

Fully absolutely summing and Hilbert-Schmidt multilinear mappings

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ABSTRACT

The space of the fully absolutely $(r; r_1, \dots, r_n)$ -summing n -linear mappings between Banach spaces is introduced along with a natural (quasi-)norm on it. If $r, r_k \in [1, +\infty]$, $k = 1, \dots, n$, this space is characterized as the topological dual of a space of virtually nuclear mappings. Other examples and properties are considered and a relationship with a topological tensor product is established. For Hilbert spaces and $r = r_1 = \dots = r_n \in [2, +\infty[$ this space is isomorphic to the space of the Hilbert-Schmidt multilinear mappings.

1. Introduction

In [13], after considering the space of the absolutely $(r; r_1, \dots, r_n)$ -summing n -linear functionals on Banach spaces, Pietsch asks if, for $n \geq 3$, it coincides with the space of the Hilbert-Schmidt n -linear functionals on Hilbert spaces for some values of r, r_1, \dots, r_n . In Matos [11], we prove that this question of Pietsch has negative answer. We also prove there that the answer to the same question, for $n \geq 2$, when infinite dimensional valued mappings are considered, is also negative. These results lead us to a natural question: can we consider a proper subspace of the space of the absolutely $(r; r_1, \dots, r_n)$ -summing n -linear mappings that coincides with the Hilbert-Schmidt multilinear mappings on Hilbert spaces for some r, r_1, \dots, r_n ? A natural condition required for these subspaces is that, in the linear case, they should coincide with the usual absolutely $(r; s)$ -summing linear operators. This question gave us the motivation for the introduction of the space of the fully absolutely $(r; r_1, \dots, r_n)$ -summing n -linear mappings between Banach spaces. This space is endowed with a

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natural norm, if $r \geq 1$, or an r -norm, if $r \in]0, 1[$. We show that it is isomorphic to the space of the n -linear Hilbert-Schmidt mappings, when $r = r_1 = \dots = r_n \in [2, +\infty[$ (see Section 5). These mappings are considered in Section 2 along with several examples and properties.

In Section 3 we consider Banach spaces E_1, \dots, E_n, F and endow $E_1 \otimes \dots \otimes E_n \otimes F$ with a (quasi-)norm in such a way that its topological dual is isometric to the space of the fully absolutely $(r; r_1, \dots, r_n)$ -summing n -linear mappings from $E_1 \times \dots \times E_n$ into F' , when $r \in [1, +\infty]$.

In Section 4 we study the virtually $(r; r_1, \dots, r_n)$ -nuclear n -linear mappings from $E_1 \times \dots \times E_n$ into F . If E'_1, \dots, E'_n have the bounded approximation property and $r, r_1, \dots, r_n \in [1, +\infty]$, we show that the vector space of these mappings, endowed with a natural linear topology, has its topological dual isometric to the space of all absolutely $(r'; r'_1, \dots, r'_n)$ -summing mappings from $E'_1 \otimes \dots \otimes E'_n$ into F' . Here, as usual, if $r \in [1, +\infty]$, r' is the element of $[1, +\infty]$ such that $1/r + 1/r' = 1$. This result is analogous to the connection between absolutely summing n -linear mappings and multilinear mappings of nuclear type proved in [10].

In Section 5 we study the space of the n -linear Hilbert-Schmidt mappings between Hilbert spaces, its properties and, as we already mentioned, its relationship with spaces of fully absolutely summing mappings. The multilinear Hilbert-Schmidt mappings were introduced by Dwyer in his doctoral dissertation [4].

For results on linear operators between Banach spaces there are some very good texts. We mention Pietsch [14], Defant-Floret [2] and Diestel-Jarchow-Tonge [3].

Now, we fix the notations used in this paper. For Banach spaces E_1, \dots, E_n and F over \mathbb{K} (either \mathbb{R} or \mathbb{C}), we denote by $\mathcal{L}(E_1, \dots, E_n; F)$ the Banach space of all continuous n -linear mappings from $E_1 \times \dots \times E_n$ into F , under the norm

$$\|T\| = \sup_{\substack{x_k \in B_{E_k} \\ k=1, \dots, n}} \|T(x_1, \dots, x_n)\|,$$

where B_{E_k} denotes the closed unit ball of E_k centered at 0. If ϕ_k is in the topological dual E'_k of E_k , $k = 1, \dots, n$, and $b \in F$, we define $\phi_1 \times \dots \times \phi_n b \in \mathcal{L}(E_1, \dots, E_n; F)$ by

$$\phi_1 \times \dots \times \phi_n b(x_1, \dots, x_n) = \phi_1(x_1) \dots \phi_n(x_n) b, \quad \forall x_k \in E_k, k = 1, \dots, n.$$

The set of all these mappings generates the vector space $\mathcal{L}_f(E_1, \dots, E_n; F)$ of the n -linear mappings of finite type.

If $r \in]0, +\infty[$, we denote by $\ell_r(\mathbb{N}^n; F)$ (or $\ell_r(\mathbb{N}^n)$, if $F = \mathbb{K}$) the vector space of all families $(y_j)_{j \in \mathbb{N}^n}$ of elements of F such that

$$\|(y_j)_{j \in \mathbb{N}^n}\|_r = \left(\sum_{j \in \mathbb{N}^n} \|y_j\|^r \right)^{1/r} < +\infty.$$

We observe that $\|\cdot\|_r$ is a norm (r -norm, if $r < 1$) on $\ell_r(\mathbb{N}^n; F)$ and defines a complete metrizable linear topology on it. We denote by $\ell_\infty(\mathbb{N}^n; F)$ (or $\ell_\infty(\mathbb{N}^n)$, if $F = \mathbb{K}$) the Banach space of all bounded families $(y_j)_{j \in \mathbb{N}^n}$ of elements of F , under the norm

$$\|(y_j)_{j \in \mathbb{N}^n}\|_\infty = \sup_{j \in \mathbb{N}^n} \|y_j\|.$$

The Banach subspace of $\ell_\infty(\mathbb{N}^n; F)$ of the families $(y_j)_{j \in \mathbb{N}^n}$ such that

$$\lim_{\substack{j_k \rightarrow \infty \\ k=1, \dots, n}} \|y_j\| = 0$$

is denoted by $c_0(\mathbb{N}^n; F)$ (or $c_0(\mathbb{N}^n)$, if $F = \mathbb{K}$). As usual, an element j of \mathbb{N}^n will be represented by (j_1, \dots, j_n) . We also consider finite families $(y_j)_{j \in \mathbb{N}_m^n}$ of elements of a Banach space. Here $\mathbb{N}_m = \{1, \dots, m\}$ and we apply the symbol $\|\cdot\|_r$ to these families as we have done in the non-finite case. If $n = 1$, it is usual to omit \mathbb{N}^n in all the preceding notations.

The vector space of all sequences $(y_j)_{j=1}^\infty$ of elements of F such that $(\phi(y_j))_{j=1}^\infty \in \ell_r$, for every $\phi \in F'$, is denoted by $\ell_r^w(F)$. This space is complete for the linear topology defined by the norm (r -norm, if $r < 1$)

$$\|(y_j)_{j=1}^\infty\|_{w,r} = \sup_{\phi \in B_{F'}} \|(\phi(y_j))_{j=1}^\infty\|_r.$$

2. Fully absolutely summing multilinear mappings

We recall the following concept introduced by Pietsch in [13] for scalar valued multilinear mappings

DEFINITION 2.1. For $r, r_1, \dots, r_n \in]0, +\infty]$, with $\frac{1}{r} \leq \frac{1}{r_1} + \dots + \frac{1}{r_n}$, a mapping $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is *absolutely $(r; r_1, \dots, r_n)$ -summing* if there is $C \geq 0$ such that

$$\|(T(x_i^1, \dots, x_i^n))_{i=1}^m\|_r \leq C \prod_{k=1}^n \|(x_i^k)_{i=1}^m\|_{w,r_k} \quad (1)$$

for every $m \in \mathbb{N}$, $x_i^k \in E_k$, $k = 1, \dots, n$ and $i = 1, \dots, m$.

We denote the vector space of all such mappings by $\mathcal{L}_{as}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ and the smallest of all C satisfying (1) by $\|T\|_{as, (r; r_1, \dots, r_n)}$. This defines a norm (r -norm, if $r < 1$) on $\mathcal{L}_{as}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$. It is easy to show that the topological vector space $(\mathcal{L}_{as}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F), \|\cdot\|_{as, (r; r_1, \dots, r_n)})$ is complete.

We introduce a more restrictive concept.

DEFINITION 2.2. For $r, r_1, \dots, r_n \in]0, +\infty]$, with $r \geq r_k$, $k = 1, \dots, n$, a mapping $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is *fully absolutely $(r; r_1, \dots, r_n)$ -summing* if there is $C \geq 0$ such that

$$\|(T(x_{j_1}^1, \dots, x_{j_n}^n))_{j \in \mathbb{N}_m^n}\|_r \leq C \prod_{k=1}^n \|(x_i^k)_{i=1}^m\|_{w, r_k} \quad (2)$$

for every $m \in \mathbb{N}$, $x_i^k \in E_k$, $k = 1, \dots, n$ and $i = 1, \dots, m$.

We denote the vector space of all such mappings by $\mathcal{L}_{fas}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ and the smallest of all C satisfying (2) by $\|T\|_{fas, (r; r_1, \dots, r_n)}$. This defines a norm (r -norm, if $r < 1$) on $\mathcal{L}_{fas}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ that makes it a complete metrizable topological vector space. It is clear that

$$\mathcal{L}_{fas}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F) \subset \mathcal{L}_{as}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$$

and

$$\|T\| \leq \|T\|_{as, (r; r_1, \dots, r_n)} \leq \|T\|_{fas, (r; r_1, \dots, r_n)},$$

for every $T \in \mathcal{L}_{fas}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$.

In order to simplify our writing, when $r_1 = \dots = r_n = s$ we replace $(r; r_1, \dots, r_n)$ by $(r; s)$ in the previous notations. If $r = s$ we replace $(r; r)$ by r and, in the case $r = 1$, we just omit $(1, 1)$.

A result of Defant and Voigt (see [1] for a proof) states that $\mathcal{L}_{as}(E_1, \dots, E_n; \mathbb{K}) = \mathcal{L}(E_1, \dots, E_n; \mathbb{K})$ isometrically.

EXAMPLES 2.3: (1) There is $T \in \mathcal{L}(c_0, c_0; \mathbb{K}) = \mathcal{L}_{as}(c_0, c_0; \mathbb{K})$ such that

$$\sum_{j,k=1}^{\infty} |T(e_j, e_k)| = +\infty,$$

for the canonical Schauder basis $(e_j)_{j=1}^{\infty}$ of c_0 (see [9]). Hence T cannot be fully absolutely summing.

(2) If E is an infinite dimensional Banach space, we fix an element $\phi \in E'$, $\phi \neq 0$, and define $T_\phi \in \mathcal{L}(E, E; E)$ by $T_\phi(x, y) = \phi(x)y$, for every $x, y \in E$.

(a) $T_\phi \in \mathcal{L}_{as}^{(r; r_1, r_2)}(E, E; E)$, if $r_1 \leq r$ and $\frac{1}{r} \leq \frac{1}{r_1} + \frac{1}{r_2}$.

In fact:

$$\begin{aligned} \|(T_\phi(x_j, y_j))_{j=1}^m\|_r &\leq \|\phi\| \|(x_j)_{j=1}^m\|_{w, r_1} \|(y_j)_{j=1}^m\|_\infty \\ &\leq \|\phi\| \|(x_j)_{j=1}^m\|_{w, r_1} \|(y_j)_{j=1}^m\|_{w, r_2}, \end{aligned}$$

for all $m \in \mathbb{N}$, $x_j, y_j \in E$, $j = 1, \dots, m$. Hence $\|T_\phi\|_{as, (r; r_1, r_2)} \leq \|\phi\|$.

(b) $T_\phi \notin \mathcal{L}_{fas}^{(r; r_1, r)}(E, E; E)$, for all $r_1 \leq r$.

In fact: if we choose $(y_j)_{j=1}^\infty \in \ell_r^w(E) \setminus \ell_r(E)$ and $(x_k)_{k=1}^\infty \in \ell_{r_1}^w(E)$, we have

$$\left(\sum_{j,k=1}^\infty \|T_\phi(x_k, y_j)\|^r \right)^{1/r} = \left(\sum_{k=1}^\infty |\phi(x_k)|^r \right)^{1/r} \left(\sum_{j=1}^\infty \|y_j\|^r \right)^{1/r} = +\infty.$$

(3) Every n -linear mapping of finite type is fully absolutely $(r; r_1, \dots, r_n)$ -summing.

This follows from the fact that $\phi_1 \times \dots \times \phi_n b$, with $\phi_k \in E'_k$, $k = 1, \dots, n$ and $b \in F$, is fully absolutely $(r; r_1, \dots, r_n)$ -summing. We have

$$\left(\sum_{\substack{j_k=1 \\ k=1, \dots, n}}^m \|\phi_1 \times \dots \times \phi_n b(x_{j_1}^1, \dots, x_{j_n}^n)\|^r \right)^{1/r} = \|b\| \prod_{k=1}^n \|(\phi_k(x_i^k))_{i=1}^m\|_{r_k}.$$

Hence

$$\|\phi_1 \times \dots \times \phi_n b\|_{fas, (r; r_1, \dots, r_n)} \leq \|\phi_1\| \dots \|\phi_n\| \|b\|.$$

Proposition 2.4

For $T \in \mathcal{L}(E_1, \dots, E_n; F)$ the following conditions are equivalent:

- (1) T is fully absolutely $(r; r_1, \dots, r_n)$ -summing.
- (2) If $(x_i^k)_{i=1}^\infty \in \ell_{r_k}^w(E_k)$, for $k = 1, \dots, n$, then $(T(x_{j_1}^1, \dots, x_{j_n}^n))_{j \in \mathbb{N}^n} \in \ell_r(\mathbb{N}^n; F)$.
- (3) The mapping T_w defined from $\ell_{r_1}^w(E_1) \times \dots \times \ell_{r_n}^w(E_n)$ into $\ell_r(\mathbb{N}^n; F)$ by

$$T_w((x_i^1)_{i=1}^\infty, \dots, (x_i^n)_{i=1}^\infty) = (T(x_{j_1}^1, \dots, x_{j_n}^n))_{j \in \mathbb{N}^n}$$

is well defined, n -linear and continuous.

In this case $\|T\|_{fas, (r; r_1, \dots, r_n)} = \|T_w\|$.

Proof. It is clear that (3) implies (2) and that (3) implies (1) with $\|T\|_{fas, (r; r_1, \dots, r_n)} \leq \|T_w\|$. If T is fully absolutely $(r; r_1, \dots, r_n)$ -summing and $(x_i^k)_{i=1}^\infty \in \ell_{r_k}^w(E_k)$, for $k = 1, \dots, n$, then we have

$$\begin{aligned} \|(T(x_{j_1}^1, \dots, x_{j_n}^n))_{j \in \mathbb{N}_m^n}\|_r &\leq \|T\|_{fas, (r; r_1, \dots, r_n)} \prod_{k=1}^m \|(x_i^k)_{i=1}^m\|_{w, r_k} \\ &\leq \|T\|_{fas, (r; r_1, \dots, r_n)} \prod_{k=1}^m \|(x_i^k)_{i=1}^\infty\|_{w, r_k}, \end{aligned}$$

for every $m \in \mathbb{N}$. Hence, by passage to the limit, for m tending to $+\infty$, we see that (1) implies (2) and (1) implies (3) with $\|T_w\| \leq \|T\|_{fas, (r; r_1, \dots, r_n)}$. If we use the Closed Graph Theorem we can show that (2) implies (3). In fact, the Closed Graph Theorem is used to show that T_w is separately continuous, hence continuous. \square

The following result has interesting consequences.

Proposition 2.5

If $r \geq r_1 \geq r_k > 0$, $k = 2, \dots, n$, and $T \in \mathcal{L}(E_1, \dots, E_n; F)$ are such that T_1 belongs to $\mathcal{L}_{as}^{(r; r_1)}(E_1; \mathcal{L}_{fas}^{(r_1; r_2, \dots, r_n)}(E_2, \dots, E_n; F))$, with T_1 defined by

$$T_1(x^1)(x^2, \dots, x^n) = T(x^1, x^2, \dots, x^n) \quad \forall x^k \in E_k, k = 1, \dots, n,$$

then T is fully absolutely $(r; r_1, \dots, r_n)$ -summing and $\|T\|_{fas, (r; r_1, \dots, r_n)} \leq \|T\|_{as, (r; r_1)}$.

Proof. For r finite, m natural and $x_i^k \in E_k$, $k = 1, \dots, n$ and $i = 1, \dots, m$ we have

$$\begin{aligned} \|(T(x_{j_1}^1, \dots, x_{j_n}^n))_{j \in \mathbb{N}_m^n}\|_r &\leq \left(\sum_{j_1=1}^m \left(\sum_{\substack{j_k=1 \\ k=2, \dots, n}}^m \|T_1(x_{j_1}^1)(x_{j_2}^2, \dots, x_{j_n}^n)\|^{r_1} \right)^{r/r_1} \right)^{1/r} \\ &\leq \left(\sum_{j_1=1}^m \left(\|T_1(x_{j_1}^1)\|_{fas, (r_1; r_2, \dots, r_n)} \prod_{k=2}^n \|(x_i^k)_{i=1}^m\|_{w, r_k}^{r_1} \right)^r \right)^{1/r} \\ &\leq \|T_1\|_{as, (r; r_1)} \prod_{k=2}^n \|(x_i^k)_{i=1}^m\|_{w, r_k}. \end{aligned}$$

The case $r = +\infty$ is trivial. \square

Consequences 2.6

(1) $\mathcal{L}_{fas}(\ell_1, \ell_2; \mathbb{K}) = \mathcal{L}(\ell_1, \ell_2; \mathbb{K})$.

This follows from 2.5 and the Grothendieck's Theorem stating that $\mathcal{L}(\ell_1; \ell_2) = \mathcal{L}_{as}(\ell_1; \ell_2)$ (see [5]).

(2) $\mathcal{L}_{fas}^2(c_0, \ell_p; \mathbb{K}) = \mathcal{L}(c_0, \ell_p; \mathbb{K})$, for every $p \in [2, +\infty[$.

(3) $\mathcal{L}_{fas}^2(c_0, c_0; \mathbb{K}) = \mathcal{L}(c_0, c_0; \mathbb{K})$.

The equalities (2) and (3) follow from 2.5 and a result of Lindenstrauss and Pelczyński proving that $\mathcal{L}_{as}^2(c_0; \ell_r) = \mathcal{L}(c_0; \ell_r)$, for $r \in [1, 2]$ (see [8]).

(4) $\mathcal{L}_{fas}^r(c_0, \ell_p; \mathbb{K}) = \mathcal{L}(c_0, \ell_p; \mathbb{K})$, for $1 < r' < p < 2$.

This follows from 2.5 and the following result of Schwartz and Kwapien: $\mathcal{L}(c_0; \ell_{p'}) = \mathcal{L}_{as}^r(c_0; \ell_{p'})$, for $2 < p' < r < +\infty$ (see [7] and [15]).

(5) $\mathcal{L}_{fas}^r(\ell_\infty, F; \mathbb{K}) = \mathcal{L}(\ell_\infty, F; \mathbb{K})$, if F' has cotype p' and $1 < r' < p < 2$.

This is a consequence of 2.5 and a result of Maurey [12] that states: $\mathcal{L}_{as}^r(\ell_\infty; F') = \mathcal{L}(\ell_\infty; F')$, if $2 < p' < r < +\infty$ and F' has cotype p' .

The following two propositions are easily proved and give ways of constructing new examples of fully absolutely summing mappings.

Proposition 2.7

If $T \in \mathcal{L}_{fas}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$, $S \in \mathcal{L}(F; G)$ and $R_k \in \mathcal{L}(D_k; E_k)$, $k = 1, \dots, n$, then $S \circ T \circ (R_1, \dots, R_n)$ is fully absolutely $(r; r_1, \dots, r_n)$ -summing and

$$\|S \circ T \circ (R_1, \dots, R_n)\|_{fas, (r; r_1, \dots, r_n)} \leq \|S\| \|T\|_{fas, (r; r_1, \dots, r_n)} \prod_{k=1}^n \|R_k\|.$$

By 2.7, we can see that, in 2.6, (4) is a consequence of (5).

Proposition 2.8

If $T \in \mathcal{L}(E_1, \dots, E_n; F)$, $R_k \in \mathcal{L}_{as}^{(s_k; r_k)}(D_k; E_k)$, $k = 1, \dots, n$, then $T \circ (R_1, \dots, R_n)$ is fully absolutely $(s; r_1, \dots, r_n)$ -summing, when $s \geq \max \{s_1, \dots, s_n\}$, and

$$\|T \circ (R_1, \dots, R_n)\|_{fas, (s; r_1, \dots, r_n)} \leq \|T\| \prod_{k=1}^n \|R_k\|_{as, (s; r_k)}.$$

We recall that a Banach space E has the *Orlicz Property* if $id_E \in \mathcal{L}_{as}^{(2;1)}(E; E)$. In this case the Orlicz constant of E is defined as $\mathcal{O}(E) = \|id_E\|_{as, (2;1)}$. If $p \in [1, 2]$, then ℓ_p has the Orlicz Property. Hence, as a consequence of 2.8 we have:

Corollary 2.9

If E_1, \dots, E_n have the Orlicz Property, then

$$\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_{fas}^{(2,1)}(E_1, \dots, E_n; F),$$

for every F . Moreover:

$$\|T\|_{fas, (2;1)} \leq \|T\| \prod_{k=1}^n \mathcal{O}(E_k),$$

for every $T \in \mathcal{L}(E_1, \dots, E_n; F)$.

As consequence of 2.8 and the results of Grothendieck, Lindenstrauss-Pelczyński, Schwartz, Kwapien and Maurey mentioned in the proof of 2.6 we can give the following examples:

EXAMPLE 2.10: If $T \in \mathcal{L}(\ell_2, \dots, \ell_2; F)$ and $S_k \in \mathcal{L}(\ell_1; \ell_2)$, $k = 1, \dots, n$, then $T \circ (S_1, \dots, S_n)$ is fully absolutely $(s; 1)$ -summing for each $s \geq 1$.

EXAMPLE 2.11: If $p \in [1, 2]$, $T \in \mathcal{L}(\ell_p, \dots, \ell_p; F)$ and $S_k \in \mathcal{L}(c_0; \ell_p)$, $k = 1, \dots, n$, then $T \circ (S_1, \dots, S_n)$ is fully absolutely $(s; 2)$ -summing for each $s \geq 2$.

EXAMPLE 2.12: If $2 < p < r < +\infty$, $T \in \mathcal{L}(\ell_p, \dots, \ell_p; F)$ and $S_k \in \mathcal{L}(c_0; \ell_p)$, $k = 1, \dots, n$, then $T \circ (S_1, \dots, S_n)$ is fully absolutely $(s; r)$ -summing for each $s \geq r$.

EXAMPLE 2.13: if $1 < r' < p < 2$, E'_k has cotype p' , $T \in \mathcal{L}(E_1, \dots, E_n; F)$ and $S_k \in \mathcal{L}(\ell_\infty; E_k)$, $k = 1, \dots, n$, then $T \circ (S_1, \dots, S_n)$ is fully absolutely $(s; r)$ -summing for each $s \geq r$.

The n -linear version of the Grothendieck-Pietsch Domination Theorem is the following result. The proof of this theorem is an adaptation of the proof for the linear case that uses Ky Fan's Lemma. We denote by $W(B_{F'})$ the set of all regular probabilities measures on the σ -algebra of the Borel subsets of $B_{F'}$, for the weak $*$ topology on F' restricted to $B_{F'}$.

Theorem 2.14

If $T \in \mathcal{L}(E_1, \dots, E_n; F)$ and $1/r = 1/r_1 + \dots + 1/r_n$, with $r, r_1, \dots, r_n \in]0, +\infty[$, then T is absolutely $(r; r_1, \dots, r_n)$ -summing, if, and only if, there are $C \geq 0$ and $\mu_k \in W(B_{E'_k})$, $k = 1, \dots, n$, such that

$$\|T(x^1, \dots, x^n)\| \leq C \left(\int_{B_{E'_1}} |\phi(x^1)|^{r_1} d\mu_1(\phi) \right)^{1/r_1} \dots \left(\int_{B_{E'_n}} |\phi(x^n)|^{r_n} d\mu_n(\phi) \right)^{1/r_n},$$

for every $x^k \in E_k$, $k = 1, \dots, n$. The infimum of all these possible C is equal to $\|T\|_{as, (r; r_1, \dots, r_n)}$.

This result is applied in the proof of the following inclusion.

Proposition 2.15

If $r, r_1, \dots, r_n \in]0, +\infty[$ are such that $1/r = 1/r_1 + \dots + 1/r_n$, then each absolutely $(r; r_1, \dots, r_n)$ -summing n -linear mapping T from $E_1 \times \dots \times E_n$ into F is fully absolutely $(s; r_1, \dots, r_n)$ -summing, with $s = \max_{k=1, \dots, n} r_k$ and

$$\|T\|_{fas, (s; r_1, \dots, r_n)} \leq \|T\|_{as, (r; r_1, \dots, r_n)}.$$

Proof. By 2.14, we can find $C \geq 0$ and $\mu_k \in W(B_{E'_k})$, $k = 1, \dots, n$, such that

$$\|T(x^1, \dots, x^n)\| \leq C \left(\int_{B_{E'_1}} |\phi(x^1)|^{r_1} d\mu_1(\phi) \right)^{1/r_1} \dots \left(\int_{B_{E'_n}} |\phi(x^n)|^{r_n} d\mu_n(\phi) \right)^{1/r_n},$$

for every $x^k \in E_k$, $k = 1, \dots, n$. Hence we can write

$$\begin{aligned} \sum_{j_1, \dots, j_n=1}^m \|T(x_{j_1}^1, \dots, x_{j_n}^n)\|^s &\leq C^s \sum_{j_1, \dots, j_n=1}^m \prod_{k=1}^n \left(\int_{B_{E'_k}} |\phi(x_{j_k}^k)|^{r_k} d\mu_k(\phi) \right)^{s/r_k} \\ &= C^s \prod_{k=1}^n \sum_{j=1}^m \left(\int_{B_{E'_k}} |\phi(x_j^k)|^{r_k} d\mu_k(\phi) \right)^{s/r_k} \\ &\leq C^s \prod_{k=1}^n \left(\sum_{j=1}^m \int_{B_{E'_k}} |\phi(x_j^k)|^{r_k} d\mu_k(\phi) \right)^{s/r_k} \\ &= \prod_{k=1}^n \left(\int_{B_{E'_k}} \sum_{j=1}^m |\phi(x_j^k)|^{r_k} d\mu_k(\phi) \right)^{s/r_k} \\ &\leq C^s \prod_{k=1}^n (\|(x_j^k)_{j=1}^m\|_{w, r_k})^s. \end{aligned}$$

This completes our proof. \square

Theorem 2.16 (Multiplication)

For $1 \leq p, q, r < +\infty$, with $1/r = 1/p + 1/q$, if $S \in \mathcal{L}_{fas}^p(E_1, \dots, E_n; F)$ and $T_k \in \mathcal{L}_{as}^q(D_k; E_k)$, $k = 1, \dots, n$, then $S \circ (T_1, \dots, T_n)$ is in $\mathcal{L}_{fas}^r(D_1, \dots, D_n; F)$.

Proof. We know that, for each $k = 1, \dots, n$, there is $\mu_k \in W(B_{D'_k})$, such that

$$\|T_k(x)\| \leq \|T_k\|_{as,q} \left(\int_{B_{D'_k}} |\phi(x)|^q d\mu_k(\phi) \right)^{1/q},$$

for every $x \in D_k$. We take

$$\rho_i^k = \left(\int_{B_{D'_k}} |\phi(x_i)|^r d\mu_k(\phi) \right)^{1/q},$$

for $k = 1, \dots, n$ and $i = 1, \dots, m$. Without loss of generality we may consider $T_k(x_i^k) \neq 0$, for all $k = 1, \dots, n$ and $i = 1, \dots, m$. Hence $\rho_i^k > 0$ and we can define $z_i^k = \frac{x_i^k}{\rho_i^k}$.

Now, for $\alpha_1, \dots, \alpha_m \in \mathbb{K}$, with $\sum_{i=1}^m |\alpha_i|^{p'} \leq 1$, since $\frac{1}{r'} + \frac{1}{p} + \frac{1}{q} = 1$, we can use Holder's inequality in order to write

$$\begin{aligned} \left| \sum_{i=1}^m \phi(\alpha_i z_i^k) \right| &\leq \sum_{i=1}^m |\alpha_i|^{p'/r'} |\alpha_i|^{p'/q} \frac{1}{\rho_i^k} |\phi(x_i^k)|^{r/q} |\phi(x_i^k)|^{r/p} \\ &\leq \left(\sum_{i=1}^m |\alpha_i|^{p'} \right)^{1/r'} \left(\sum_{i=1}^m |\alpha_i|^{p'} \frac{1}{(\rho_i^k)^q} |\phi(x_i^k)|^r \right)^{1/q} \left(\sum_{i=1}^m |\phi(x_i^k)|^r \right)^{1/p} \\ &\leq \left(\sum_{i=1}^m |\alpha_i|^{p'} \frac{1}{(\rho_i^k)^q} |\phi(x_i^k)|^r \right)^{1/q} \left(\sum_{i=1}^m |\phi(x_i^k)|^r \right)^{1/p}. \end{aligned}$$

Thus

$$\begin{aligned} \left\| \sum_{i=1}^m \alpha_i T_k(z_i^k) \right\| &= \left\| T_k \left(\sum_{i=1}^m \alpha_i z_i^k \right) \right\| \\ &\leq \|T_k\|_{as,q} \left(\int_{B_{D'_k}} \left| \sum_{i=1}^m \phi(\alpha_i z_i^k) \right|^q d\mu_k(\phi) \right)^{1/q} \\ &\leq \|T_k\|_{as,q} \left(\sum_{i=1}^m |\alpha_i|^{p'} \frac{1}{(\rho_i^k)^q} \int_{B_{D'_k}} |\phi(x_i^k)|^r d\mu_k(\phi) \right)^{1/q} (\| (x_i^k)_{i=1}^m \|_{w,r})^{r/p} \\ &\leq \|T_k\|_{as,q} (\| (x_i^k)_{i=1}^m \|_{w,r})^{r/p}. \end{aligned}$$

Hence

$$\| (T_k(z_i^k))_{i=1}^m \|_{w,p} \leq \|T_k\|_{as,q} (\| (x_i^k)_{i=1}^m \|_{w,r})^{r/p}.$$

Now we have:

$$\begin{aligned}
& \left(\sum_{i_1, \dots, i_n=1}^m \|S(T_1(x_{i_1}^1), \dots, T_n(x_{i_n}^n))\|^r \right)^{1/r} \\
&= \left(\sum_{i_1, \dots, i_n=1}^m (\rho_{i_1}^1 \dots \rho_{i_n}^n)^r \|S(T_1(z_{i_1}^1), \dots, T_n(z_{i_n}^n))\|^r \right)^{1/r} \\
&\leq \left(\sum_{i_1, \dots, i_n=1}^m (\rho_{i_1}^1 \dots \rho_{i_n}^n)^q \right)^{1/q} \left(\sum_{i_1, \dots, i_n=1}^m \|S(T_1(z_{i_1}^1), \dots, T_n(z_{i_n}^n))\|^p \right)^{1/p} \\
&\leq \prod_{k=1}^n \|(\rho_i^k)_{i=1}^m\|_q \|S\|_{fas,p} \prod_{k=1}^n \|T_k(z_i^k)_{i=1}^m\|_{w,p} \\
&\leq \prod_{k=1}^n \|(\rho_i^k)_{i=1}^m\|_q \|S\|_{fas,p} \prod_{k=1}^n \|T_k\|_{as,q} (\|(x_i^k)_{i=1}^m\|_{w,r})^{r/p} \\
&\leq \|S\|_{fas,p} \prod_{k=1}^n \|T_k\|_{as,q} (\|(x_i^k)_{i=1}^m\|_{w,r})^{r/p+r/q} \\
&\leq \|S\|_{fas,p} \prod_{k=1}^n \|T_k\|_{as,q} \|(x_i^k)_{i=1}^m\|_{w,r}.
\end{aligned}$$

Therefore $S \circ (T_1, \dots, T_n)$ is fully absolutely r -summing and

$$\|S \circ (T_1, \dots, T_n)\|_{fas,r} \leq \|S\|_{fas,p} \prod_{k=1}^n \|T_k\|_{as,q},$$

as we wanted to prove. \square

We recall that, for $p \in [1, +\infty]$ and $\lambda > 1$, a Banach space E is called an $\mathcal{L}_{p,\lambda}$ -space if every finite dimensional subspace D of E is contained in a finite dimensional subspace F of E for which there is an isomorphism v from F onto $\ell_p^{\dim(F)}$ with $\|v\|\|v^{-1}\| < \lambda$. It is said that E is an \mathcal{L}_p -space if it is an $\mathcal{L}_{p,\lambda}$ -space for every $\lambda > 1$. If (Ω, Σ, μ) is any measure space, then $L_p(\mu)$ is an $\mathcal{L}_{p,\lambda}$ -space for every $\lambda > 1$. If K is a compact Hausdorff space, then $C(K)$ is an $\mathcal{L}_{\infty,\lambda}$ -space for every $\lambda > 1$.

See Theorem 3.7, page 64, in [3] for a proof of the following linear result.

Theorem 2.17

If $p \in [1, 2]$, E is an $\mathcal{L}_{\infty,\lambda}$ -space, F is an $\mathcal{L}_{p,\lambda'}$ -space, then every continuous linear operator u from E into F is absolutely 2-summing, with $\|u\|_{as,2} \leq \lambda\lambda' K_G \|u\|$ (K_G is the Grothendieck constant of the Grothendieck's inequality).

Theorem 2.18

If $p \in [1, 2]$, F is a Banach space and E_k is a subspace of an $\mathcal{L}_{p,\lambda_k}$ -space, for $k = 1, \dots, n$, then

$$\mathcal{L}_{fas}^q(E_1, \dots, E_n; F) \subset \mathcal{L}_{fas}^1(E_1, \dots, E_n; F),$$

for all $q \in [1, 2]$. In this case

$$\|S\|_{fas,1} \leq \|S\|_{fas,q} K_G \prod_{k=1}^n \lambda_k,$$

for all $S \in \mathcal{L}_{fas}^q(E_1, \dots, E_n; F)$.

Proof. If $q = 1$ the result is trivial. If $q \in]1, 2]$, we have $q' \in [2, +\infty[$. If $x_1^k, \dots, x_m^k \in E_k$, we can define a continuous linear operator v_k from ℓ_∞^m into E_k , by $v^k(e_j) = x_j^k$, for all $j = 1, \dots, m$. By 2.17 v^k is absolutely 2-summing and $\|v^k\|_{as,2} \leq K_G \lambda_k \|v^k\|$. Since $\|v^k\| \leq \|(x_j^k)_{j=1}^m\|_{w,1}$ and $q' \geq 2$, we have v^k absolutely q' -summing and $\|v^k\|_{as,q'} \leq K_G \lambda_k \|(x_j^k)_{j=1}^m\|_{w,1}$. By 2.16 we have

$$\|S \circ (v^1, \dots, v^n)\|_{fas,1} \leq \|S\|_{fas,q} \prod_{k=1}^n \|v^k\|_{as,q'} \leq \|S\|_{fas,q} K_G \prod_{k=1}^n \lambda_k \|(x_j^k)_{j=1}^m\|_{w,1}.$$

This implies

$$\sum_{j_1, \dots, j_n=1}^m \|S(x_{j_1}^1, \dots, x_{j_n}^n)\| \leq \|S\|_{fas,q} K_G \prod_{k=1}^n \lambda_k \|(x_j^k)_{j=1}^m\|_{w,1},$$

as we wanted to show. \square

3. Connection with tensor products

For $r \in [1, +\infty]$, $0 < r_k \leq r, k = 1, \dots, n$ and $u \in E_1 \otimes \dots \otimes E_n \otimes F$, we consider

$$\rho(r; r_1, \dots, r_n)(u) = \inf \|(\lambda_j)_{j \in \mathbb{N}_m^n}\|_{r'} \| (b_j)_{j \in \mathbb{N}_m^n} \|_\infty \prod_{k=1}^n \|(x_i^k)_{i=1}^m\|_{w, r_k},$$

where the infimum is taken over all representations of u of the form

$$u = \sum_{j \in \mathbb{N}_m^n} \lambda_j x_{j_1}^1 \otimes \dots \otimes x_{j_n}^n b_j,$$

with $\lambda_j \in \mathbb{K}, x_i^k \in E_k, b_j \in F, i = 1, \dots, m, j \in \mathbb{N}_m^n$ and $m \in \mathbb{N}$. We denote by s_n the element of $[0, 1]$ given by

$$\frac{1}{s_n} = \frac{1}{r'} + \frac{1}{r_1} + \dots + \frac{1}{r_n}.$$

Proposition 3.1

$\rho(r; r_1, \dots, r_n)$ is an s_n -norm and $\epsilon \leq \rho(r; r_1, \dots, r_n)$, where ϵ denotes the injective tensor norm on $E_1 \otimes \dots \otimes E_n \otimes F$.

Proof. If

$$u = \sum_{j \in \mathbb{N}_m^n} \lambda_j x_{j_1}^1 \otimes \dots \otimes x_{j_n}^n b_j,$$

we have

$$\begin{aligned} \epsilon(u) &= \sup \left\{ \left| \sum_{j \in \mathbb{N}_m^n} \lambda_j \phi_1(x_{j_1}^1) \dots \phi_n(x_{j_n}^n) \psi(b) \right|; \phi_k \in B_{E'_k}, \psi \in B_{F'} \right\} \\ &\leq \|(\lambda_j)_{j \in \mathbb{N}_m^n}\|_{r'} \sup_{\phi_k \in B_{E'_k}} \|(\phi_1(x_{j_1}^1) \dots \phi_n(x_{j_n}^n))_{j \in \mathbb{N}_m^n}\|_r \| (b_j)_{j \in \mathbb{N}_m^n} \|_\infty \\ &\leq \|(\lambda_j)_{j \in \mathbb{N}_m^n}\|_{r'} \| (b_j)_{j \in \mathbb{N}_m^n} \|_\infty \prod_{k=1}^n \| (x_i^k)_{i=1}^m \|_{w, r_k}. \end{aligned}$$

Hence $\epsilon(u) \leq \rho_{(r; r_1, \dots, r_n)}(u)$.

For $u, v \in E_1 \otimes \dots \otimes E_n \otimes F$ and $\delta > 0$, we can find representations of u and v of the form

$$u = \sum_{j \in \mathbb{N}_m^n} \lambda_j x_{j_1}^1 \otimes \dots \otimes x_{j_n}^n b_j, v = \sum_{j \in \mathbb{N}_p^n} \eta_j y_{j_1}^1 \otimes \dots \otimes x_{j_n}^n c_j,$$

such that

$$\begin{aligned} \|(\lambda_j)_{j \in \mathbb{N}_m^n}\|_{r'} &\leq ((1 + \delta) \rho_{(r; r_1, \dots, r_n)}(u))^{s_n/r'}, \\ \|(\eta_j)_{j \in \mathbb{N}_p^n}\|_{r'} &\leq ((1 + \delta) \rho_{(r; r_1, \dots, r_n)}(v))^{s_n/r'}, \\ \| (x_i^k)_{i=1}^m \|_{w, r_k} &\leq ((1 + \delta) \rho_{(r; r_1, \dots, r_n)}(u))^{s_n/r_k}, \\ \| (y_i^k)_{i=1}^p \|_{w, r_k} &\leq ((1 + \delta) \rho_{(r; r_1, \dots, r_n)}(v))^{s_n/r_k}, \\ \| (b_j)_{j \in \mathbb{N}_m^n} \|_\infty &= 1 = \| (c_j)_{j \in \mathbb{N}_p^n} \|_\infty. \end{aligned}$$

Thus we can write

$$\begin{aligned} &(\rho_{(r; r_1, \dots, r_n)}(u + v))^{s_n} \\ &\leq \left(\sum_{j \in \mathbb{N}_m^n} |\lambda_j|^{r'} + \sum_{j \in \mathbb{N}_p^n} |\eta_j|^{r'} \right)^{s_n/r'} \prod_{k=1}^n \left(\sup_{\phi \in B_{E'_k}} \left(\sum_{i=1}^m |\phi(x_i^k)|^{r_k} + \sum_{i=1}^p |\phi(y_i^k)|^{r_k} \right) \right)^{s_n/r_k} \\ &\leq (1 + \delta)^{s_n} ((\rho_{(r; r_1, \dots, r_n)}(u))^{s_n} + (\rho_{(r; r_1, \dots, r_n)}(v))^{s_n}). \end{aligned}$$

For $r' = +\infty$, the same inequality can be deduced in a similar way. Hence the triangular inequality is proved for $\rho_{(r; r_1, \dots, r_n)}$. The other conditions are easily verified. \square

Proposition 3.2

The topological dual $(E_1 \otimes \dots \otimes E_n \otimes F, \rho_{(r; r_1, \dots, r_n)})'$ of $(E_1 \otimes \dots \otimes E_n \otimes F, \rho_{(r; r_1, \dots, r_n)})$ is isometric to $\mathcal{L}_{fas}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F')$ through the mapping B defined by

$$B(\psi)(x^1, \dots, x^n)(b) = \psi(x^1 \otimes \dots \otimes x^n \otimes b),$$

for every $\rho_{(r; r_1, \dots, r_n)}$ -continuous functional ψ on $E_1 \otimes \dots \otimes E_n \otimes F$, $x^k \in E_k$, $k = 1, \dots, n$ and $b \in F$.

Proof. (1) We consider $B(\psi)$ defined as above. It is clear that $B(\psi) \in \mathcal{L}(E_1, \dots, E_n; F')$. For $\delta > 0$, $x_i^k \in E_k$, $k = 1, \dots, n$ and $i = 1, \dots, m$, we can find $b_j = b_{(j_1, \dots, j_n)} \in F$, $\|b_j\| = 1$, such that

$$\sum_{j \in \mathbb{N}_m^n} \|B(\psi)(x_{j_1}^1, \dots, x_{j_n}^n)\|^r \leq \delta + \sum_{j \in \mathbb{N}_m^n} |B(\psi)(x_{j_1}^1, \dots, x_{j_n}^n)(b_j)|^r = (*).$$

For a convenient choice of $\lambda_j \in \mathbb{K}$, $|\lambda_j| = 1$, we can write

$$\begin{aligned} (*) &= \delta + \psi \left(\sum_{j \in \mathbb{N}_m^n} \lambda_j |\psi(x_{j_1}^1 \otimes \dots \otimes x_{j_n}^n \otimes b_j)|^{r-1} x_{j_1}^1 \otimes \dots \otimes x_{j_n}^n \otimes b_j \right) \\ &\leq \delta + \|\psi\| \left(\sum_{j \in \mathbb{N}_m^n} |\psi(x_{j_1}^1 \otimes \dots \otimes x_{j_n}^n \otimes b_j)|^{(r-1)r'} \right)^{1/r'} \prod_{k=1}^n \|(x_i^k)_{i=1}^m\|_{w, r_k} \\ &= \delta + \|\psi\| \left(\sum_{j \in \mathbb{N}_m^n} \|B(\psi)(x_{j_1}^1, \dots, x_{j_n}^n)\|^r \right)^{1/r'} \prod_{k=1}^n \|(x_i^k)_{i=1}^m\|_{w, r_k}. \end{aligned}$$

Since $\delta > 0$ is arbitrary these inequalities imply

$$\|(B(\psi)(x_{j_1}^1, \dots, x_{j_n}^n))_{j \in \mathbb{N}_m^n}\|_r \leq \|\psi\| \prod_{k=1}^n \|(x_i^k)_{i=1}^m\|_{w, r_k}.$$

The same inequality is true for $r = +\infty$. Thus $B(\psi)$ is fully absolutely $(r; r_1, \dots, r_n)$ -summing and

$$\|B(\psi)\|_{fas, (r; r_1, \dots, r_n)} \leq \|\psi\|.$$

(2) If T is fully absolutely $(r; r_1, \dots, r_n)$ -summing from $E_1 \times \dots \times E_n$ into F' , we define a linear functional on $E_1 \otimes \dots \otimes E_n \otimes F$ by

$$\psi_T(u) = \sum_{j \in \mathbb{N}_m^n} \lambda_j T(x_{j_1}^1, \dots, x_{j_n}^n)(b_j),$$

when

$$u = \sum_{j \in \mathbb{N}_m^n} \lambda_j x_{j_1}^1 \otimes \dots \otimes x_{j_n}^n \otimes b_j.$$

We have

$$\begin{aligned} |\psi_T(u)| &\leq \|(\lambda_j)_{j \in \mathbb{N}_m^n}\|_{r'} \|T(x_{j_1}^1, \dots, x_{j_n}^n)_{j \in \mathbb{N}_m^n}\|_r \|b_j\|_{j \in \mathbb{N}_m^n}^\infty \\ &\leq \|T\|_{fas, (r, r_1, \dots, r_n)} \|(\lambda_j)_{j \in \mathbb{N}_m^n}\|_{r'} \prod_{k=1}^n \|(x_i^k)_{i=1}^m\|_{w, r_k} \|b_j\|_{j \in \mathbb{N}_m^n}^\infty. \end{aligned}$$

This shows that ψ_T is $\rho_{(r; r_1, \dots, r_n)}$ -continuous and $\|\psi_T\| \leq \|T\|_{fas, (r, r_1, \dots, r_n)}$. \square

Remark 3.3. The s_n -norm $\rho_{(r; r_1, \dots, r_n)}$ is a norm if $\frac{1}{r} = \frac{1}{r_1} + \dots + \frac{1}{r_n}$. In this case we have $\rho_{(r; r_1, \dots, r_n)} \leq \pi$, where π denotes the projective tensor norm on $E_1 \otimes \dots \otimes E_n \otimes F$.

4. Virtually nuclear multilinear mappings

In this section, unless it is stated explicitly otherwise, we consider $r \in]0, +\infty]$ and $r_k \in [1, +\infty]$, with $r \leq r_k$, $k = 1, \dots, n$. We also write

$$\frac{1}{t_n} = \frac{1}{r} + \frac{1}{r'_1} + \dots + \frac{1}{r'_n}.$$

Hence $t_n \in]0, 1]$.

DEFINITION 4.1. A mapping $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is *virtually $(r; r_1, \dots, r_n)$ -nuclear* if it has a representation of the form

$$T = \sum_{j \in \mathbb{N}^n} \lambda_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n b_j,$$

with $(\lambda_j)_{j \in \mathbb{N}^n} \in \ell_r(\mathbb{N}^n)$, if $r < +\infty$, or $(\lambda_j)_{j \in \mathbb{N}^n} \in c_0(\mathbb{N}^n)$, if $r = +\infty$, $(\phi_i^k)_{i=1}^\infty \in \ell_{r_k}^w(E'_k)$, for $k = 1, \dots, n$ and $(b_j)_{j \in \mathbb{N}^n} \in \ell_\infty(\mathbb{N}^n; F)$.

The vector space of all such mappings is denoted by $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ and we consider on it the t_n -norm

$$\|T\|_{VN, (r; r_1, \dots, r_n)} = \inf \|(\lambda_j)_{j \in \mathbb{N}^n}\|_r \| (b_j)_{j \in \mathbb{N}^n} \|_\infty \prod_{k=1}^n \|(\phi_i^k)_{i=1}^\infty\|_{w, r'_k},$$

where the infimum is taken over all possible representations of T as described in 4.1. As usual, we replace $(r; r_1, \dots, r_n)$ by $(r; s)$, if $s = r_1 = \dots = r_n$, and $(r; s)$ by r , when $r = s$, in all the preceding notations. If $r = s = 1$, we omit 1 in the notations. In all cases we have complete metrizable topological vector spaces.

In order to justify the use of the term “virtually nuclear” we recall the following concept considered in [10].

DEFINITION 4.2. A mapping $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is of *nuclear type $(r; r_1, \dots, r_n)$* if it has a representation of the form

$$T = \sum_{j=1}^\infty \lambda_j \phi_j^1 \times \dots \times \phi_j^n b_j,$$

with $(\lambda_j)_{j=1}^\infty \in \ell_r$, if $r < +\infty$, or $(\lambda_j)_{j=1}^\infty \in c_0$, if $r = +\infty$, $(\phi_j^k)_{j=1}^\infty \in \ell_{r_k}^w(E'_k)$, for $k = 1, \dots, n$ and $(b_j)_{j=1}^\infty \in \ell_\infty(F)$.

The vector space of all such mappings is denoted by $\mathcal{L}_N^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ and we consider on it the t_n -norm

$$\|T\|_{N, (r; r_1, \dots, r_n)} = \inf \|(\lambda_j)_{j=1}^\infty\|_r \| (b_j)_{j=1}^\infty \|_\infty \prod_{k=1}^n \|(\phi_j^k)_{j=1}^\infty\|_{w, r'_k},$$

where the infimum is taken over all possible representations of T as described in 4.2. The simplification of the notations is made as in the virtually nuclear case.

REMARKS 4.3:

(1) $\mathcal{L}_N^{(r;r_1,\dots,r_n)}(E_1,\dots,E_n;F) \subset \mathcal{L}_{VN}^{(r;r_1,\dots,r_n)}(E_1,\dots,E_n;F)$ and

$$\|T\| \leq \|T\|_{VN,(r;r_1,\dots,r_n)} \leq \|T\|_{N,(r;r_1,\dots,r_n)},$$

for every T of nuclear type $(r;r_1,\dots,r_n)$.

(2) $\mathcal{L}_f(E_1,\dots,E_n;F)$ is dense in $\mathcal{L}_{VN}^{(r;r_1,\dots,r_n)}(E_1,\dots,E_n;F)$ and

$$\|\phi^1 \times \dots \times \phi^n b\|_{VN,(r;r_1,\dots,r_n)} = \|\phi^1\| \dots \|\phi^n\| \|b\|,$$

for every $\phi^k \in E'_k$, $k = 1, \dots, n$ and $b \in F$.

(3) $\mathcal{L}_N(E_1,\dots,E_n;F) = \mathcal{L}_{VN}(E_1,\dots,E_n;F)$ isometrically.

(4) For $T \in \mathcal{L}_{VN}^{(r;r_1,\dots,r_n)}(E_1,\dots,E_n;F)$, $S_k \in \mathcal{L}(D_k;E_k)$, $k = 1, \dots, n$ and $R \in \mathcal{L}(F;G)$, it follows that $R \circ T \circ (S_1, \dots, S_n)$ is virtually $(r;r_1,\dots,r_n)$ -nuclear and

$$\|R \circ T \circ (S_1, \dots, S_n)\|_{VN,(r;r_1,\dots,r_n)} \leq \|R\| \|T\|_{VN,(r;r_1,\dots,r_n)} \prod_{k=1}^n \|S_k\|.$$

(5) If $(\lambda_j)_{j \in \mathbb{N}^n}$ is in $\ell_r(\mathbb{N}^n)$, if $r < +\infty$, or in $c_0(\mathbb{N}^n)$, if $r = +\infty$, then the n -linear mapping $D_{(\lambda_j)_{j \in \mathbb{N}^n}}$, defined on $\ell_{r'_1} \times \dots \times \ell_{r'_n}$, with values in $\ell_1(\mathbb{N})$, by

$$D_{(\lambda_j)_{j \in \mathbb{N}^n}}((\xi_j^1)_{j=1}^\infty, \dots, (\xi_j^n)_{j=1}^\infty) = (\lambda_j \xi_{j_1}^1 \dots \xi_{j_n}^n)_{j \in \mathbb{N}^n},$$

is virtually $(r;r_1,\dots,r_n)$ -nuclear and $\|D_{(\lambda_j)_{j \in \mathbb{N}^n}}\|_{VN,(r;r_1,\dots,r_n)} \leq \|(\lambda_j)_{j \in \mathbb{N}^n}\|_r$.

Now we have another characterization of virtually nuclear mappings.

Proposition 4.4

For $T \in \mathcal{L}(E_1, \dots, E_n; F)$, the following conditions are equivalent

- (1) T is virtually $(r;r_1,\dots,r_n)$ -nuclear.
- (2) There are $A_k \in \mathcal{L}(E_k; l_{r'_k})$, $k = 1, \dots, n$, $Y \in \mathcal{L}(\ell_1(\mathbb{N}^n); F)$ and $(\lambda_j)_{j \in \mathbb{N}^n} \in \ell_r(\mathbb{N}^n)$ such that

$$T = Y \circ D_{(\lambda_j)_{j \in \mathbb{N}^n}} \circ (A_1, \dots, A_n).$$

In this case

$$\|T\|_{VN,(r;r_1,\dots,r_n)} \leq \inf \|Y\| \|(\lambda_j)_{j \in \mathbb{N}^n}\|_r \prod_{k=1}^n \|A_k\|,$$

where the infimum is taken over all such possible factorizations.

Proof. It is clear that (2) implies (1) by 4.3.(4) and 4.3.(5).

In order to show that (1) implies (2), we consider a representation of T as in 4.1 and define

$$\begin{aligned} A_k(x) &= (\phi_i^k(x))_{i=1}^\infty & \forall x \in E_k, k = 1, \dots, n, \\ Y((\xi_j)_{j \in \mathbb{N}^n}) &= \sum_{j \in \mathbb{N}^n} \xi_j b_j & \forall (\xi_j)_{j \in \mathbb{N}^n} \in \ell_1(\mathbb{N}^n). \end{aligned}$$

Now, the result follows. \square

Remark 4.5. By definition every T in $\mathcal{L}_f(E_1, \dots, E_n; F)$ has a finite representation

$$T = \sum_{j \in \mathbb{N}_m^n} \lambda_j \phi_{j_1}^1 \dots \phi_{j_n}^n b_j.$$

It is clear that we have a t_n -norm on $\mathcal{L}_f(E_1, \dots, E_n; F)$ defined by

$$\|T\|_{VN_f, (r; r_1, \dots, r_n)} = \inf \|(\lambda_j)_{j \in \mathbb{N}_m^n}\|_r \| (b_j)_{j \in \mathbb{N}_m^n} \prod_{k=1}^n \|(\phi_i^k)_{i=1}^m\|_{w, r_k},$$

where the infimum is taken over all finite representations of T . It is obvious that

$$\|T\|_{VN, (r; r_1, \dots, r_n)} \leq \|T\|_{VN_f, (r; r_1, \dots, r_n)},$$

for every $T \in \mathcal{L}_f(E_1, \dots, E_n; F)$. We would like to know cases where there is equality for these t_n -norms.

Proposition 4.6

If E_1, \dots, E_n are finite dimensional, then

$$\|T\|_{VN, (r; r_1, \dots, r_n)} \geq \|T\|_{VN_f, (r; r_1, \dots, r_n)},$$

for every $T \in \mathcal{L}_f(E_1, \dots, E_n; F)$.

Proof. In this case $\mathcal{L}(E_1, \dots, E_n; F) = \mathcal{L}_f(E_1, \dots, E_n; F)$ is complete for both t_n -norms. Hence, by the open mapping theorem, these two t_n -norms are equivalent and there is $C \geq 0$, such that

$$\|T\|_{VN_f, (r; r_1, \dots, r_n)} \leq C \|T\|_{VN, (r; r_1, \dots, r_n)},$$

for every $T \in \mathcal{L}_f(E_1, \dots, E_n; F)$. For each $\epsilon > 0$, we choose a representation

$$T = \sum_{j \in \mathbb{N}^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j,$$

such that

$$\|(\sigma_j)_{j \in \mathbb{N}^n}\|_r \| (b_j)_{j \in \mathbb{N}^n} \|_\infty \prod_{k=1}^n \|(\phi_i^k)_{i=1}^\infty\|_{w, r_k} \leq (1 + \epsilon) \|T\|_{VN, (r; r_1, \dots, r_n)}.$$

We can write

$$\begin{aligned}
& (\|T\|_{VN_f, (r; r_1, \dots, r_n)})^{t_n} \\
& \leq \left(\left\| \sum_{j \in \mathbb{N}_m^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j \right\|_{VN_f, (r; r_1, \dots, r_n)} \right)^{t_n} \\
& \quad + \left(\left\| \sum_{j \in \mathbb{N}^n \setminus \mathbb{N}_m^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j \right\|_{VN_f, (r; r_1, \dots, r_n)} \right)^{t_n} \\
& \leq (1 + \epsilon)^{t_n} (\|T\|_{VN, (r; r_1, \dots, r_n)})^{t_n} \\
& \quad + C^{t_n} \left(\left\| \sum_{j \in \mathbb{N}^n \setminus \mathbb{N}_m^n} \sigma_j \phi_{j_1}^1 \times \dots \times \phi_{j_n}^n y_j \right\|_{VN, (r; r_1, \dots, r_n)} \right)^{t_n} \\
& \leq [(1 + \epsilon)^{t_n} + \epsilon^{t_n}] (\|T\|_{VN, (r; r_1, \dots, r_n)})^{t_n},
\end{aligned}$$

if m is large enough. \square

Proposition 4.7

If $T \in \mathcal{L}_{VN}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ and $S_k \in \mathcal{L}_f(D_k; E_k)$, for $k = 1, \dots, n$, then

$$\|T \circ (S_1, \dots, S_n)\|_{VN_f, (r; r_1, \dots, r_n)} \leq \|T\|_{VN, (r; r_1, \dots, r_n)} \prod_{K=1}^n \|S_k\|.$$

Proof. If J_k denotes the natural injection of $S_k(D_k)$ into E_k , we can write $S_k = J_k \circ \tilde{S}_k$, with $\|\tilde{S}_k\| = \|S_k\|$. Hence, $T \circ (J_1, \dots, J_n) \in \mathcal{L}_f(S_1(D_1), \dots, S_n(D_n); F)$. Now we apply 4.6 and 4.3.(4) in order to have the result proved. \square

Proposition 4.8

If E'_1, \dots, E'_n have the bounded approximation property, then

$$\|T\|_{VN, (r; r_1, \dots, r_n)} \geq \|T\|_{VN_f, (r; r_1, \dots, r_n)},$$

for every $T \in \mathcal{L}_f(E_1, \dots, E_n; F)$.

Proof. We note that the mapping $T_k \in \mathcal{L}(E_k; \mathcal{L}(E_1, \dots, E_{k-1}, E_{k+1}, \dots, E_n; F))$, defined by

$$T_k(x^k)(x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^n) = T(x^1, \dots, x^{k+1}, x^k, x^{k+1}, \dots, x^n),$$

is of finite type. Since E'_k has the λ_k -approximation property for some $\lambda_k > 0$, for each $\epsilon > 0$, we can find $S_k \in \mathcal{L}_f(E_k; E_k)$, such that $T_k = T_k \circ S_k$ and $\|S_k\| \leq (1 + \epsilon)\lambda_k$. Therefore, for all $x^j \in E_j$, with $j = 1, \dots, n$, we have

$$T(x^1, \dots, x^{k-1}, S_k(x^k), x^{k+1}, \dots, x^n) = T(x^1, \dots, x^{k-1}, x^k, x^{k+1}, \dots, x^n).$$

Now, we can write

$$T(x^1, \dots, x^n) = T(S_1(x^1), \dots, S_n(x^n)), \quad \forall x^j \in E_j, j = 1, \dots, n.$$

Thus, by 4.7, we have

$$\begin{aligned} \|T\|_{VN_f, (r; r_1, \dots, r_n)} &\leq \|T\|_{VN, (r; r_1, \dots, r_n)} \prod_{k=1}^n \|S_k\| \\ &\leq \|T\|_{VN, (r; r_1, \dots, r_n)} (1 + \epsilon)^n \prod_{k=1}^n \lambda_k. \end{aligned}$$

Hence

$$\|T\|_{VN_f, (r; r_1, \dots, r_n)} \leq \left(\prod_{k=1}^n \lambda_k \right) \|T\|_{VN, (r; r_1, \dots, r_n)}.$$

With the same argument used in the proof of 4.6, we finally have:

$$\|T\|_{VN_f, (r; r_1, \dots, r_n)} \leq \|T\|_{VN, (r; r_1, \dots, r_n)}. \quad \square$$

Corollary 4.9

If E'_1, \dots, E'_n have the bounded approximation property, then

$$\mathcal{L}_{VN}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$$

and the completion of $(E'_1 \otimes \dots \otimes E'_n \otimes F, \rho_{(r; r'_1, \dots, r'_n)})$ are isometric, when $r, r_k \in [1, +\infty]$, $k = 1, \dots, n$.

Proposition 4.10

If E'_1, \dots, E'_n have the bounded approximation property, then the topological dual of $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ is isometric to $\mathcal{L}_{fas}^{(r'; r'_1, \dots, r'_n)}(E'_1, \dots, E'_n; F')$, for $r, r_k \in [1, +\infty]$, $k = 1, \dots, n$ through the mapping \mathcal{B} defined by

$$\mathcal{B}(\Psi)(\phi^1, \dots, \phi^n)(b) = \Psi(\phi^1 \times \dots \times \phi^n b),$$

when Ψ is in the topological dual of $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$, $\phi^k \in E'_k$, $k = 1, \dots, n$ and $b \in F$.

Proof. It is a consequence of 4.9 and 3.2. \square

Remark 4.11. We recall that in [10] we proved that, when E'_1, \dots, E'_n have the bounded approximation property, the topological dual of $\mathcal{L}_N^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ is isometric to $\mathcal{L}_{as}^{(r'; r'_1, \dots, r'_n)}(E'_1, \dots, E'_n; F')$, for $r, r_k \in [1, +\infty]$, $k = 1, \dots, n$ through the mapping \mathcal{B} defined by

$$\mathcal{B}(\Psi)(\phi^1, \dots, \phi^n)(b) = \Psi(\phi^1 \times \dots \times \phi^n b),$$

when Ψ is in the topological dual of $\mathcal{L}_N^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$, $\phi^k \in E'_k$, $k = 1, \dots, n$ and $b \in F$. This fact, 4.10 and 2.2.(2) show that the spaces $\mathcal{L}_N^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ and $\mathcal{L}_{VN}^{(r; r_1, \dots, r_n)}(E_1, \dots, E_n; F)$ are different in general.

5. Hilbert-Schmidt multilinear mappings

In this section E_1, \dots, E_n, F are Hilbert spaces. We are going to show that there is a close relationship between the Hilbert-Schmidt and the fully absolutely summing multilinear mappings.

Proposition 5.1

If $T \in \mathcal{L}(E_1, \dots, E_n; F)$, then the (finite or infinite) value

$$\sum_{\substack{j_k \in J_k \\ k=1, \dots, n}} \|T(u_{j_1}^1, \dots, u_{j_n}^n)\|^2$$

is independent of the orthonormal basis $(u_j^k)_{j \in J_k}$ chosen in E_k , $k = 1, \dots, n$

Proof. For $n = 1$, Parseval's equality gives

$$\sum_{j \in J_1} \|T(u_j^1)\|^2 = \sum_{j \in J} \|T^*(v_j)\|^2,$$

where $(v_j)_{j \in J}$ is an orthonormal basis in F . The case $n > 1$ is proved by fixing $n - 1$ variables and applying the linear result to the remaining variable. \square

DEFINITION 5.2. A mapping $T \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be *Hilbert-Schmidt* if there is an orthonormal basis $(u_j^k)_{j \in J_k}$ for E_k , for each $k = 1, \dots, n$, such that

$$\|T\|_{HS} = \left(\sum_{\substack{j_k \in J_k \\ k=1, \dots, n}} \|T(u_{j_1}^1, \dots, u_{j_n}^n)\|^2 \right)^{1/2} < +\infty.$$

We denote by $\mathcal{L}_{HS}(E_1, \dots, E_n; F)$ the vector space of all such mappings. It is easy to show that it is a Hilbert space under the norm $\|\cdot\|_{HS}$ defined by the inner product

$$(T|S) = \sum_{\substack{j_k \in J_k \\ k=1, \dots, n}} (T(u_{j_1}^1, \dots, u_{j_n}^n) | S(u_{j_1}^1, \dots, u_{j_n}^n)).$$

Proposition 5.3

The mapping

$$T \in \mathcal{L}_{HS}(E_1, \dots, E_n; F) \longrightarrow T_1 \in \mathcal{L}_{HS}(E_1; \mathcal{L}_{HS}(E_2, \dots, E_n; F)),$$

where $T_1(x^1)(x^2, \dots, x^n) = T(x^1, x^2, \dots, x^n)$, for $x^k \in E_k$, $k = 1, \dots, n$, is an isometric isomorphism.

Proof. If $T \in \mathcal{L}_{HS}(E_1, \dots, E_n; F)$, $(u_j^k)_{j \in J_k}$ is an orthonormal basis of E_k , for each $k = 1, \dots, n$ and $(v_j)_{j \in J}$ is an orthonormal basis of F , we can write

$$\begin{aligned} \sum_{\substack{j_k \in J_k \\ k=2, \dots, n}} \|T_1(x)(u_{j_2}^2, \dots, u_{j_n}^n)\|^2 &= \sum_{\substack{j_k \in J_k, j \in J \\ k=2, \dots, n}} \left| \sum_{j_1 \in J_1} (x|u_{j_1}^1)(T(u_{j_1}^1, u_{j_2}^2, \dots, u_{j_n}^n)|v_j) \right|^2 \\ &\leq (\|T\|_{HS})^2 \|x\|^2, \end{aligned}$$

for every $x \in E_1$. This shows that $T_1(x)$ is Hilbert-Schmidt and $\|T_1(x)\|_{HS} \leq \|T\|_{HS} \|x\|$. Now it is also clear that

$$\sum_{j_1 \in J_1} (\|T_1(u_{j_1}^1)\|_{HS})^2 = (\|T\|_{HS})^2.$$

This proves that T_1 is Hilbert-Schmidt and $\|T_1\|_{HS} = \|T\|_{HS}$. It is easy to see that the mapping $T \longrightarrow T_1$ is onto $\mathcal{L}_{HS}(E_1; \mathcal{L}_{HS}(E_2, \dots, E_n; F))$. \square

Corollary 5.4

(a) *The mapping*

$$T \in \mathcal{L}_{HS}(E_1, \dots, E_n; F) \longrightarrow T_k \in \mathcal{L}_{HS}(E_1 \dots E_k; \mathcal{L}_{HS}(E_{k+1}, \dots, E_n; F)),$$

where $T_k(x^1, \dots, x^k)(x^{k+1}, \dots, x^n) = T(x^1, x^2, \dots, x^n)$, for $x^j \in E_j$, $j = 1, \dots, n$, is an isometric isomorphism.

(b) *The mapping*

$$T \in \mathcal{L}_{HS}(E_1, \dots, E_n, F; \mathbb{K}) \longrightarrow T_n \in \mathcal{L}_{HS}(E_1, \dots, E_n; F')$$

is an isometric isomorphism.

Proposition 5.5

$\mathcal{L}_{fas}^2(E_1, \dots, E_n; F)$ and $\mathcal{L}_{HS}(E_1, \dots, E_n; F)$ are identically isometric.

Proof. (a) If $T \in \mathcal{L}_{fas}^2(E_1, \dots, E_n; F)$ and $(u_j^k)_{j \in J_k}$ is an orthonormal basis of E_k , for each $k = 1, \dots, n$, we know that

$$\left(\sum_{j \in I_1 \times \dots \times I_n} \|T(u_{j_1}^1, \dots, u_{j_n}^n)\|^2 \right)^{1/2} \leq \|T\|_{fas,2} \prod_{k=1}^n \|(u_i^k)_{i \in I_k}\|_{w,2} \leq \|T\|_{fas,2},$$

for every finite subset I_k of J_k , with m elements, $k = 1, \dots, n$ and m natural. Hence T is Hilbert-Schmidt and $\|T\|_{HS} \leq \|T\|_{fas,2}$.

(b) We consider $T \in \mathcal{L}_{HS}(E_1, \dots, E_n; F)$.

(i) If $n = 1$, we consider, m natural, $x_i \in E_1$, $i = 1, \dots, m$ and an orthonormal basis $(v_j)_{j \in J}$ of F . Then

$$\begin{aligned} \left(\sum_{i=1}^m \|T(x_i)\|^2 \right)^{1/2} &= \left(\sum_{i=1}^m \sum_{j \in J} |(x_i | T^*(v_j))|^2 \right)^{1/2} \\ &\leq \left(\sum_{j \in J} \|T^*(v_j)\|^2 \right)^{1/2} \sup_{\phi \in B_{E'_1}} \left(\sum_{i=1}^m |(x_i | \phi)|^2 \right)^{1/2} \\ &= \|T\|_{HS} \|(x_i)_{i=1}^m\|_{w,2}. \end{aligned}$$

Hence $T \in \mathcal{L}_{fas}^2(E_1; F)$ and $\|T\|_{fas,2} \leq \|T\|_{HS}$.

(ii) If $n > 1$, we assume that the result is true for $k \leq n - 1$. Since, by 5.3, $T_1 \in \mathcal{L}_{HS}(E_1; \mathcal{L}_{HS}(E_2, \dots, E_n; F))$, we have

$$T_1 \in \mathcal{L}_{fas}^2(E_1; \mathcal{L}_{HS}(E_2, \dots, E_n; F)) \subset \mathcal{L}_{fas}^2(E_1; \mathcal{L}_{fas}^2(E_2, \dots, E_n; F)),$$

with $\|T_1\|_{fas,2} \leq \|T_1\|_{HS} = \|T\|_{HS}$, via the induction hypothesis. By 2.5, we obtain $T \in \mathcal{L}_{fas}^2(E_1, \dots, E_n; F)$ and $\|T\|_{fas,2} \leq \|T_1\|_{fas,2} \leq \|T\|_{HS}$. \square

Proposition 5.6

If $p \in]0, +\infty[$, then

$$\mathcal{L}_{HS}(E_1, \dots, E_n; F) \subset \mathcal{L}_{fas}^p(E_1, \dots, E_n; F),$$

and there is a constant $d_p > 0$, such that

$$(d_p)^n \|T\|_{fas,p} \leq \|T\|_{HS}, \quad \forall T \in \mathcal{L}_{HS}(E_1, \dots, E_n; F)$$

Proof. We use induction on n .

(i) For $n = 1$, we know that T has a Schmidt representation

$$T(u) = \sum_{i \in I} \lambda_i (u | x_i) y_i,$$

for every $u \in E$, with $(x_i)_{i \in I}$ orthonormal in E_1 , $(y_i)_{i \in I}$ orthonormal in F and $(\lambda_i)_{i \in I} \in \ell_2(I)$. In this case $\|T\|_{HS} = \|(\lambda_i)_{i \in I}\|_2 < +\infty$. Since this implies that $\lambda_i \neq 0$ only for i in a denumerable subset of I , we may consider $I = \mathbb{N}$. We consider the sequence $(r_i)_{i \in \mathbb{N}}$ of the Rademacher functions and define $v(t)$ by

$$\|T\|_{HS} v(t) = \sum_{i=1}^{\infty} r_i(t) \lambda_i x_i \in E_1,$$

for each $t \in [0, 1]$. By Khintchine's inequality (see [4] and [10]), for every finite sequence $(u_j)_{j=1}^m$ of elements of E_1 , we have

$$\begin{aligned} (\|T\|_{HS} \|(u_j)_{j=1}^m\|_{w,p})^p &\geq (\|T\|_{HS})^p \sup_{t \in [0,1]} \sum_{j=1}^m |(u_j|v(t))|^p \\ &\geq \sum_{j=1}^m \int_0^1 \left| \sum_{i=1}^{\infty} r_i(t) \overline{\lambda_i}(u_j|x_i) \right|^p dt \\ &\geq (d_p)^p \sum_{j=1}^m \left(\sum_{i=1}^{\infty} |\lambda_i(u_j|x_i)|^2 \right)^{p/2} = (d_p)^p \sum_{j=1}^m \|T(u_j)\|^p. \end{aligned}$$

Hence $T \in \mathcal{L}_{fas}^p(E_1; F)$ and $d_p \|T\|_{fas,p} \leq \|T\|_{HS}$.

(ii) If $n > 1$, we assume that the result is true for $k \leq n-1$. We want to prove the result for n . By 5.3 we know that $T_1 \in \mathcal{L}_{HS}(E_1; \mathcal{L}_{HS}(E_2, \dots, E_n; F))$. By the induction hypothesis, we have $T_1 \in \mathcal{L}_{fas}^p(E_1; \mathcal{L}_{HS}(E_2, \dots, E_n; F))$, with $d_p \|T_1\|_{fas,p} \leq \|T_1\|_{HS}$. If J is the inclusion from $\mathcal{L}_{HS}(E_2, \dots, E_n; F)$ into $\mathcal{L}_{fas}^p(E_2, \dots, E_n; F)$, the induction hypothesis show that $(d_p)^{n-1} \|J\| \leq 1$. Thus we have $J \circ T_1 \in \mathcal{L}_{fas}^p(E_1; \mathcal{L}_{fas}^p(E_2, \dots, E_n; F))$. We can write

$$(d_p)^n \|J \circ T_1\|_{fas,p} \leq d_p ((d_p)^{n-1} \|J\|) \|T_1\|_{fas,p} \leq d_p \|T_1\|_{fas,p} \leq \|T_1\|_{HS} = \|T\|_{HS}.$$

By 2.5, we have $T \in \mathcal{L}_{fas}^p(E_1, \dots, E_n; F)$ and

$$(d_p)^n \|T\|_{fas,p} \leq (d_p)^n \|J \circ T_1\|_{fas,p} \leq \|T\|_{HS}. \quad \square$$

In order to prove next result, we consider $m \in \mathbb{N}$ and $D_m = \{1, -1\}^m$. A measure μ is considered on the set of the parts of D_m . It is defined by $\mu(e) = 2^{-m}$, for every $e = (e_1, \dots, e_m) \in D_m$. We denote by π_k the k -th projection from D_m onto $\{1, -1\}$. It follows that

$$\int_{D_m} \pi_j(e) \pi_k(e) d\mu(e) = \delta_{j,k}.$$

We recall that $\delta_{j,k} = 1$, if $j = k$, and $\delta_{j,k} = 0$, if $j \neq k$.

Proposition 5.7

For $p \in [2, +\infty[$,

$$\mathcal{L}_{fas}^p(E_1, \dots, E_n; F) = \mathcal{L}_{HS}(E_1, \dots, E_n; F),$$

and there is a constant $b_p > 0$ such that

$$(d_p)^n \|T\|_{fas,p} \leq \|T\|_{HS} \leq (b_p)^n \|T\|_{fas,p},$$

for every fully absolutely p -summing n -linear mapping T .

Proof. One part of this result is Proposition 5.6. In order to prove the other part we consider T in $\mathcal{L}_{fas}^p(E_1, \dots, E_n; F)$ and an orthonormal basis $(u_j^k)_{j \in I_k}$ of E_k , for each $k = 1, \dots, n$. For every finite subset J_k of I_k with m elements, we consider $(u_j^k)_{j \in J_k}$ ordered linearly and write u_1^k, \dots, u_m^k , $k = 1, \dots, n$. We denote

$$w_k(e) = \sum_{j=1}^m \pi_j(e) u_j^k \quad \text{for } e \in D_m, k = 1, \dots, n.$$

We can write:

$$\begin{aligned} & \left(\sum_{\substack{j_k=1 \\ k=1, \dots, n}}^m \|T(u_{j_1}^1, \dots, u_{j_n}^n)\|^2 \right)^{1/2} \\ &= \left(\int_{D_m^n} \|T(w_1(e^1), \dots, w_n(e^n))\|^2 d\mu(e^1) \dots d\mu(e^n) \right)^{1/2} \\ &\leq \left(\int_{D_m^n} \|T(w_1(e^1), \dots, w_n(e^n))\|^p d\mu(e^1) \dots d\mu(e^n) \right)^{1/p} = (*). \end{aligned}$$

Since the integral we have used is a finite sum, we can use the fact that T is fully absolutely p -summing and the Khintchine's inequality in order to write

$$\begin{aligned} (*) &\leq \|T\|_{fas,p} \prod_{k=1}^n \sup_{\phi \in B_{E'_k}} \left(\int_{D_m} |\phi(w_k(e))|^p d\mu(e) \right)^{1/p} \\ &\leq \|T\|_{fas,p} \prod_{k=1}^n b_p \sup_{\phi \in B_{E'_k}} \left(\sum_{j=1}^m |\phi(u_j^k)|^2 \right)^{1/2} = \|T\|_{fas,p} (b_p)^n. \end{aligned}$$

Hence T is Hilbert-Schmidt and $\|T\|_{HS} \leq (b_p)^n \|T\|_{fas,p}$. \square

DEFINITION 5.8. It is considered on $E_1 \otimes \dots \otimes E_n$ the inner product

$$(u|v)_H = \sum_{j=1}^p \sum_{k=1}^q (x_j^1 | y_k^1) \dots (x_j^n | y_k^n),$$

where

$$u = \sum_{j=1}^p x_j^1 \otimes \dots \otimes x_j^n \quad \text{and} \quad v = \sum_{k=1}^q y_k^1 \otimes \dots \otimes y_k^n.$$

The space $E_1 \otimes \dots \otimes E_n$ with this inner product is denoted by $E_1 \otimes_H \dots \otimes_H E_n$ and its completion by $E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n$. The corresponding norm is denoted by $\|\cdot\|_H$.

Remark 5.9. If $(e_j^k)_{j \in J_k}$ is an orthonormal basis for E_k , $k = 1, \dots, n$, then

$$(e_{j_1}^1 \otimes \dots \otimes e_{j_n}^n)_{\substack{j_k \in J_k \\ k=1, \dots, n}}$$

is an orthonormal basis for $E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n$.

As a consequence of this remark we can prove

Proposition 5.10

If $T \in \mathcal{L}(E_1, \dots, E_n; F)$ and T_{\otimes} denotes the corresponding linear mapping from $E_1 \otimes \dots \otimes E_n$ into F , then the following conditions are equivalent:

- (1) T is Hilbert-Schmidt.
- (2) $T_{\hat{\otimes}} \in \mathcal{L}(E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n; F)$, where $T_{\hat{\otimes}}$ denotes the extension of T_{\otimes} to $E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n$.

In this case $\|T\|_{HS} = \|T_{\hat{\otimes}}\|_{HS}$.

Proposition 5.11

The Hilbert space $\mathcal{L}_{HS}(E'_1, \dots, E'_n; F')$ is isometric to $(\mathcal{L}_{HS}(E_1, \dots, E_n; F))'$ through the mapping \mathcal{B} given by

$$\mathcal{B}(\psi)(x'_1, \dots, x'_n) = \sum_{j \in J} \psi(x'_1, \dots, x'_n) f'_j,$$

where $\psi \in (\mathcal{L}_{HS}(E_1, \dots, E_n; F))'$, $x'_k \in E'_k$, $k = 1, \dots, n$, and $(f_j)_{j \in J}$ is an orthonormal basis of F , with $(f'_j)_{j \in J}$ being the corresponding dual orthonormal basis of F' .

Proof. If we prove the result for $F = \mathbb{K}$, we can use it and 5.4.(b) in order to obtain the isometries:

$$\begin{aligned} (\mathcal{L}_{HS}(E_1, \dots, E_n; F))' &\cong (\mathcal{L}_{HS}(E_1, \dots, E_n, F'; \mathbb{K}))' \\ &\cong \mathcal{L}_{HS}(E'_1, \dots, E'_n, F; \mathbb{K}) \cong \mathcal{L}_{HS}(E'_1, \dots, E'_n; F'). \end{aligned}$$

This shows that we have to prove only the case $F = \mathbb{K}$. It is clear that

$$(e'_{1,j_1} \times \dots \times e'_{n,j_n})_{(j_1, \dots, j_n) \in J_1 \times \dots \times J_n}$$

is an orthonormal basis for $\mathcal{L}_{HS}(E_1, \dots, E_n; \mathbb{K})$, when $(e'_{k,j})_{j \in J_k}$ denotes the dual basis of the orthonormal basis $(e_{k,j})_{j \in J_k}$ of E_k , $k = 1, \dots, n$. We have

$$T = \sum_{(j_1, \dots, j_n) \in J_1 \times \dots \times J_n} T(e_{1,j_1}, \dots, e_{n,j_n}) e'_{1,j_1} \times \dots \times e'_{n,j_n},$$

for every $T \in \mathcal{L}_{HS}(E_1, \dots, E_n; \mathbb{K})$. Hence, for $\psi \in (\mathcal{L}_{HS}(E_1, \dots, E_n; \mathbb{K}))'$, we have

$$\sum_{(j_1, \dots, j_n) \in J_1 \times \dots \times J_n} |\mathcal{B}(\psi)(e'_{1,j_1}, \dots, e'_{n,j_n})|^2 = \|\psi\|^2$$

This shows that $\mathcal{B}(\psi)$ is Hilbert-Schmidt and $\|\mathcal{B}(\psi)\|_{HS} = \|\psi\|$.

On the other hand, if $S \in \mathcal{L}_{HS}(E'_1, \dots, E'_n; \mathbb{K})$, we define

$$\psi_S \in (\mathcal{L}_{HS}(E_1, \dots, E_n; \mathbb{K}))'$$

by

$$\psi_S(T) = \sum_{(j_1, \dots, j_n) \in J_1 \times \dots \times J_n} T(e_{1,j_1}, \dots, e_{n,j_n}) S(e'_{1,j_1}, \dots, e'_{n,j_n}).$$

Thus, we have $\mathcal{B}(\psi_S) = S$ and $|\psi_S(T)| \leq \|T\|_{HS} \|S\|_{HS}$. Therefore, $\|\psi_S\| \leq \|S\|_{HS}$. In fact we have equality, since $\|\psi_S\| = \|\mathcal{B}(\psi_S)\|_{HS} = \|S\|_{HS}$. \square

Corollary 5.12

The Hilbert spaces $(E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n)'$ and $E'_1 \hat{\otimes}_H \dots \hat{\otimes}_H E'_n$ are isometric.

Proof. By 5.10 and 5.11 we have the isometries:

$$\begin{aligned} E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n &\cong (E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n)'' \cong (\mathcal{L}_{HS}(E_1, \dots, E_n; \mathbb{K}))' \\ &\cong \mathcal{L}_{HS}(E'_1, \dots, E'_n; \mathbb{K}) \cong (E'_1 \hat{\otimes}_H \dots \hat{\otimes}_H E'_n)'. \end{aligned}$$

The result follows by duality. \square

Proposition 5.13

The following spaces are isometric:

$$\mathcal{L}_{HS}(E_1, \dots, E_n; F), \mathcal{L}_{HS}(E_1, \dots, E_n; \mathbb{K}) \hat{\otimes}_H F \text{ and } (E'_1 \hat{\otimes}_H \dots \hat{\otimes}_H E'_n) \hat{\otimes}_H F.$$

Proof. By 5.4.(b), 5.10 and 5.12 we have the isometries:

$$\begin{aligned} \mathcal{L}_{HS}(E_1, \dots, E_n; F) &\cong \mathcal{L}_{HS}(E_1, \dots, E_n, F'; \mathbb{K}) \cong (E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n \hat{\otimes}_H F)' \\ &\cong (E_1 \hat{\otimes}_H \dots \hat{\otimes}_H E_n)' \hat{\otimes}_H F \end{aligned}$$

and this space is isometric to the space $\mathcal{L}_{HS}(E_1, \dots, E_n; \mathbb{K}) \hat{\otimes}_H F$, as well as to $(E'_1 \hat{\otimes}_H \dots \hat{\otimes}_H E'_n) \hat{\otimes}_H F$. \square

Remark 5.14. The results obtained in this section and in Section 4 give a linear homeomorphism between $(\mathcal{L}_{HS}(E_1, \dots, E_n; F))'$ and $(\mathcal{L}_{VN}^p(E_1, \dots, E_n; F))'$, for $p \in]1, 2]$. But we observe that $(\mathcal{L}_{VN}^p(E_1, \dots, E_n; F))'$ is not normed for $n \geq 2$.

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