Collectanea Mathematica (electronic version): http://www.imub.ub.es/collect

Collect. Math. **54**, 1 (2003), 87–98

Extreme positive definite double sequences which are not moment sequences

Torben Maack Bisgaard

Nandrupsvej 7 st. th., DK-2000 Frederiksberg C, Denmark

E-mail: torben.bisgaard@get2net.dk

Received February 19, 2002

Abstract

From the fact that the two-dimensional moment problem is not always solvable, one can deduce that there must be extreme ray generators of the cone of positive definite double sequences which are not moment sequences. Such an argument does not lead to specific examples. In this paper it is shown how specific examples can be constructed if one is given an example of an N-extremal indeterminate measure in the one-dimensional moment problem (such examples exist in the literature). Konrad Schmüdgen had an example similar to ours about 10 years ago, but did not publish it.

1. Introduction

Suppose $(S, +)$ is a countable abelian semigroup with zero. This is what we always have in mind when we use the term 'semigroup'. A function $\varphi: S \to \mathbb{R}$ is *positive definite* if

$$\sum_{j,k=1}^{n} c_j c_k \varphi(s_j + s_k) \geq 0$$

for every choice of $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbb{R}$. Denote by $\mathcal{P}(S)$ the set of all such functions. A *character* on S is a function $\sigma: S \to \mathbb{R}$ such that $\sigma(0) = 1$ and $\sigma(s + t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. Denote by S^* the set of all characters on S. We equip S^* with the trace topology of the topology of pointwise convergence on $\mathbb{R}^S$, cf. [9], 4.2.2. A function $\varphi: S \to \mathbb{R}$ is called a *moment function* if there is a Radon measure μ on S^* such that

$$\varphi(s) = \int_{S^*} \sigma(s)\, d\mu(\sigma), \quad s \in S. \tag{1}$$

Keywords: Extreme, positive definite, double sequence, moment sequence.
MSC2000: 43A35, 44A60.

(Recall that a Radon measure is a measure μ defined on the Borel σ-field $\mathcal{B}(S^*)$, finite on compact sets and inner regular. In our present situation, inner regularity is automatic since S^* is a Polish space.) Denote by $\mathcal{H}(S)$ the set of all moment functions on S. We have $\mathcal{H}(S) \subset \mathcal{P}(S)$ since if (1) holds and if $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbb{R}$ then

$$\sum_{j,k=1}^{n} c_j c_k \varphi(s_j + s_k) = \int_{S^*} \left(\sum_{j=1}^{n} c_j \sigma(s_j) \right)^2 d\mu(\sigma) \geq 0.$$

The semigroup S is called *semiperfect* if $\mathcal{H}(S) = \mathcal{P}(S)$.

Denote by $\mathcal{P}_e(S)$ the set of all generators of extreme rays in $\mathcal{P}(S)$. Jens Peter Reus Christensen informed us that for each $\varphi \in \mathcal{P}(S)$ there is a Radon measure μ on $\mathcal{P}_e(S)$ such that

$$\varphi(s) = \int_{\mathcal{P}_e(S)} \omega(s)\, d\mu(\omega), \quad s \in S. \tag{2}$$

(This is by a general property of proper closed convex cones in the space $\mathbb{R}^{\mathbb{N}}$, cf. [12].) If S is non-semiperfect then we can choose $\varphi \in \mathcal{P}(S) \setminus \mathcal{H}(S)$. Then in the integral representation (2) the measure μ cannot be concentrated on $\mathcal{H}(S)$ (or else φ would be a moment function). Thus the set $\mathcal{P}_e(S) \setminus \mathcal{H}(S)$ must be nonempty.

With the possible exception of finite groups, the oldest example of a semiperfect semigroup is $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, which is semiperfect by Hamburger's Theorem ([13], see the monograph by Akhiezer [1]). A sequence $(s_n)_{n=0}^{\infty}$ of reals is positive definite if and only if it is a moment sequence, i.e.,

$$s_n = \int_{\mathbb{R}} x^n\, d\mu, \quad n \in \mathbb{N}_0$$

for some measure μ on $\mathbb{R}$. We here denoted, as we shall do later also, by x^n the function $t \mapsto t^n$ on $\mathbb{R}$. The moment sequence (s_n) is *determinate* if there is only one such μ; otherwise, *indeterminate*. The measure μ is called determinate or indeterminate according as (s_n) is determinate or indeterminate. The measure μ is *N-extremal* if the polynomial algebra $\mathbb{R}[x]$ is dense in $L^2(\mu)$. It is well-known that every determinate measure is N-extremal. In fact, if μ is determinate then $\mathbb{R}[x]$ is even dense in $L^2\big((1 + x^2)\mu\big)$, and this condition is characteristic of determinate measures (cf. [1]). (We said 'the oldest example'. Herglotz' Theorem is older than that of Hamburger, but in the former, positive definiteness of a function on the group of integers is defined using $s_j - s_k$ instead of $s_j + s_k$, so it is outside our scope.)

The first semigroup to be shown to be non-semiperfect was $\mathbb{N}_0^2$. The non-semiperfectness of $\mathbb{N}_0^2$ (and more generally, of $\mathbb{N}_0^k$ for all $k \geq 2$) was shown by Berg, Christensen, and Jensen [8] and independently by Schmüdgen [15]. A double sequence $\varphi\colon \mathbb{N}_0^2 \to \mathbb{R}$ is a moment function if and only if there is a measure μ on $\mathbb{R}^2$ such that

$$\varphi(m, n) = \int_{\mathbb{R}^2} x^m y^n\, d\mu(x, y), \quad (m, n) \in \mathbb{N}_0^2.$$

Some time ago, Jens Peter Reus Christensen pointed out to us that although the set $\mathcal{P}_e(\mathbb{N}_0^2) \setminus \mathcal{H}(\mathbb{N}_0^2)$ is necessarily nonempty, no one knew a specific example of a function in this set. The purpose of the present paper is to show that if one can give a specific example of an N-extremal indeterminate measure then one can also give a specific example of a function in $\mathcal{P}_e(\mathbb{N}_0^2)\backslash\mathcal{H}(\mathbb{N}_0^2)$. Specific examples of N-extremal indeterminate measures can be found in [2]. After the present manuscript was completed, Jens Peter Reus Christensen informed us that Konrad Schmüdgen has an explicit example of a function in $\mathcal{P}_e(\mathbb{N}_0^2)\setminus\mathcal{H}(\mathbb{N}_0^2)$. We asked Professor Schmüdgen, who informed us that his example is from 1991 or 1992. His example uses the curve $y^2 = x^{2k+1}$ with $k \in \mathbb{N}$, and for $k = 1$ this curve can be identified with S^* for our semigroup $S = \mathbb{N}_0 \setminus \{1\}$. Also, Professor Schmüdgen's example used N-extremal indeterminate measures.

Our first observation is the following. Suppose h is a homomorphism of a semigroup S onto a semigroup T. Then

$$\varphi \in \mathcal{P}_e(T) \Rightarrow \varphi \circ h \in \mathcal{P}_e(S). \tag{3}$$

To see this, note that it is trivial to verify that $\varphi \circ h$ is positive definite. It remains to be shown that if $\varphi\circ h = \omega_1+\omega_2$ for some $\omega_1, \omega_2 \in \mathcal{P}(S)$ then ω_1 and ω_2 are nonnegative multiples of $\varphi \circ h$. For $s, t \in S$ such that $h(s) = h(t)$ we have

$$\begin{aligned}0 &= \varphi \circ h(2s) + \varphi \circ h(2t) - 2\varphi \circ h(s + t)\\ &= [\omega_1(2s) + \omega_1(2t) - 2\omega_1(s + t)] + [\omega_2(2s) + \omega_2(2t) - 2\omega_2(s + t)].\end{aligned}$$

The numbers in brackets being nonnegative by positive definiteness, they both vanish. By the Cauchy-Schwarz inequality,

$$\big(\omega_1(s) - \omega_1(t)\big)^2 \le \omega_1(0)[\omega_1(2s) + \omega_1(2t) - 2\omega_1(s + t)] = 0,$$

so $\omega_1(s) = \omega_1(t)$. (The Cauchy-Schwarz inequality in its most general form asserts that if $\psi \in \mathcal{P}(S)$ then

$$\Big|\sum_{j=1}^{n}\sum_{p=1}^{r} c_j d_p \psi(s_j + t_p)\Big|^2 \le \sum_{j,k=1}^{n} c_j c_k \psi(s_j + s_k) \sum_{p,q=1}^{r} d_p d_q \psi(t_p + t_q)$$

for all

$$n, r \in \mathbb{N} := \{1, 2, 3, \ldots\},\ c_j \in \mathbb{R},\ s_j \in S(j = 1, \ldots, n),\ d_p \in \mathbb{R}, t_p \in S(p = 1, \ldots, r).$$

It is an application of the inequality $|f(x)|^2 \le \langle f, f\rangle \cdot \varphi(x, x)$ on p. 81 l. 6 from below in [9]. We are here applying the special case that $n = 1$ and $r = 2$.) Similarly, $\omega_2(s) = \omega_2(t)$. Thus there exist real-valued functions φ_1 and φ_2 on T such that $\omega_i = \varphi_i \circ h$ for $i = 1, 2$. Using the fact that $h(S) = T$, one easily sees that φ_1 and φ_2 are positive definite. Now $\varphi \circ h = \omega_1 + \omega_2 = \varphi_1 \circ h + \varphi_2 \circ h = (\varphi_1 + \varphi_2) \circ h$. Since $h(S) = T$ it follows that $\varphi = \varphi_1 + \varphi_2$. Since $\varphi \in \mathcal{P}_e(T)$ it follows that φ_1 and φ_2 are nonnegative multiples of φ. It follows that ω_1 and ω_2 are nonnegative multiples of $\varphi \circ h$, as desired. This proves (3). Now by [11], Proposition 1, we also have the implication $\varphi \circ h \in \mathcal{H}(S) \Rightarrow \varphi \in \mathcal{H}(T)$. Combining this with (3), we get

$$\varphi \in \mathcal{P}_e(T) \setminus \mathcal{H}(T) \Rightarrow \varphi \circ h \in \mathcal{P}_e(S) \setminus \mathcal{H}(S).$$

We see from this that one can find an element ω of $\mathcal{P}_e(\mathbb{N}_0^2) \setminus \mathcal{H}(\mathbb{N}_0^2)$ by choosing a non-semiperfect semigroup S generated by two elements a and b, choosing $\varphi \in \mathcal{P}_e(S)\setminus\mathcal{H}(S)$, and defining $\omega(m,n) = \varphi(ma+nb)$ for $(m,n) \in \mathbb{N}_0^2$.

So far, we have merely reduced the original problem to the corresponding problem on a different semigroup. We shall see, however, that if S is a subsemigroup of $\mathbb{N}_0$ then methods from the rich theory of the classical moment problem can be applied in the study of positive definite functions on S.

It follows from the main result of [10] that if $p, q \in \mathbb{N}$ are such that $p < q$ then the subsemigroup of $\mathbb{N}_0$ generated (as a semigroup with zero) by p and q is semiperfect if and only if p divides q. We shall consider only the case that $(p,q) = (2,3)$. In the remainder of this paper, S always denotes the subsemigroup of $\mathbb{N}_0$ generated (as a semigroup with zero) by $\{2,3\}$, that is, $S = \mathbb{N}_0 \setminus \{1\}$. Although the main theorem of [10] was discovered in 1986, it was not published quickly, so the first published proof of the non-semiperfectness of S is due to Nakamura and Sakakibara [14]. We are going to describe certain elements of $\mathcal{P}_e(S) \setminus \mathcal{H}(S)$ in terms of N-extremal indeterminate measures. As above, one gets similar functions on $\mathbb{N}_0^2$ by applying a homomorphism of $\mathbb{N}_0^2$ onto S, such as $(m,n) \mapsto 2m+3n$.

Thus a new proof of the non-semiperfectness of S is afforded. This conclusion, however, can be proved in a much more explicit way. Indeed, if $\varphi\colon S \to \mathbb{R}$ is defined by $\varphi(2) = -1$, $\varphi(n) = 0$ for odd n, and $\varphi(2n) = 4^{n^2}$ for $n \in S$ then φ can be shown to be in $\mathcal{P}(S) \setminus \mathcal{H}(S)$. We omit the proof, which is rather simple.

2. Results

We first identify S^*. If $x \in \mathbb{R}$ then clearly the function $n \mapsto x^n$ is a character on S. Conversely, suppose $\sigma \in S^*$. Define $x = \sqrt[3]{\sigma(3)}$. Then $\sigma(2)^3 = \sigma(6) = \sigma(3)^2 = x^6$, so $\sigma(2) = x^2$. Since the set $\{2,3\}$ generates S, from the fact that the equation $\sigma(n) = x^n$ holds for $n \in \{2,3\}$ we can infer that it holds for all $n \in S$. This proves that S^* can be identified with $\mathbb{R}$ by identifying $x \in \mathbb{R}$ with the character $n \mapsto x^n$ on S. Thus, a function $\varphi\colon S \to \mathbb{R}$ is a moment function if and only if there is a measure μ on $\mathbb{R}$ such that

$$\varphi(n) = \int_{\mathbb{R}} x^n \, d\mu, \quad n \in S. \tag{4}$$

We are going to consider a positive definite function φ on S. We shall introduce certain objects λ, M, Λ, and f which are said to be *associated* with φ. Since φ can mostly be considered to be fixed, we feel no need to say in each theorem what λ, etc., are.

For $n \in \mathbb{N}_0$ define $s_n = \varphi(n+4)$. Then the sequence $(s_n)_{n=0}^{\infty}$ is positive definite. To see this, note that for $n \in \mathbb{N}_0$ and $c_0, \ldots, c_n \in \mathbb{R}$ we have

$$\sum_{j,k=0}^{n} c_j c_k s_{j+k} = \sum_{j,k=0}^{n} c_j c_k \varphi(t_j + t_k) \geq 0$$

where $t_j = j + 2 \in S$. Since (s_n) is positive definite, by Hamburger's Theorem it follows that there is a measure λ on $\mathbb{R}$ such that

$$\varphi(n+4) = \int_{\mathbb{R}} x^n \, d\lambda, \quad n \in \mathbb{N}_0. \tag{5}$$

We say that λ is *associated* with φ, noting that there may be several such λ. At the end of the paper, however, we shall assume that there is only one.

Let A be the linear subspace of the polynomial algebra $\mathbb{R}[x]$ spanned by monomials of the form x^n with $n \in S$. Then A is just the set of those polynomials in which the coefficient of x vanishes. Since S is a semigroup then A is an algebra. Define a linear form M on A by $M(x^n) = \varphi(n)$ for $n \in S$. Then M is *positive* in the sense that $M(p^2) \geq 0$ for all $p \in A$. This is just a more compact way of stating the positive definiteness of φ. It is well-known that from the positivity of M it follows that the *Cauchy-Schwarz inequality* $M(pq)^2 \leq M(p^2)M(q^2)$ holds for all $p, q \in A$. Equation (5) can be written in the form

$$M(x^{n+4}) = \int_{\mathbb{R}} x^n \, d\lambda,$$

which extends by linearity to

$$M(x^4 p) = \int_{\mathbb{R}} p \, d\lambda, \quad p \in \mathbb{R}[x]. \tag{6}$$

Define a linear form Λ on $\mathbb{R}[x]$ by $\Lambda(p) = M(x^2 p)$ for $p \in \mathbb{R}[x]$.

Theorem 1

The linear form Λ is bounded and of norm less than or equal to $\sqrt{\varphi(0)}$ on the linear subspace $\mathbb{R}[x]$ of $L^2(\lambda)$.

Proof. By the Cauchy-Schwarz inequality, for $p \in \mathbb{R}[x]$ we have $M(x^2 p)^2 \leq M(1)M(x^4 p^2)$, that is (using (6)), $\Lambda(p)^2 \leq \varphi(0) \int p^2 \, d\lambda$. □

Corollary 1

There is a function $f \in L^2(\lambda)$ such that

$$\int f^2 \, d\lambda \leq \varphi(0) \tag{7}$$

and

$$\Lambda(p) = \int pf \, d\lambda, \quad p \in \mathbb{R}[x] \tag{8}$$

or equivalently,

$$\varphi(n+2) = \int x^n f \, d\lambda, \quad n \in \mathbb{N}_0. \tag{9}$$

Moreover,

$$\int p(1 - x^2 f) \, d\lambda = 0, \quad p \in \mathbb{R}[x]. \tag{10}$$

Proof. By Theorem 1, Λ extends to a bounded linear form L, of norm less than or equal to $\sqrt{\varphi(0)}$, on all of $L^2(\lambda)$. Now there is a unique $f \in L^2(\lambda)$ such that $L(g) = \int fg\,d\lambda$ for all $g \in L^2(\lambda)$. Applied to $g = p \in \mathbb{R}[x]$, this gives (8), which is obviously equivalent to (9). Since

$$\int p\,d\lambda = M(x^4 p) = \Lambda(x^2 p) = \int x^2 pf\,d\lambda,$$

we get (10). □

The function f will be said to be *associated* with φ (and with λ). In general, f need not be uniquely determined. Note, however, that f is uniquely determined if $\mathbb{R}[x]$ is dense in $L^2(\lambda)$ (in particular, if λ is determinate).

Theorem 2

If $x^2 f = 1$ λ-a.e. then $\varphi \in \mathcal{H}(S)$.

Proof. Clearly $\lambda(\{0\}) = 0$ and $f = x^{-2}$ λ-a.e. By (7) it follows that $\int x^{-4}\,d\lambda \le \varphi(0)$, so the measure $\mu = \big(\varphi(0) - \int x^{-4}\,d\lambda\big)\varepsilon_0 + x^{-4}\lambda$ is a well-defined positive measure. (For $x \in \mathbb{R}$ we denote by ε_x the Dirac measure at x.) This measure satisfies (4) since by (9) we have $\varphi(n) = \int x^{n-2} f\,d\lambda = \int x^{n-4}\,d\lambda$ for $n \ge 2$. □

Corollary 2

If $\mathbb{R}[x]$ is dense in $L^2\big((1+x^4)\lambda\big)$ then $\varphi \in \mathcal{H}(S)$.

Proof. The equation (10) can be written in the form

$$\int p\frac{1-x^2 f}{1+x^4}(1+x^4)\,d\lambda = 0, \quad p \in \mathbb{R}[x]. \tag{11}$$

Since the measure $(1+x^4)\lambda$ is bounded then the function $1/(1+x^4)$ is in $L^2\big((1+x^4)\lambda\big)$. Moreover,

$$\int \left(\frac{x^2 f}{1+x^4}\right)^2 (1+x^4)\,d\lambda = \int \frac{x^4 f^2}{1+x^4}\,d\lambda \le \int f^2\,d\lambda \le \varphi(0)$$

by (7), so $x^2 f/(1+x^4) \in L^2\big((1+x^4)\lambda\big)$. Thus $(1-x^2 f)/(1+x^4) \in L^2\big((1+x^4)\lambda\big)$. It now follows from (11) that $(1-x^2 f)/(1+x^4)$ is orthogonal to $\mathbb{R}[x]$, which is dense in $L^2\big((1+x^4)\lambda\big)$, so $1 - x^2 f = 0$ λ-a.e. □

Theorem 3

Suppose λ is a determinate measure. In order that a function $\varphi\colon S \to \mathbb{R}$ belong to $\mathcal{P}(S) \setminus \mathcal{H}(S)$ and that λ be associated with φ, it is necessary and sufficient that there be a function $f \in L^2(\lambda)$ satisfying (7), (9), *and* (10), *but such that it is not the case that $x^2 f = 1$ λ-a.e.*

Proof. First suppose $\varphi \in \mathcal{P}(S)\setminus\mathcal{H}(S)$ and that λ is associated with φ. By Corollary 1, there is a function f satisfying (7), (9), and (10). The last condition follows from Theorem 2.

Conversely, suppose the conditions are satisfied. We first have to show that $\varphi \in \mathcal{P}(S)$. The linear form M is of course well-defined even though φ is not assumed to be positive definite. Showing that φ is positive definite is now equivalent to showing that M is positive, i.e., $M(p^2) \geq 0$ for $p \in A$. By the definition of A we can write $p = a + x^2 q$ with $a \in \mathbb{R}$ and $q \in \mathbb{R}[x]$, and we then have to show

$$0 \leq \varphi(0)a^2 + 2a\Lambda(q) + \int q^2\,d\lambda. \tag{12}$$

Since $\int q^2\,d\lambda \geq 0$, showing that (12) holds for all $a \in \mathbb{R}$ for a fixed $q \in \mathbb{R}[x]$ is equivalent to showing $\Lambda(q)^2 \leq \varphi(0)\int q^2\,d\lambda$. But by (8) (which is equivalent to (9)) and by Hölder's inequality,

$$\Lambda(q)^2 = \left(\int qf\,d\lambda\right)^2 \leq \int f^2\,d\lambda \cdot \int q^2\,d\lambda \leq \varphi(0)\int q^2\,d\lambda$$

where we used (7). This proves $\varphi \in \mathcal{P}(S)$. By (9) and (10) we have

$$\varphi(n+4) = \int x^{n+2} f d\lambda = \int x^n\;d\lambda \quad \text{for} \quad n \in \mathbb{N}_0,$$

which shows that λ is associated with φ.

It remains to be shown that $\varphi \notin \mathcal{H}(S)$. Suppose $\varphi \in \mathcal{H}(S)$. Choose a measure μ such that (4) holds. In particular, $\varphi(n+4) = \int x^{n+4}\,d\mu$, and since λ is determinate it follows that $\lambda = x^4\mu$. In particular, $\lambda(\{0\}) = 0$. Since the measure μ is bounded (the total mass being $\varphi(0)$) then $x^{-2} \in L^2(\lambda)$. Now

$$\varphi(n+2) = \int x^{n+2} d\mu = \int x^{n-2} d\lambda \quad \text{for} \quad n \in \mathbb{N}_0,$$

which shows that the function x^{-2} could replace f in (9). Since λ is determinate then $\mathbb{R}[x]$ is dense in $L^2(\lambda)$, which shows that the extension of Λ to a bounded linear form on $L^2(\lambda)$ is uniquely determined. Thus f is uniquely determined, and it follows that $f = x^{-2}$ λ-a.e., a contradiction. □

Theorem 4

In order that the conditions of Theorem 3 can be satisfied, it is necessary that the measure $\sigma = (1+x^2)\lambda$ be indeterminate.

Proof. Suppose σ is determinate. Then $\mathbb{R}[x]$ is dense in $L^2\big((1+x^2)\sigma\big) = L^2\big((1+x^2)^2\lambda\big)$, or equivalently, in $L^2\big((1+x^4)\lambda\big)$. Reference to Corollary 2 completes the proof. $\square$

In the following, σ always denotes the measure $(1+x^2)\lambda$. We have seen that it is necessary that σ is indeterminate. Since λ is determinate then $\mathbb{R}[x]$ is dense in $L^2\big((1+x^2)\lambda\big) = L^2(\sigma)$, that is, σ is N-extremal. An N-extremal indeterminate measure σ has the form $\sigma = \sum_{n=1}^{\infty} a_n \varepsilon_{x_n}$ for some $a_n > 0$ and $x_n \in \mathbb{R}$ such that the set $\{\, x_n \mid n \in \mathbb{N}\,\}$ is discrete in $\mathbb{R}$, cf. [1].

The class of determinate measures λ such that the measure $\sigma = (1+x^2)\lambda$ is indeterminate has been studied intensively and for several purposes by Berg and Duran, cf. [3–6].

Theorem 5

In order that $\varphi \in \mathcal{P}_e(S) \setminus \mathcal{H}(S)$, it is necessary that equality hold in (7).

Proof. If equality does not hold in (7), write $a = \varphi(0) - \int f^2\, d\lambda > 0$, $\varphi_1 = a1_{\{0\}}$, and $\varphi_2 = \varphi - \varphi_1$. Trivially, $\varphi_1 \in \mathcal{P}(S)$. Since φ_2 coincides with φ on $S \setminus \{0\}$ and $\varphi_2(0) = \int f^2\, d\lambda$, it follows from the proof of Theorem 3 that $\varphi_2 \in \mathcal{P}(S)$. Since $\varphi = \varphi_1 + \varphi_2$ and since φ_1 and φ_2 are not nonnegative multiples of φ then $\varphi \notin \mathcal{P}_e(S)$. $\square$

The story of the next Lemma is the following. At first we did not even notice that such a result was needed. Reading through the first version of the manuscript, we noted that the proof of Theorem 6 was incomplete. We had to formulate the Lemma as a Conjecture. We asked Christian Berg, and he produced a proof. We later found a proof of our own. We give both proofs.

Lemma 1

If σ is an N-extremal indeterminate measure with $\sigma(\{0\}) = 0$ and if $k \in L^2(\sigma)$ is orthogonal to the space $x\mathbb{R}[x] = \{\, xp \mid p \in \mathbb{R}[x]\,\}$ and not identically zero then k is nowhere zero on the support of σ.

First proof. (Christian Berg) The reproducing kernel $K\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of the indeterminate moment problem associated with σ satisfies

$$p(x) = \int K(x,y)p(y)\, d\sigma(y)$$

for all $p \in \mathbb{R}[x]$, cf. [1]. Replacing $p(x)$ by $xp(x)$ and setting $x = 0$ we obtain

$$0 = \int K(0,y)yp(y)\, d\sigma(y)$$

which shows that the function $y \mapsto K(0,y)$ is orthogonal to $x\mathbb{R}[x]$. Since $x\mathbb{R}[x]$ has codimension 1 in $\mathbb{R}[x]$, which is dense in $L^2(\sigma)$ (σ being N-extremal), then the closure of $x\mathbb{R}[x]$ has codimension 1 in $L^2(\sigma)$, so a function orthogonal to $x\mathbb{R}[x]$ is uniquely determined up to a scalar factor. Therefore our function k is proportional to the function $y \mapsto K(0,y)$. It is a fact (see [7]) that if μ is any N-extremal indeterminate measure and x is in the support of μ then for $y \in \mathbb{R}$ we have $K(x, y) = 0$ if and only if y is in the support of μ and distinct from x. Choose an N-extremal measure μ which is equivalent to σ (i.e., has the same moment sequence) and satisfies $\mu(\{0\}) > 0$ (see [1]). Since $\sigma(\{0\}) = 0$ then $\mu \neq \sigma$. It is a fact (see [1]) that distinct equivalent N-extremal measures have disjoint supports. Since $\mu(\{0\}) > 0$ then $K(0, y) = 0$ if and only if $y \neq 0$ is in the support of μ, which is disjoint with the support of σ. Thus the function $y \mapsto K(0, y)$ (that is, the function k) is nowhere zero on the support of σ, as desired.

Second proof. Let B be the set of those x in the support of σ such that $k(x) \neq 0$, and define $\mu = 1_B\sigma$. Then k is obviously orthogonal to $x\mathbb{R}[x]$ in $L^2(\mu)$. Since k is not identically zero then $x\mathbb{R}[x]$ is not dense in $L^2(\mu)$. Equivalently, $\mathbb{R}[x]$ is not dense in $L^2(x^2\mu)$. (This equivalence is where we use the hypothesis that $\sigma(\{0\}) = 0$.) So much the less is $\mathbb{R}[x]$ dense in $L^2\big((1 + x^2)\mu\big)$. This means that the measure μ is indeterminate. Now if B had been a proper subset of the support of σ then μ would have been determinate by the result in [7] to the effect that if one removes the mass in one point of the support of an N-extremal indeterminate measure then the remaining measure is determinate. Thus B is all of the support of σ. That is, k is nowhere zero on the support of σ. □

Theorem 6

If λ is determinate, σ is indeterminate, and $\lambda(\{0\}) = 0$ then the conditions in Theorem 3 can be satisfied. If φ is so chosen that equality holds in (7) then $\varphi \in \mathcal{P}_e(S) \setminus \mathcal{H}(S)$.

Proof. The problem is to find $f \in L^2(\lambda)$ such that (10) holds but the function $1 - x^2 f$ does not vanish λ-a.e. Now (10) can be written in the form

$$\int p\left(\frac{1}{1 + x^2} - \frac{x^2 f}{1 + x^2}\right) d\sigma = 0, \quad p \in \mathbb{R}[x].$$

Since σ is an N-extremal indeterminate measure with $\sigma(\{0\}) = 0$ then 0 is not in the support of σ. Hence the function $x^2/(1 + x^2)$ and its inverse are both bounded on the support of σ. It follows that if we write

$$g = \frac{x^2 f}{1 + x^2}$$

then the problem is to choose $g \in L^2(\lambda)$ such that

$$\int p\left(\frac{1}{1 + x^2} - g\right) d\sigma = 0, \quad p \in \mathbb{R}[x]$$

but such that the function $1/(1 + x^2) - g$ does not vanish λ-a.e. Writing

$$h = g/x,$$

we have $g \in L^2(\lambda)$ if and only if $h \in L^2(\sigma)$. Thus the problem is to find $h \in L^2(\sigma)$ such that

$$\int xp\left(\frac{1}{x(1+x^2)} - h\right) d\sigma = 0, \quad p \in \mathbb{R}[x]$$

but such that the function $1/x(1+x^2) - h$ does not vanish λ-a.e. Since σ is a bounded measure, the support of which does not contain zero, then $1/x(1+x^2) \in L^2(\sigma)$. Thus, writing

$$k = \frac{1}{x(1+x^2)} - h,$$

the problem is to find $k \in L^2(\sigma)$, orthogonal to the space $x\mathbb{R}[x] = \{\, xp \mid p \in \mathbb{R}[x] \,\}$ but not identically zero. The function k given by $k(x) = K(0,x)$ is a solution, cf. the first proof of Lemma 1.

We note that k is uniquely determined up to a scalar factor. This is because k has to be orthogonal to $x\mathbb{R}[x]$, which has codimension 1 in $\mathbb{R}[x]$, which is dense in $L^2(\sigma)$ (since λ is determinate), so that the closure of $x\mathbb{R}[x]$ has codimension 1 in $L^2(\sigma)$.

It remains to be shown that if equality holds in (7) then $\varphi \in \mathcal{P}_e(S)$. Suppose $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1, \varphi_2 \in \mathcal{P}(S)$; we have to show that φ_1 and φ_2 are nonnegative multiples of φ. Let λ_i be the measure which is to φ_i what λ is to φ. For $n \in \mathbb{N}_0$ we have

$$\int x^n \, d(\lambda_1 + \lambda_2) = \varphi_1(n+4) + \varphi_2(n+4) = \varphi(n+4) = \int x^n \, d\lambda.$$

By the determinacy of λ it follows that $\lambda_1 + \lambda_2 = \lambda$. Thus λ_1 and λ_2 are both absolutely continuous with respect to λ, so there exist $h_1, h_2 \geq 0$ in $L^\infty(\lambda)$ such that $\lambda_i = h_i\lambda$ for $i = 1, 2$. Since $\lambda_1 + \lambda_2 = \lambda$ then $h_1 + h_2 = 1$ λ-a.e. Choose functions f_i that are to φ_i and λ_i what f is to φ and λ. Then

$$\begin{aligned}\int x^n f d\lambda &= \varphi(n+2) = \varphi_1(n+2) + \varphi_2(n+2)\\ &= \int x^n f_1 d\lambda_1 + \int x^n f_2 d\lambda_2 = \int x^n (f_1h_1 + f_2h_2)\, d\lambda \ \text{ for } n \in \mathbb{N}_0,\end{aligned}$$

hence by linearity $\int pf \; d\lambda = \int p(f_1h_1 + f_2h_2)\, d\lambda$ for $p \in \mathbb{R}[x]$. Since $\mathbb{R}[x]$ is dense in $L^2(\lambda)$ (λ being determinate) it follows that $f = f_1h_1 + f_2h_2$ λ-a.e. Now

$$\int f^2 d\lambda = \varphi(0) = \varphi_1(0) + \varphi_2(0) \geq \int f_1^2 \, d\lambda_1 + \int f_2^2 \, d\lambda_2 = \int (f_1^2h_1 + f_2^2h_2)\, d\lambda.$$

Noting that $f^2 \leq f_1^2h_1 + f_2^2h_2$ λ-a.e. (by convexity of the function $x \mapsto x^2$), we infer

$$(f_1h_1 + f_2h_2)^2 = f_1^2h_1 + f_2^2h_2$$

λ-a.e. By the strict convexity of the function $x \mapsto x^2$ it follows that on the set of those points where h_1 and h_2 are both positive we have $f_1 = f_2$ λ-a.e. Now f_1 is merely an equivalence class in the space $L^2(\lambda_1) = L^2(h_1\lambda)$, so we are at liberty to redefine f_1 on the set of those points where h_1 vanishes. Similarly for f_2. Thus we may as well assume $f_1 = f_2$ λ-a.e. Since $f = f_1h_1 + f_2h_2$ and $h_1 + h_2 = 1$ λ-a.e., it follows that

$f_1 = f_2 = f$ λ-a.e. By that part of the proof of Theorem 3 which is independent of the condition $\varphi \notin \mathcal{H}(S)$, we have $0 = \int p(1 - x^2 f_i)\, d\lambda_i = \int p(1 - x^2 f) h_i \, d\lambda$ for $p \in \mathbb{R}[x]$ and $i = 1, 2$. In order to get the desired conclusion it suffices to infer that the h_i are constants (λ-a.e.) Thus, writing $E = (1 - x^2 f)\mathbb{R}[x] \subset L^1(\lambda)$ and denoting by $E^\perp$ the set of those $h \in L^\infty(\lambda)$ which are orthogonal to E under the canonical duality, we have to show that each element of $E^\perp$ is a constant. Since $E^\perp$ contains the constant 1 (by (10)), it suffices to show that $E^\perp$ is 1-dimensional. With g, h, and k as in the first paragraph of the present proof, we have

$$E = (1 - x^2 f)\mathbb{R}[x] = \big(1 - (1 + x^2)g\big)\mathbb{R}[x] = \big(1 - x(1 + x^2)h\big)\mathbb{R}[x] = x(1 + x^2)k\mathbb{R}[x].$$

Now suppose $l \in E^\perp$. Since $kl \in L^2(\sigma) = L^2\big((1+x^2)\lambda\big)$ then $xkl \in L^2(\lambda)$. Since $l \in E^\perp$ then xkl is orthogonal to $(1+x^2)\mathbb{R}[x]$ in $L^2(\lambda)$. To see that xkl must necessarily belong to the same 1-dimensional space for all such l (whence the desired conclusion since k is nowhere zero on the support of σ, by the Lemma), it now suffices to show that the closure of $(1 + x^2)\mathbb{R}[x]$ has codimension at most 1 in $L^2(\lambda)$. For this we need to refer to complex L^2-spaces. For a measure κ, denote by $L^2_{\mathbf{C}}(\kappa)$ the complex L^2-space. Since λ is determinate then $\mathbb{R}[x]$ is dense in $L^2(\sigma)$, which is equivalent to saying that $\mathbf{C}[x]$ is dense in $L^2_{\mathbf{C}}(\sigma) = L^2_{\mathbf{C}}\big((1 + x^2)\lambda\big)$. Since $1 + x^2 = |x - i|^2$, an equivalent statement is that $(x - i)\mathbf{C}[x]$ is dense in $L^2_{\mathbf{C}}(\lambda)$. Since $(1+x^2)\mathbf{C}[x]$ has codimension 1 in $(x - i)\mathbf{C}[x]$, it follows that the closure of $(1 + x^2)\mathbf{C}[x]$ has codimension 1 in $L^2_{\mathbf{C}}(\lambda)$, or equivalently, the closure of $(1 + x^2)\mathbb{R}[x]$ has codimension 1 in $L^2(\lambda)$, as desired. □

Remark 1. We sum up the construction of elements of $\mathcal{P}_e(\mathbb{N}_0^2) \setminus \mathcal{H}(\mathbb{N}_0^2)$. Choose an N-extremal indeterminate measure σ with $\sigma(\{0\}) = 0$. (Such measures exist, cf. [1]; see [2] for specific examples.) Choose $k \in L^2(\sigma)$ orthogonal to $x\mathbb{R}[x]$ and nonzero. (We saw in the preceding proof that k is uniquely determined up to a scalar factor.) Define $f = 1/x^2 - (1 + x^2)k/x \in L^2(\lambda)$ where λ is the determinate measure $(1 + x^2)^{-1}\sigma$. Define $\varphi\colon S \to \mathbb{R}$ by $\varphi(0) = \int f^2\, d\lambda$ and $\varphi(n + 2) = \int x^n f \, d\lambda$ for $n \in \mathbb{N}_0$. Then $\varphi \in \mathcal{P}_e(S) \setminus \mathcal{H}(S)$. Define $\omega\colon \mathbb{N}_0^2 \to \mathbb{R}$ by $\omega(m, n) = \varphi(2m + 3n)$ for $(m, n) \in \mathbb{N}_0^2$. Then $\omega \in \mathcal{P}_e(\mathbb{N}_0^2) \setminus \mathcal{H}(\mathbb{N}_0^2)$.

Remark 2. In order to complete a characterization of those $\varphi \in \mathcal{P}_e(S)\setminus\mathcal{H}(S)$ for which the associated measure λ is determinate, it remains to determine whether it is possible that $\lambda(\{0\}) > 0$, and if so, to find necessary and sufficient conditions for that case.

Acknowledgment. Running expenses connected with this piece of research were covered by the Carlsberg Foundation.

References

1. N.I. Akhiezer, *The classical moment problem and some related questions in analysis*, Hafner Publishing Co., New York, 1965.
2. C. Berg, Indeterminate moment problems and the theory of entire functions, *J. Comput. Appl. Math.* **65** (1995), 27–55.

3. C. Berg and A.J. Duran, The index of determinacy for measures and the ℓ^2-norm of orthonormal polynomials, *Trans. Amer. Math. Soc.* **347** (1995), 2795–2811.
4. C. Berg and A.J. Duran, When does a discrete differential perturbation of a sequence of orthonormal polynomials belong to ℓ^2?, *J. Funct. Anal.* **136** (1996), 127–153.
5. C. Berg and A.J. Duran, Orthogonal polynomials, L^2-spaces and entire functions, *Math. Scand.* **79** (1996), 209–223.
6. C. Berg and A.J. Duran, Measures with finite index of determinacy or a mathematical model for Dr. Jekyll and Mr. Hyde, *Proc. Amer. Math. Soc.* **125** (1997), 523–530.
7. C. Berg and J.P.R. Christensen, Density questions in the classical theory of moments, *Ann. Inst. Fourier (Grenoble)* **31** (1981), 99–114.
8. C. Berg, J.P.R. Christensen, and C.U. Jensen, A remark on the multidimensional moment problem, *Math. Ann.* **243** (1979), 163–169.
9. C. Berg, J.P.R. Christensen, and P. Ressel, *Harmonic analysis on semigroups. Theory of positive definite and related functions*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1984.
10. T.M. Bisgaard, Extensions of Hamburger's theorem, *Semigroup Forum* **57** (1998), 397–429.
11. T.M. Bisgaard and P. Ressel, Unique disintegration of arbitrary positive definite functions on $*$-divisible semigroups, *Math. Z.* **200** (1989), 511–525.
12. G. Choquet, *Lectures on analysis*, W.A. Benjamin, Inc., New York-Amsterdam, 1969.
13. H.L. Hamburger, Über eine Erweiterung des Stieltjesschen Momentenproblemes, *Math. Ann.* **81** (1920), 235–319; **82** (1921), 120–164, 168–187.
14. Y. Nakamura and N. Sakakibara, Perfectness of certain subsemigroups of a perfect semigroup, *Math. Ann.* **287** (1990), 213–220.
15. K. Schmüdgen, An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional, *Math. Nachr.* **88** (1979), 385–390.