\right)^{-\tilde{M}}$$
$$\leq C \sum_{j=-\infty}^{i-1} |\det A|^{i/2+(j-i)(1+\langle a,\gamma\rangle+K)+|j-i|\triangle} \left(\frac{|\det A|^{-j}}{|\det A|^{-j} + \rho_A(A^{-i}x - x_Q)}\right)^{\tilde{M}}$$

and; hence,

$$II \leq C \sum_{j=-\infty}^{i-1} |\det A|^{i/2+(j-i)(1+\langle a,\gamma\rangle+K-\tilde{M})-(j-i)\triangle} \left(\frac{|\det A|^{-i}}{|\det A|^{-i} + \rho_A(A^{-i}x - x_Q)}\right)^{\tilde{M}}$$
$$\leq C|\det A|^{i/2} \left(\frac{|\det A|^{-i}}{|\det A|^{-i} + \rho_A(A^{-i}x - x_Q)}\right)^{\tilde{M}} \leq C|\det A|^{i/2} \frac{1}{(1 + \rho_A(x - A^i x_Q))^{\tilde{M}}}.$$

We use $|P| > |Q|$ and choose $\triangle > 0$ small enough, so that $1 + K > \tilde{M} + \triangle$ in the summation over j. Hence,

$$\left|\partial^\gamma (h_Q(A^{-i}x))\right| \leq C|\det A|^{i/2} \frac{1}{(1 + \rho_A(x - A^i x_Q))^{\tilde{M}}}$$

if $\langle a,\gamma\rangle \leq L$.

The vanishing moment conditions,

$$\int x^\gamma h_Q(x)dx = 0 \quad \text{if} \quad \langle a, \gamma \rangle \le L \tag{6.3}$$

are well defined because $\tilde{M} > 1 + L$ and

$$\int |x^\gamma||h_Q(x)|dx = \int |x^\gamma| \sum_P |a_{PQ}||k_P(x)|dx < \infty \quad \text{if} \quad \langle a, \gamma \rangle \le L.$$

The vanishing moment is inherited by h_Q form the corresponding conditions on k_Q . $\square$

Proof of Theorem 4.2: Fix a $\varepsilon < \frac{\ln|\lambda_-|}{\ln|\det A|}$. Let $\{\phi_P\}_P$ be strong molecules of order K', M'. From Theorem 4.1, we have, for $P = B_{\gamma,h}$, $\gamma \in \mathbb{Z}$ and $h \in \mathbb{Z}^n$ (recall that $\mathbb{Z}^n_k(\beta,\eta) = \{l \in \mathbb{Z}^n : B_{\beta-\eta,l} \cap B_{\beta,k} \neq \emptyset\}$),

$$\begin{aligned}
&(I - |\det B|\mathcal{F}_\eta)\phi_P(x) \\
&= \sum_{\beta\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} \big(\tilde{\varphi}_\beta * \phi_P(y)\big)\big[\varphi_\beta(x-y) - \varphi_\beta(x - A^{-\beta+\eta}Bl)\big]dy \\
&+ \sum_{\beta\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} \big[\tilde{\varphi}_\beta * \phi_P(y) - \tilde{\varphi}_\beta * \phi_P(A^{-\beta+\eta}Bk)\big]\varphi_\beta\big(x - A^{-\beta+\eta}Bl\big)dy.
\end{aligned}$$

For $Q = B_{\beta,k}$, we have

$$(I - |\det B|\mathcal{F}_\eta)\phi_P(x) = \sum_{Q=B_{\beta,k}} |\det A|^{\eta\varepsilon} s_{QP} m_Q(x) + \sum_{Q=B_{\beta,k}} t_{QP} n_Q(x)$$

where

$$\begin{aligned}
s_{QP} &= |\det A|^{\beta/2} |\det B| \int_{B_{\beta,k}} |\tilde{\varphi}_\beta * \phi_P(y)|dy, \\
m_Q(x) &= \frac{|\det A|^{-\eta\varepsilon}}{s_{QP}|\det B|^\varepsilon} \sum_{l\in\mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} \tilde{\varphi}_\beta * \phi_P(y)[\varphi_\beta(x-y) - \varphi_\beta(x - A^{-\beta+\eta}Bl)]dy;
\end{aligned}$$

and

$$\begin{aligned}
t_{QP} &= |\det A|^{\beta/2} \sum_{l\in\mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} |\tilde{\varphi}_\beta * \phi_P(y) - \tilde{\varphi}_\beta * \phi_P(A^{-\beta+\eta}Bl)|dy, \\
n_Q(x) &= \frac{1}{t_{QP}} \sum_{l\in\mathbb{Z}^n_k(\beta,\eta)} \int_{B_{\beta-\eta,l}\cap B_{\beta,k}} [\tilde{\varphi}_\beta * \phi_P(y) - \tilde{\varphi}_\beta * \phi_P(A^{-\beta+\eta}Bk)] \\
&\quad \times \varphi_\beta(x - A^{-\beta+\eta}Bl)dy.
\end{aligned}$$

First of all, since $\tilde{\varphi}_Q(y-x)$ is a strong molecule for any $y \in Q = B_{\beta,k}$, we have

$$\|\{\tilde{\varphi}_Q(y-\cdot)\}\|_{\mathcal{M}_{K',M'}} \sim \|\{\tilde{\varphi}_Q(\cdot)\}\|_{\mathcal{M}_{K',M'}}.$$

Therefore, by Theorem 3.1,

$$\begin{aligned} s_{QP} &= |\det A|^{\beta/2}|\det B| \int_{B_{\beta,k}} |\tilde{\varphi}_\beta * \phi_P(y)|dy \le |\det A|^{-\beta/2} \sup_{y\in B_{\beta,k}} |\tilde{\varphi}_\beta * \phi_P(y)| \\ &\le \sup_{y\in B_{\beta,k}} \left| \int \tilde{\varphi}_Q(y-x)\phi_P(x)dx \right| \le C\|\{\varphi_Q\}\|_{\mathcal{M}_{K',M'}} \|\{\phi_P\}\|_{\mathcal{M}_{K',M'}} s\kappa_{QP}(K,M). \end{aligned}$$

Similarly, we have, for any $y \in B_{\beta,l}$,

$$\begin{aligned} \|t_{QP}\|_{s\kappa(K,M)} &\le C\|\{\tilde{\varphi_Q}(y-\cdot) - \tilde{\varphi_Q}(A^{-\beta}Bl-\cdot)\}_Q\|_{\mathcal{M}_{K',M'}} \|\{\phi_P\}_P\|_{\mathcal{M}_{K',M'}} \\ &\le C|\det A|^{\eta\varepsilon} \|\{\varphi_Q\}_Q\|_{\mathcal{M}_{K'+a_n,M'}} \|\{\phi_P\}_P\|_{\mathcal{M}_{K',M'}}. \end{aligned}$$

Similar to the estimates for I and II in the proof of Theorem 4.1, $\{m_Q\}$ and $\{n_Q\}$ are families of strong molecules of order K', M' with

$$\|\{m_Q(x)\}\|_{\mathcal{M}_{K',M'}} \le C\|\{\phi_P\}_P\|_{\mathcal{M}_{K',M'}} \quad \text{and} \quad \|\{n_Q(x)\}\|_{\mathcal{M}_{K',M'}} \le C\|\{\phi_P\}_P\|_{\mathcal{M}_{K',M'}}.$$

Hence, by Theorem 6.4, we have

$$\begin{aligned} &\|\{(I - |\det B|\mathcal{F}_\eta)\phi_P\}_P\|_{\mathcal{M}_{L,\tilde{M}}} \\ &\le C|\det A|^{\eta\varepsilon} \|\{\varphi_Q\}_Q\|^2_{\mathcal{M}_{K'+a_n,M'}} \|\{\phi_P\}_P\|_{\mathcal{M}_{K',M'}} + C\|t_Q\|_{s\kappa(K,M)} \|\{\varphi_Q\}_Q\|_{\mathcal{M}_{K'+a_n,M'}} \\ &\le C|\det A|^{\eta\varepsilon} \|\{\varphi_Q\}_Q\|^2_{\mathcal{M}_{K'+a_n,M'}} \|\{\phi_P\}_P\|_{\mathcal{M}_{K',M'}}. \end{aligned}$$

Therefore, we have shown that $\{(I - |\det B|\mathcal{F}_\eta)\phi_P\}_P$ is a family of strong molecules of order $L, \tilde{M}$.

For the last result, we are going to estimate $(I - |\det B|\mathcal{F}_\eta)^i\phi_P(x)$, $i > 0$. For $i = 2$, we have

$$\begin{aligned} (I - |\det B|\mathcal{F}_\eta)^2\phi_P(x) = &\sum_{Q=B_{\beta,k}} |\det A|^{\eta\varepsilon} s_{QP}(I - |\det B|\mathcal{F}_\eta)m_Q(x) \\ &+ \sum_{Q=B_{\beta,k}} t_{QP}(I - |\det B|\mathcal{F}_\eta)n_Q(x). \end{aligned}$$

Since m_Q and n_Q are strong molecules, by Theorem 4.1, there exist strongly almost diagonal matrices $\{s^{[2]}_{QP}\}_{QP}$ and $\{t^{[2]}_{QP}\}_{QP}$ satisfying

$$\begin{aligned} \|\{s^{[2]}_{QP}\}_{QP}\|_{s\kappa(K,M)} &\le C\|\{\varphi_Q\}\|^2_{\mathcal{M}_{K'+a_n,M'}} \\ \|\{t^{[2]}_{QP}\}_{QP}\|_{s\kappa(K,M)} &\le C|\det A|^{\eta\varepsilon}\|\{\varphi_Q\}\|^2_{\mathcal{M}_{K'+a_n,M'}} \end{aligned}$$

and strong molecules $m^{[2]}_Q$ and $n^{[2]}_Q$ satisfying

$$\|m^{[2]}_Q\|_{\mathcal{M}_{K',M'}} \le C|\det A|^{\eta\varepsilon}\|\varphi_Q\|_{\mathcal{M}_{K'+a_n,M'}} \quad \text{and} \quad \|n^{[2]}_Q\|_{\mathcal{M}_{K',M'}} \le C\|\varphi_Q\|_{\mathcal{M}_{K'+a_n,M'}}$$

such that

$$(I - |\det B|\mathcal{F}_\eta)m_Q(x) = \sum_{Q'} s^{[2]}_{QQ'} m^{[2]}_{Q'}(x) \quad \text{and}$$
$$(I - |\det B|\mathcal{F}_\eta)n_Q(x) = \sum_{Q'} t^{[2]}_{QQ'} n^{[2]}_{Q'}(x).$$

Hence,

$$\begin{aligned}(I - |\det B|\mathcal{F}_\eta)^2\phi_P(x) =& \sum_Q |\det A|^{\eta\varepsilon} s_{QP} \sum_{Q'} s^{[2]}_{QQ'} m^{[2]}_{Q'}(x) + \sum_Q t_{QP} \sum_{Q'} t^{[2]}_{QQ'} n^{[2]}_{Q'}(x)\\ =& \sum_{Q'} |\det A|^{\eta\varepsilon} \left(\sum_Q s_{QP} s^{[2]}_{QQ'} \right) m^{[2]}_{Q'}(x)\\ &+ \sum_{Q'} \left(\sum_Q t_{QP} t^{[2]}_{QQ'} \right) n^{[2]}_{Q'}(x).\end{aligned}$$

Therefore, by Corollary 6.3 with $K + K' + 1 > M$, the matrices

$$\left\{ \sum_Q s_{QP} s^{[2]}_{QQ'} \right\}_{PQ'} \quad \text{and} \quad \left\{ \sum_Q t_{QP} t^{[2]}_{QQ'} \right\}_{PQ'}$$

are strongly almost diagonal matrices with order K, M. Hence, we have

$$\begin{aligned}&\|\{(I - |\det B|\mathcal{F}_\eta)^2 \varphi_P\}_P\|_{\mathcal{M}_{L,\tilde{M}}}\\ &\leq C|\det A|^{\eta\varepsilon} \|\{s_{QP}\}_{QP}\|_{s\kappa(K',M')} \|\{s^{[2]}_{QP}\}_{QP}\|_{s\kappa(K',M')} \|m^{[2]}_Q\|_{\mathcal{M}_{K',M'}}\\ &\quad + C\|\{t_{QP}\}_{QP}\|_{s\kappa(K',M')} \|\{t^{[2]}_{QP}\}_{QP}\|_{s\kappa(K',M')} \|n^{[2]}_Q\|_{\mathcal{M}_{K',M'}}\\ &\leq C|\det A|^{2\eta\varepsilon} \|\{\varphi\}_Q\|^2_{\mathcal{M}_{K'+a_n,M'}} \|\{\phi\}_P\|_{\mathcal{M}_{K',M'}} \|\varphi_Q\|^2_{\mathcal{M}_{K',M'}}.\end{aligned}$$

Similarly, for any $i > 0$, we have

$$\|\{(I - |\det B|\mathcal{F}_\eta)^i \varphi_P\}_P\|_{\mathcal{M}_{L,\tilde{M}}} \leq C^i |\det A|^{\eta\varepsilon i} \|\{\varphi_Q\}_Q\|^{2i}_{\mathcal{M}_{K'+a_n,M'}} \|\{\phi\}_P\|_{\mathcal{M}_{K',M'}}.$$

We know that, if the operator norm of $I - |\det B|\mathcal{F}_\eta$ is strictly less than one, we have

$$\mathcal{F}_\eta^{-1} = |\det B| \sum_{i=0}^{\infty} (I - |\det B|\mathcal{F}_\eta)^i$$

by the Neumann series.

Hence, we can estimate the operator norm of $\mathcal{F}_\eta^{-1}$,

$$\begin{aligned}\|\{\mathcal{F}_\eta^{-1}\varphi_P\}_P\|_{\mathcal{M}_{L,\tilde{M}}} &\leq |\det B| \sum_{i=0}^{\infty} \|\{(I - |\det B|\mathcal{F}_\eta)^i \varphi_P\}_P\|_{\mathcal{M}_{L,\tilde{M}}}\\ &\leq |\det B| \sum_{i=0}^{\infty} C^i |\det A|^{\eta\varepsilon i} \|\{\varphi_P\}_P\|^{2i}_{\mathcal{M}_{K'+a_n,M'}} \|\{\phi\}_P\|_{\mathcal{M}_{K',M'}}.\end{aligned}$$

Hence, if we choose an η_0 such that

$$|\det A|^{\eta_0 \varepsilon} < C^{-1} \|\{\varphi_P\}_P\|^{-2}_{\mathcal{M}_{K'+a_n,M'}} \|\{\phi\}_P\|^{-2}_{\mathcal{M}_{K',M'}},$$

then $\mathcal{F}_\eta$ is invertible because $\mathcal{M}_{L,\tilde{M}}$ is a Banach space. □

7. Convergence of the frame operator

We postpone the proof until we establish the following results. We study the kernel $T(x,y)$ defined by,

$$T(x,y) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \phi_{j,k}(x)\overline{\varphi_{j,k}(y)}.$$

Define

$$\Phi(x,y) = \sum_{k\in\mathbb{Z}^n} \phi_{0,k}(x)\overline{\varphi_{0,k}(y)} = \sum_{k\in\mathbb{Z}^n} \phi_{0,k}(x)\overline{\varphi(y - Bk)}.$$

Since $\mathcal{D}_A\mathcal{F} = \mathcal{F}\mathcal{D}_A$ (see discussion preceding Theorem 4.3), we can write

$$|\det A|^j \Phi(A^j x, A^j y) = |\det A|^j \sum_{k\in\mathbb{Z}^n} \phi_{0,k}(A^j x)\overline{\varphi_{0,k}(A^j y)} = \sum_{k\in\mathbb{Z}^n} \phi_{j,k}(x)\overline{\varphi_{j,k}(y)}.$$

Therefore, we have $T(x,y) = \sum_{j\in\mathbb{Z}} |\det A|^j \Phi(A^j x, A^j y)$. By Theorem 4.2, we may assume that $\{\phi_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ is a family of strong molecules of order N, J for any N, J satisfying $1 + K > N > 1 + J$ and $\phi_{j,k}$ has the same center as $\varphi_{j,k}$.

Lemma 7.1

Let φ,ϕ be as above, we then have, for any N satisfying $1 + K > N$,

$$|\Phi(x,y)| \le \frac{C\|\{\varphi_P\}_P\|_{\mathcal{M}_{M,K}} \|\{\phi_P\}_P\|_{\mathcal{M}_{N,J}}}{(1 + \rho_A(x - y))^N}$$

for a constant $C > 0$ depending only on A.

Proof. This is a simple consequence of Lemma 6.1 with the size conditions

$$|\varphi_{0,k}(y)| \le \|\{\varphi_P\}_P\|_{\mathcal{M}_{M,K}} (1 + \rho_A(y - k))^{-N} \quad \text{and}$$
$$|\phi_{0,k}(x)| \le \|\{\phi_P\}_P\|_{\mathcal{M}_{N,J}} (1 + \rho_A(x - k))^{-N}. \ \square$$

Define the operator,

$$\mathcal{P}f(x) = \sum_{j<0} \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle \phi_{j,k}(x) = \int_{\mathbb{R}^n} \left(\sum_{j<0} \sum_{k\in\mathbb{Z}^n} \phi_{j,k}(x)\overline{\varphi_{j,k}(y)} \right) f(y)dy$$

$$\mathcal{Q}f(x) = \sum_{j\ge 0} \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{j,k}\rangle \phi_{j,k}(x) = \int_{\mathbb{R}^n} \left(\sum_{j\ge 0} \sum_{k\in\mathbb{Z}^n} \phi_{j,k}(x)\overline{\varphi_{j,k}(y)} \right) f(y)dy.$$

Denote the kernels of the operators $\mathcal{P}$ and $\mathcal{Q}$ by $P(x,y)$ and $Q(x,y)$ respectively, so that we have

$$P(x,y) = \sum_{j<0} |\det A|^j \Phi(A^j x, A^j y) \quad \text{and}$$
$$Q(x,y) = \sum_{j\geq 0} |\det A|^j \Phi(A^j x, A^j y).$$

Next, we are going to show $P(x,y)$ is well-defined on $\mathbb{R}^n \times \mathbb{R}^n$ and $Q(x,y)$ is well-defined in $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\}$.

Theorem 7.2

The sum defining $Q(x,y)$ converges uniformly to a continuous function bounded by $C\rho_A(x-y)^{-N}$ with constant $C>0$, and the sum defining $P(x,y)$ converges uniformly to a bounded continuous function on sets at a positive distance from the diagonal. Furthermore,

$$P(x,y) + Q(x,y) = 0 \qquad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\}.$$

Proof. For any $m \in \mathbb{Z}$, let

$$P_m(x,y) = \sum_{-m\leq j<0} |\det A|^j \Phi(A^j x, A^j y) \text{ and}$$
$$Q_m(x,y) = \sum_{0\leq j\leq m} |\det A|^j \Phi(A^j x, A^j y).$$

Hence,

$$\begin{aligned}|P_m(x,y)| &\leq \sum_{-m\leq j<0} |\det A|^j |\Phi(A^j x, A^j y)| \leq C \sum_{-m\leq j<0} \frac{|\det A|^j}{(1+\rho_A(A^j(x-y)))^N} \\ &\leq C \sum_{-m\leq j<0} |\det A|^j \leq C\end{aligned}$$

uniformly on $\mathbb{R}^n \times \mathbb{R}^n$, for a constant $C>0$. Since m is arbitrary, we show that $P_m(x,y)$ converges uniformly to $P(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n$ and $|P(x,y)| \leq C$. For $Q_m(x,y)$, we have

$$\begin{aligned}|Q_m(x,y)| &\leq \sum_{0\leq j\leq m} |\det A|^j |\Phi(A^j x, A^j y)| \leq C \sum_{0\leq j\leq m} \frac{|\det A|^j}{(1+\rho_A(A^j(x-y)))^N} \\ &\leq C \sum_{0\leq j\leq m} \frac{|\det A|^{-jN}}{(|\det A|^{-j}+\rho_A(x-y))^N} \\ &\leq C \sum_{0\leq j\leq m} |\det A|^{-jN} \rho_A(x-y)^N \\ &\leq C\rho_A(x-y)^{-N}.\end{aligned}$$

Therefore $Q_m(x,y)$ converges uniformly to $Q(x,y)$ on any compact subset of $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\}$ and

$$|Q(x,y)| \leq C\rho_A(x-y)^{-N}.$$

Finally, we only need to show that

$$P(x,y) + Q(x,y) = 0 \qquad x \neq y.$$

Let $B_1, B_2 \subseteq \mathbb{R}^n$ be closed balls with $B_1 \bigcap B_2 = \emptyset$. Let $f_1(x), f_2(x)$ be bounded non-negative functions supported in B_1 and B_2 respectively. Let

$$T_m(x,y) = P(x,y) + Q_m(x,y).$$

By the assumption that $\{\varphi_{j,k}\}$ and $\{\phi_{j,k}\}$ are a frame and its dual frame for $L^2(\mathbb{R}^n)$, we have

$$\int_{R^n} T_m(x,y) f_2(y) dy \longrightarrow f_2(y)$$

on $L^2(\mathbb{R}^n)$ as $m \to \infty$. Therefore,

$$\begin{aligned}\iint f_1(x) T(x,y) f_2(y) dy dx &= \lim_{m\to\infty} \iint f_1(x) T_m(x,y) f_2(y) dy dx \\ &= \lim_{m\to\infty} \iint f_1(x) f_2(y) dy dx = 0\end{aligned}$$

because $\int T_m(x,y) f_2(y) dy \to f_2(y)$ in $L^2(\mathbb{R}^n)$.

Since $f_1(x)$ and $f_2(x)$ are arbitrary functions on B_1 and B_2 and we may pick $B_1, B_2 \subset \mathbb{R}^n$ to be disjoint closed balls, we have

$$T(x,y) = 0 \qquad x \neq y.$$

That is,

$$P(x,y) + Q(x,y) = 0 \qquad x \neq y. \ \square$$

Corollary 7.3

Let $P(x,y)$ be the above function, then,

$$|P(x,y)| \leq \frac{C}{(1+\rho_A(x-y))^N}.$$

Proof. We already know $|P(x,y)| \leq C$ on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$|Q(x,y)| \leq \frac{C}{\rho_A(x-y)^N} \qquad \text{for} \quad x \neq y.$$

Since

$$P(x,y) = -Q(x,y) \qquad \text{on} \quad x \neq y,$$

we have

$$|P(x,y)| = |Q(x,y)| \le \frac{C}{\rho_A(x-y)^N} \qquad \text{on} \quad x \ne y.$$

Combining this with $|P(x,y)| \le C$, we have

$$|P(x,y)| \le \frac{C}{(1+\rho_A(x-y))^N} \qquad \text{on} \quad \mathbb{R}^n \times \mathbb{R}^n. \ \square$$

Recall that the operator $\mathcal{T}_m$, for any $m \in \mathbb{Z}$, satisfies

$$\mathcal{T}_m f(x) = \sum_{j \le m} \sum_{k \in \mathbb{Z}^n} \langle f, \varphi_{j,k} \rangle \phi_{j,k}(x).$$

Then the kernel of $\mathcal{T}_m$ is

$$T_m(x,y) = \sum_{j \le m} |\det A|^j \Phi(A^j x, A^j y) = |\det A|^m P(A^m x, A^m y) \qquad \forall m \in \mathbb{N}.$$

In order to prove Theorem 4.3, we need the following lemma:

Lemma 7.4

There is a constant $c \in \mathbb{R}$ such that, for every $x \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} P(x,y) dy = c.$$

Proof. Let $a(x) = \int_{\mathbb{R}^n} P(x,y)dy$. We want to show $a(x) \equiv c$ for a constant c. We have already shown that $P(x,y)$ is the uniform limit of continuous functions, it follows that $a(x)$ is continuous. We are going to show that $a(Ax) = a(x)$. Since

$$P(x,y) + \Phi(x,y) = \sum_{j \le 0} |\det A|^j \Phi(A^j x, A^j y) = |\det A| P(Ax, Ay)$$

and both $P(x,y)$ and $\Phi(x,y)$ are majorized by L^1 functions. So we may take the integral with respect to y on both sides and we see that

$$\int_{\mathbb{R}^n} (P(x,y) + \Phi(x,y))\, dy = \int_{\mathbb{R}^n} |\det A| P(Ax, Ay) dy = \int_{\mathbb{R}^n} P(Ax, y) dy$$

by the change of variable $Ay \to y$. Since

$$\int_{\mathbb{R}^n} \Phi(x,y) dy = \sum_{k \in \mathbb{Z}^n} \phi_{j,k}(x) \int_{\mathbb{R}^n} \overline{\varphi(y-k)} dy$$

and $\varphi(x)$ satisfies the vanishing moment conditions (4.3); in particular $\varphi(x)$ has the first vanishing moment condition. Therefore,

$$\int_{\mathbb{R}^n} \Phi(x,y) dy = 0.$$

Hence,

$$a(x) = \int_{\mathbb{R}^n} P(x,y)dy = \int_{\mathbb{R}^n} P(Ax,y)dy = a(Ax).$$

Furthermore, we have

$$a(x) = a(A^j x) \qquad \forall j \in \mathbb{Z}.$$

Fix a $x \in \mathbb{R}^n$ and let $j \to -\infty$ in the above identity, we see that $a(x) = a(0)$, that is, $a(x)$ is a constant function. □

Having make these preparations, we are now ready to prove Theorem 4.3.

Proof of Theorem 4.3: By the definition of $\mathcal{T}_j$, we have

$$\begin{aligned} |\mathcal{T}_j f(x)| &= \left| \int_{\mathbb{R}^n} |\det A|^j P(A^j x, A^j y) f(y) dy \right| \\ &\leq C \int_{\mathbb{R}^n} \frac{|\det A|^j}{(1+\rho_A(A^j(x-y)))^N} |f(y)| dy \end{aligned}$$

Therefore,

$$\begin{aligned} |\mathcal{T}_j f(x)| \leq C \Bigg(&\int_{\rho_A(A^j(x-y)) \leq 1} |\det A|^j |f(y)| dy \\ &+ \sum_{k=2}^{\infty} \int_{|\det A|^{k-1} \leq \rho_A(A^j(x-y)) \leq |\det A|^k} \frac{|\det A|^j}{\rho_A(A^j(x-y))^N} |f(y)| dy \Bigg) \\ \leq\ & C |\det A|^{-j} \int_{\rho_A((x-y)) \leq |\det A|^{-j}} |f(y)| dy \\ &+ C \sum_{k=2}^{\infty} |\det A|^{-k(N-1)} \int_{\rho_A((x-y)) \leq |\det A|^{k-j}} \frac{1}{|\det A|^{k-j}} |f(y)| dy \end{aligned}$$

and hence,

$$|\mathcal{T}_j f(x)| \leq C(\mathrm{M}_{\rho_A}(f)(x) + \sum_{k=2}^{\infty} |\det A|^{-k(N-1)} \mathrm{M}_{\rho_A}(f)(x)).$$

Summing over k, by the L^1-majorization of $P(x,y)$, we have

$$|\mathcal{T}_j f(x)| \leq C \mathrm{M}_{\rho_A}(f)(x)$$

for a constant $C > 0$ depending only on A. It is easy to see that $\mathcal{T}_j f(x) \to c f(x)$ (c is the constant in Lemma 7.4) if $f(x)$ is a continuous function. Since the continuous functions are dense in L^p and $\mathcal{T}_j f(x)$ is majorized by the maximal function $\mathrm{M}_{\rho_A}(f)(x)$, it follows that $\mathcal{T}_j f(x)$ converges to $c f(x)$ for any $f \in L^p$. On the other hand, since $\{\varphi_{j,k}\}$ is a frame and $\{\phi_{j,k}\}$ is its dual-frame, we have

$$f = \sum_{j,k} \langle f, \varphi_{j,k} \rangle \phi_{j,k},$$

therefore $c = 1$ and, hence, obtain our desired result. □

Remark 7.1. Notice that $c = 1$ is a consequence of the assumption that $\{\varphi_{j,k}\}$ is a frame and $\{\phi_{j,k}\}$ is its dual-frame. If we define the operator $\mathcal{P}$ and $\mathcal{Q}$ by $\{c\varphi_{j,k}\}$ and $\{\phi_{j,k}\}$ with constant $c \neq 1$, then our argument is still valid but, in this case, the convergence $\mathcal{T}_j f(x) \to cf(x)$ with $c \neq 1$.

8. Appendix

Lemma 8.1

Suppose A is an expansive matrix, $R > 1$, $i, j \in \mathbb{Z}$, $i \geq j$ and $x_0 \in \mathbb{R}^n$. Suppose $g, h \in L^1(\mathbb{R}^n)$ satisfy

$$|g(x)| \leq |\det A|^{j/2} \left(1 + \rho_A(A^j x)\right)^{-R} \tag{8.1}$$

and

$$|h(x)| \leq |\det A|^{i/2} \left(1 + \rho_A(A^i(x - x_0))\right)^{-R}, \tag{8.2}$$

then

$$|g * h(x)| \leq C|\det A|^{-(i-j)/2} \left(1 + \rho_A(A^j(x - x_0))\right)^{-R} \tag{8.3}$$

for a constant $C > 0$.

Proof. It is a simple modification of the one for the isotropic case [6]. □

Lemma 8.2

Suppose A is an expansive matrix, $L \in \mathbb{Z}_+$, $\delta > 0$, $R > L + \delta + 1 + a_n$, $i, j \in \mathbb{Z}$, $i \geq j$, and $x_0 \in \mathbb{R}^n$. Suppose $g, h \in L^1(\mathbb{R}^n)$ and they satisfy

$$\left|\partial^\gamma (g(A^{-j}x))\right| \leq |\det A|^{j/2} \left(1 + \rho_A(x)\right)^{-R} \tag{8.4}$$

if $\langle a, \gamma \rangle \leq L + \delta + a_n$,

$$\left|h(A^{-i}x)\right| \leq |\det A|^{i/2} \left(1 + \rho_A(x - A^i x_0)\right)^{-R}; \tag{8.5}$$

$$\int x^\gamma h(x) dx = 0 \quad \textit{if} \quad \langle a, \gamma \rangle \leq L + \delta. \tag{8.6}$$

Then for any $0 < \triangle < \min(\delta, a_n)$, there exists a constant $C_\triangle > 0$, depending only on $\triangle$, such that,

$$|g * h(x)| \leq C_\triangle |\det A|^{-(i-j)(L+\epsilon_0+1/2)} \left(1 + \rho_A(A^j(x - x_0))\right)^{-R} \tag{8.7}$$

where $\epsilon_0 = \delta - \triangle$.

Proof. In order to have a better understanding of the following proof, the reader is recommended to read the corresponding estimates for the isotropic case, [6] p.150–152. The proof for the anisotropic case is much more tedious than the isotropic case. We provide a detailed proof for the completeness for the anisotropic case and indicate the difference between the anisotropic and the isotropic cases. The first difference is found on the Taylor expansion:

Theorem 8.3 (Anisotropic Taylor's series).

Let $a_i > 0$, $1 \le i \le n$ satisfying $a_i \le a_j$ if $i \le j$ and $a = (a_1, a_2, \cdots, a_n)$. For any $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{Z}^n$, recall that $\langle a, \gamma\rangle$ denotes the "inner product" of a and γ, that is, $\langle a, \gamma\rangle = \sum_{i=1}^n a_i\gamma_i$. Then, for any $L > 0$, $y \in \mathbb{R}^n$ and $f(x)$ such that $\partial^\gamma f$ exist for $\langle a, \gamma\rangle \le L + \delta + a_n$, there exists a collection $\{\xi_\gamma\}_{L<\langle a,\gamma\rangle \le L+a_n}$, $\xi_\gamma \in \mathbb{R}^n$, satisfying $|x - \xi_\gamma| \le |x - y|$ and a collection of constants $\{C_\gamma\}_{\langle a,\gamma\rangle \le L+a_n}$ independent on $f(x)$ and $y \in \mathbb{R}^n$ such that

$$f(x) = \sum_{\langle a,\gamma\rangle \le L} C_\gamma (x-y)^\gamma (\partial^\gamma f)(y) + R_L$$

where

$$R_L = \sum_{L<\langle a,\gamma\rangle \le L+a_n} C_\gamma (x-y)^\gamma (\partial^\gamma f)(\xi_\gamma).$$

Sketch of the proof. First, for any fixed $x_2, \cdots x_n$, we consider $f(x_1, x_2, \cdots x_n)$ as a function of x_1 and represent it by the one-variable Taylor series to the order $[L/a_1]$. That is, for any $y_1 \in \mathbb{R}$, there exist a $\xi_1 \in \mathbb{R}$ with $|x_1 - \xi_1| \le |x_1 - y_1|$ such that

$$f(x_1, x_2, \cdots, x_n) = \sum_{|\gamma_1| \le [L/a_1]} \frac{1}{\gamma_1!}(x_1 - y_1)^{\gamma_1} (\partial_{x_1}^{\gamma_1} f)(y_1, x_2, \ldots, x_n) + R \tag{8.8}$$

where

$$R = \frac{1}{([L/a_1]+1)!}(x_1 - y_1)^{[L/a_1]+1} (\partial_{x_1}^{[L/a_1]+1} f)(\xi_1, x_2, \ldots, x_n).$$

Therefore, for $\gamma = ([L/a_1]+1, 0, \cdots, 0)$, $\xi_\gamma = (\xi_1, x_2, \ldots, x_n)$ is the variable for the expansion for R_L.

Next, for each term $(\partial_{x_1}^{\gamma_1} f)(y_1, x_2, \ldots, x_n)$ on the right hand side of (8.8), we consider it as a function of x_2 and expand it by the one-variable Taylor series with order $[(L - a_1\gamma_1)/a_2]$. Hence, for any $y_2 \in \mathbb{R}$, we have

$$\begin{aligned}&(\partial_{x_1}^{\gamma_1} f)(x_1, x_2, \cdots, x_n)\\ &= \sum_{|\gamma_2| \le [(L-a_1\gamma_1)/a_2]} \frac{1}{\gamma_2!}(x_2 - y_2)^{\gamma_2} (\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2} f)(y_1, y_2, x_3, \ldots, x_n) + R\end{aligned} \tag{8.9}$$

where

$$\begin{aligned}R = &\frac{1}{([(L-a_1\gamma_1)/a_2]+1)!}(x_2 - y_2)^{[(L-a_1\gamma_1)/a_2]+1}\\ &\times (\partial_{x_1}^{\gamma_1}\partial_{x_2}^{[(L-a_1\gamma_1)/a_2]+1} f)(y_1, \xi_2, x_3, \ldots, x_n)\end{aligned}$$

for some $\xi_2 \in \mathbb{R}$ with $|\xi_2 - x_2| \leq |x_2 - y_2|$. Hence, for any $\gamma_1 \leq [L/a_1]$, if $\gamma = (\gamma_1, [(L - a_1\gamma_1)/a_2] + 1, 0, \cdots, 0)$, then $\xi_\gamma = (y_1, \xi_2, \ldots, x_n)$.

We expand each term $(\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2} f)(y_1, y_2, x_3, \ldots, x_n)$ on the right hand side of (8.9) by Taylor's series with respect to the variable x_3 and repeat the above procedure until we reach the variable x_n. This, then, give us our desired result. $\square$

The proof of Lemma 8.2 is based on the decomposition of $\mathbb{R}^n$ into three domains, D_1, D_2 and D_3. On D_1 the estimate of the remainder term in the Taylor expansion provides the desired result. On D_2 and D_3, the estimates rely on the decay satisfied by molecules.

Using translation and dilation, we may assume $j = 0$ and $x_0 = 0$. Let $D_1 = \{y \in \mathbb{R}^n : \rho_A(y - x) < 1\}$, $D_2 = \{y \in \mathbb{R}^n : \rho_A(y - x) \geq 1$ and $\rho_A(y) \leq \rho_A(x)/2H\}$ and $D_3 = \{y \in \mathbb{R}^n : \rho_A(y - x) \geq 1 \quad$ and $\quad \rho_A(y) > \rho_A(x)/2H\}$ where H is the constant of the quasi-subadditivity inequality. Then, for any collection of constants C_γ, using the vanishing moment conditions satisfied by $h(x)$, we have

$$\begin{aligned} |g * h(x)| &\leq \int_{\mathbb{R}^n} |g(y) - \sum_{\langle a,\gamma\rangle \leq L+\delta} C_\gamma (y - x)^\gamma (\partial^\gamma g)(x)||h(x - y)|dy \\ &= \int_{D_1} + \int_{D_2} + \int_{D_3}. \end{aligned}$$

Case 1: Estimate on D_1

For $y \in D_1$, using the Anisotropic Taylor expansion, Theorem 8.3, to the order $L + \delta$, we have a collection of constants C_γ such that

$$g(y) - \sum_{\langle a,\gamma\rangle \leq L+\delta} C_\gamma (y - x)^\gamma (\partial^\gamma g)(x) = \sum_{L+\delta < \langle a,\gamma\rangle \leq L+\delta+a_n} C_\gamma (y - x)^\gamma (\partial^\gamma g)(w_\gamma) \quad (8.10)$$

for some $w_\gamma \in \mathbb{R}^n$ that satisfies $\rho_A(w_\gamma - x) \leq C$ for a constant $C > 0$. We have $|w_\gamma - x| \leq |x - y|$ by the Anisotropic Taylor's Theorem, therefore, by Lemma 2.2, we have

$$\begin{aligned} \rho_A(w_\gamma - x) &\leq C \max(|x - w_\gamma|^\zeta, |x - w_\gamma|^\tau) \leq C \max(|x - y|^\zeta, |x - y|^\tau) \\ &\leq C \max(\rho_A(x - y)^{\zeta\tau}, \rho_A(x - y)^{\tau^2}) \leq C \end{aligned}$$

because $\rho_A(x - y) < 1$.

By (2.4) or (3.6), for any $\triangle > 0$, there is a constant $C_\triangle > 0$ such that

$$|x^\gamma| \leq C_\triangle \rho_A(x)^{\langle a,\gamma\rangle - \triangle} \quad \text{for any} \quad \rho_A(x) < 1 \quad \text{and} \quad \langle a, \gamma\rangle \leq L + \delta + a_n.$$

Therefore, we may replace $|(y - x)^\gamma|$ by $C_\triangle \rho_A(x - y)^{\langle a,\gamma\rangle - \triangle}$. Furthermore, we can replace $\rho_A(w_\gamma)$ by $\rho_A(x)$ because

$$\rho_A(x) \leq H[\rho_A(x - w_\gamma) + \rho_A(w_\gamma)] \leq H[C + \rho_A(w_\gamma)] \leq C[1 + \rho_A(w_\gamma)].$$

Therefore, for any $\triangle > 0$, there exists $C_\triangle > 0$ such that,

$$|g(y) - \sum_{\langle a,\gamma\rangle \le L+\delta} C_\gamma (y-x)^\gamma (\partial^\gamma g)(x)| \le C_\triangle \Big(\sum_{\substack{L+\delta<\langle a,\gamma\rangle \\ \langle a,\gamma\rangle \le L+\delta+a_n}} \rho_A(x-y)^{\langle a,\gamma\rangle - \triangle}(1+\rho_A(x))^{-R} \Big)$$

by conditions (8.4). Hence, we have

$$\int_{D_1} \le C_\triangle |\det A|^{i/2}(1+\rho_A(x))^{-R}\Big[\sum_{\substack{L+\delta<\langle a,\gamma\rangle \\ \langle a,\gamma\rangle \le L+\delta+a_n}} \int_{D_1} \rho_A(x-y)^{\langle a,\gamma\rangle-\triangle}(1+\rho_A(A^i(x-y)))^{-R} dy\Big].$$

Furthermore, replacing the domain of integration by $\mathbb{R}^n$ and using a change of variable in the integral, we have

$$\int_{D_1} \le C_\triangle |\det A|^{-i/2}(1+\rho_A(x))^{-R}\Big[\sum_{\substack{L+\delta<\langle a,\gamma\rangle \\ \langle a,\gamma\rangle \le L+\delta+a_n}} \int_{\mathbb{R}^n} \rho_A(A^{-i}(x-y))^{\langle a,\gamma\rangle-\triangle}(1+\rho_A(x-y))^{-R} dy\Big].$$

Therefore,

$$\begin{aligned} \int_{D_1} &\le C_\triangle (1+\rho_A(x))^{-R}\Big[\sum_{\substack{L+\delta<\langle a,\gamma\rangle \\ \langle a,\gamma\rangle \le L+\delta+a_n}} |\det A|^{-i/2-i\langle a,\gamma\rangle+i\triangle} \\ &\qquad \times \int_{\mathbb{R}^n} \rho_A(x-y)^{\langle a,\gamma\rangle-\triangle}(1+\rho_A(x-y))^{-R} dy\Big] \\ &\le C_\triangle |\det A|^{-i(L+\delta-\triangle+1/2)}(1+\rho_A(x))^{-R} \end{aligned}$$

because $R > L+\delta+a_n+1$, $i \ge 0$.

Case 2: Estimate on D_2

For $y \in D_2$, we have

$$\rho_A(x-y) \ge \rho_A(x)/H - \rho_A(y) \ge \rho_A(x)/H - \rho_A(x)/2H = \rho_A(x)/2H.$$

On the other hand,

$$\rho_A(x-y) \le H\big(\rho_A(x)+\rho_A(y)\big) \le H\rho_A(x) + \rho_A(x)/2 = (H+1/2)\rho_A(x);$$

therefore,

$$\begin{aligned} \int_{D_2} \le C|\det A|^{i/2} \int_{D_2} \Big[&(1+\rho_A(y)|)^{-R}\big(|\det A|^i \rho_A(x-y)\big)^{-R} \\ &+ \sum_{\langle a,\gamma\rangle \le L+\delta} \frac{|(x-y)^\gamma|}{(1+\rho_A(x))^R}\big(|\det A|^i \rho_A(x-y)\big)^{-R}\Big] dy \end{aligned}$$

and, hence, replacing $|(y-x)^\gamma|$ by $C_\triangle \rho_A(x-y)^{\langle a,\gamma\rangle+\triangle}$ (since $\rho_A(x-y) \ge 1$), we have

$$\int_{D_2} \leq C_\triangle |\det A|^{i/2}\Big[\int_{D_2}(1+\rho_A(y))^{-R}\big(|\det A|^i\rho_A(x-y)\big)^{-R}$$
$$+\sum_{\langle a,\gamma\rangle\leq L+\delta}\frac{\rho_A(x-y)^{\langle a,\gamma\rangle+\triangle}}{(1+\rho_A(x))^R}\big(|\det A|^i\rho_A(x)\big)^{-R}\Big]dy,$$

where we use $\rho_A(x-y)\sim\rho_A(x)$ for the last term. Because $\rho_A(x)\sim\rho_A(x-y)\geq 1$, we have

$$\int_{D_2}\leq C_\triangle|\det A|^{-i(R-1/2)}(1+\rho_A(x))^{-R}$$
$$\times\left[\int\frac{1}{(1+\rho_A(y))^R}dy+\sum_{\langle a,\gamma\rangle\leq L+\delta}\frac{\rho_A(x)^{\langle a,\gamma\rangle+\triangle}}{(1+\rho_A(x))^R}\int_{\rho_A(y)\leq\rho_A(x)/2H}dy\right]$$
$$\leq C_\triangle|\det A|^{-i(R-1/2)}(1+\rho_A(x))^{-R}\leq C_\triangle|\det A|^{-i(L+\delta+1/2)}(1+\rho_A(x))^{-R}$$

as needed. Since $R>L+\delta+a_n+1$, and $a_n>\triangle$, hence, $\frac{\rho_A(x)^{\langle a,\gamma\rangle+\triangle+1}}{(1+\rho_A(x))^R}\leq 1$ for those γ satisfies $\langle a,\gamma\rangle\leq L+\delta$.

Case 3: Estimate on D_3

For $y\in D_3$, we have $\rho_A(y)\geq\rho_A(x)/2H$ and, hence,

$$\int_{D_3}\leq C\int_{D_3}\left[\frac{1}{(1+\rho_A(y))^R}+\sum_{\langle a,\gamma\rangle\leq L+\delta}\frac{|(x-y)^\gamma}{(1+\rho_A(x))^R}\right]$$
$$\times|\det A|^{i/2}\big(1+|\det A|^i\rho_A(x-y)\big)^{-R}dy$$
$$\leq C_\triangle\int_{D_3}\left[\frac{1}{(1+\rho_A(y))^R}+\sum_{\langle a,\gamma\rangle\leq L+\delta}\frac{\rho_A(x-y)^{\langle a,\gamma\rangle+\triangle}}{(1+\rho_A(x))^R}\right]$$
$$\times|\det A|^{i/2}\big(|\det A|^i\rho_A(x-y)\big)^{-R}dy$$
$$\leq C_\triangle|\det A|^{-i(R-1/2)}(1+\rho_A(x))^{-R}$$
$$\times\int_{1\leq\rho_A(x-y)}\left(\sum_{\langle a,\gamma\rangle\leq L+\delta}\rho_A(x-y)^{\langle a,\gamma\rangle-R+\triangle}\right)dy.$$

Since $\triangle<\min(\delta,a_n)$, $\gamma\in\mathbb{Z}^n$, above, satisfies $\langle a,\gamma\rangle\leq L+\delta$, $R>L+\delta+a_n+1$ and $i\geq 0$ we have,

$$\int_{D_3}\leq C_\triangle|\det A|^{-i(R-1/2)}(1+\rho_A(x))^{-R}\leq C_\triangle|\det A|^{-i(L+\delta+1/2)}(1+\rho_A(x))^{-R}.\ \Box$$

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