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Wavelets on the integers

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Abstract

In this paper the theory of wavelets on the integers is developed. For this, one needs to first find analogs of translations and dyadic dilations which appear in the classical theory. Translations in $\ell^2(\mathbb{Z})$ are defined in the obvious way, taking advantage of the additive group structure of the integers. Dyadic dilations, on the other hand, pose a greater problem. In the classical theory of wavelets on the real line, translation T and dyadic dilation D obey the "commutativity" relation $DT^2 = TD$. We choose to define dyadic dilations on the integers in terms of this functional equation. All such dyadic dilations are characterized and the corresponding multiresolution structures they generate are introduced and examined. The main results of this paper focus on connecting multiresolution structures and wavelets on the integers with their counterparts on the line and include the fact that every wavelet on the integers is an MRA wavelet.

1. Introduction

The goal of this paper is to develop the theory of wavelets in $\ell^2(\mathbb{Z})$. First of all, the notion of a dyadic dilation on square-summable sequences is defined and these dyadic dilations (there are infinitely many of them) are characterized. In the process, several very important properties of discrete dyadic dilations will be derived. Following this, we will define discrete wavelets. Many of the initial results can be found in [1]; these results will be extended in this paper and new results on MRA-type structures on the integers will be presented. These results will show that MRAs on $\ell^2(\mathbb{Z})$ arise very naturally from the choice of dyadic dilation and in many cases are connected to traditional MRAs on the real line via an isometry. One conclusion of this comparison

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will be that all discrete wavelets are discrete MRA wavelets. After all of this has been accomplished, there will be some discussion of other results in the theory which may be of interest. This work is the result of a senior honors project done at Washington University in St. Louis under the supervision and guidance of Guido L. Weiss and Edward N. Wilson.

2. Dilations

Let τ be the translation operator on $\ell^2(\mathbb{Z})$ ($(\tau f)_k := f_{k-1}$). This operator is unitary:

$$\tau^{-1} = \tau^*. \tag{1}$$

Furthermore, if we let $\Delta = \{\Delta_j\}_{j\in\mathbb{Z}}$ be the sequence with entries

$$\Delta_j := \delta_{j,0} \qquad \forall j \in \mathbb{Z}, \tag{2}$$

a basis of $\ell^2(\mathbb{Z})$ is given by all the translates of Δ; i.e., $\{\tau^k\Delta\}_{k\in\mathbb{Z}}$ is a complete orthonormal system.

For functions in $L^2(\mathbb{R})$, the meaning of dyadic dilation is clear: $f(\cdot) \mapsto \sqrt{2}f(2\cdot)$. Let D denote the map which takes f to its dyadic dilation. If we now fix $t \in \mathbb{R}$ and let T_t be the map which sends $f(\cdot) \mapsto f(\cdot - t)$, we observe that

$$\begin{aligned} T_tD(f) &= \sqrt{2}f(2(\cdot - t)) \\ &= \sqrt{2}f(2\cdot -2t) = DT_t^2(f). \end{aligned}$$

This commutativity ($DT_t^2 = T_tD$) is the key property of the dyadic dilation on $L^2(\mathbb{R})$ which we would like dyadic dilation operators on $\ell^2(\mathbb{Z})$ to possess as well. However, for reasons which will hopefully become clear as we progress, it is advantageous to reverse the commutativity relationship (technically making our operators contraction operators rather than dilation operators).

DEFINITION 1. A linear operator ρ on $\ell^2(\mathbb{Z})$ will be called a dyadic dilation operator if and only if

$$\rho\tau = \tau^2\rho \tag{3}$$

$$||\rho f|| = ||f|| \qquad \forall f \in \ell^2(\mathbb{Z}) \tag{4}$$

Observe that property (4) ensures that every dyadic dilation operator is an isometry into $\ell^2(\mathbb{Z})$ but does not guarantee that a dyadic dilation is unitary. To obtain an idea of what these dyadic dilation operators do, we will present two simple examples. The first example takes a sequence to its upsampled counterpart:

$$\rho_u\big(\{\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots\}\big) := \big\{\ldots, 0, f_{-2}, 0, f_{-1}, 0, f_0, 0, f_1, 0, f_2, 0, \ldots\big\}.$$

Clearly this function preserves the norm of f, and it is also true that it has the desired commutativity with τ. Both upsampling and downsampling play crucial roles in the

discrete wavelet transform and other discrete wavelet algorithms. See [4] for more details. Observe that ρ_u is not invertible.

Another example is the "Haar" dyadic dilation:

$$\rho_{Haar}\big(\{\ldots, f_{-1}, f_0, f_1, \ldots\}\big) := \Big\{\ldots, \frac{f_{-2}}{\sqrt{2}}, \frac{f_{-1}}{\sqrt{2}}, \frac{f_{-1}}{\sqrt{2}}, \frac{f_0}{\sqrt{2}}, \frac{f_0}{\sqrt{2}}, \frac{f_1}{\sqrt{2}}, \frac{f_1}{\sqrt{2}}, \ldots\Big\}.$$

Suppose we define a map Φ from $\ell^2(\mathbb{Z})$ to the class of functions $F \subset L^2(\mathbb{R})$ which are constant on $(k, k+1]$ for all $k \in \mathbb{Z}$ by

$$(\Phi f)(x) = f_{\lfloor x \rfloor},$$

where $\lfloor x \rfloor$ is the greatest integer less than x. Clearly Φ is invertible and an isometry. Via Φ we may "pull back" the standard dyadic dilation operator on $L^2(\mathbb{R})$ (or rather its inverse):

$$\Phi^{-1} D^{-1} \Phi \quad \text{takes} \quad f \mapsto \frac{1}{\sqrt{2}} \Phi^{-1} \Big[(\Phi f)\big(\frac{x}{2}\big)\Big].$$

This map is simply the Haar dilation ρ_{Haar}. We call it the Haar dilation because it arises naturally (as we will see later) from the Haar wavelet. For the time being, simply observe that any function in F is easily expressed in terms of dilations and translations of the Haar wavelet.

These are by no means the only possible dyadic dilation operators (Theorem 1 near the end of this section characterizes all dyadic dilations). They do however, seem to indicate that a dyadic dilation is, in some sense, a way to "stretch-out" a sequence into one of "twice the length."

The first important property of ρ is that it preserves the inner product of two sequences:

Proposition 1

$\langle \rho f, \rho g \rangle = \langle f, g \rangle \quad \forall f, g \in \ell^2(\mathbb{Z})$.

Proof. This follows immediately from polarization:

$$\langle f, g \rangle = \frac{1}{4} \sum_{j=0}^{3} i^j \big\| f + i^j g \big\|^2.$$

As ρ preserves norms, it must also preserve inner products. □

Using the above proposition, we can begin to understand how ρ must behave on the sequence level.

Proposition 2

The following equation is true for all $f \in \ell^2(\mathbb{Z})$:

$$\big\langle \rho f, \tau^k \Delta \big\rangle = \sum_{l \in \mathbb{Z}} \big\langle f, \tau^l \Delta \big\rangle \big\langle \rho\Delta, \tau^{k-2l} \Delta \big\rangle. \tag{5}$$

Proof. Let $f \in \ell^2(\mathbb{Z})$.

$$\begin{aligned}\left\langle \rho f, \tau^k \Delta \right\rangle &= \left\langle \rho \left(\sum_{l\in\mathbb{Z}} \left\langle f, \tau^l \Delta \right\rangle \tau^l \Delta \right), \tau^k \Delta \right\rangle \\ &= \sum_{l\in\mathbb{Z}} \left\langle f, \tau^l \Delta \right\rangle \left\langle \rho\tau^l \Delta, \tau^k \Delta \right\rangle .\end{aligned}$$

We need only observe that $\left\langle \rho\tau^l \Delta, \tau^k \Delta \right\rangle = \left\langle \tau^{2l}\rho\Delta, \tau^k \Delta \right\rangle = \left\langle \rho\Delta, \tau^{k-2l}\Delta \right\rangle$ and we obtain the desired conclusion. □

Notice that $(\rho f)_k = \left\langle \rho f, \tau^k \Delta \right\rangle$. Therefore, with respect to the basis $\{\tau^k \Delta\}_{k\in\mathbb{Z}}$, the matrix form of $\rho = (\rho_{k,l})_{k,l\in\mathbb{Z}}$ has entries $\rho_{k,l} = \left\langle \rho\Delta, \tau^{k-2l}\Delta \right\rangle$. For the matrix $(\rho_{k,l})$, then, all entries with a fixed value of $k-2l$ are equal. Also as one would expect, the columns of this matrix form an orthonormal system, demonstrated by the following proposition:

Proposition 3

The sequence $r_k = \left\langle \rho\Delta, \tau^k \Delta \right\rangle$ *satisfies*

$$\sum_{k\in\mathbb{Z}} r_{k-2l}\overline{r_{k-2l'}} = \delta_{l,l'}. \tag{6}$$

Proof. As observed in the proof of (5), $(\rho\Delta)_k = \left\langle \rho\Delta, \tau^k \Delta \right\rangle = r_k$. Therefore $\rho\Delta = \sum_{k\in\mathbb{Z}} r_k \tau^k \Delta$ and hence $\tau^{2l}\rho\Delta = \sum_{k\in\mathbb{Z}} r_k \tau^{k+2l}\Delta = \sum_{k\in\mathbb{Z}} r_{k-2l}\tau^k \Delta$

$$\begin{aligned}\delta_{l,l'} &= \left\langle \tau^l \Delta, \tau^{l'} \Delta \right\rangle = \left\langle \rho\tau^l \Delta, \rho\tau^{l'} \Delta \right\rangle \\ &= \left\langle \tau^{2l}\rho\Delta, \tau^{2l'}\rho\Delta \right\rangle .\end{aligned}$$

This last inner product is equal to $\sum_{k\in\mathbb{Z}} r_{k-2l}\overline{r_{k-2l'}}$ as desired. □

These properties of ρ, i.e., that $\rho\tau = \tau^2\rho$ and equation (6), characterize all dyadic dilations. Their sufficiency is shown by the following proposition.

Proposition 4

Suppose a sequence $\{r_k\}_{k\in\mathbb{Z}}$ *satisfies (6). Then the operator defined by*

$$\rho f := \sum_{l\in\mathbb{Z}} \left\langle f, \tau^l \Delta \right\rangle \left(\sum_{k\in\mathbb{Z}} r_k \tau^{k+2l}\Delta \right) \tag{7}$$

is a (dyadic) dilation operator.

Proof. First of all, the sum $\sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta$ exists and has norm 1 since

$$\left|\left|\sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta\right|\right|^2 = \sum_{k\in\mathbb{Z}} r_{k-2l}\overline{r_{k-2l}} = 1.$$

Condition (6) not only implies that the sum converges, but that, as a function of l produces an orthonormal system:

$$\left\langle \sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta \ , \ \sum_{k\in\mathbb{Z}} r_k\tau^{k+2l'}\Delta \right\rangle = \sum_{k\in\mathbb{Z}} r_{k-2l}\overline{r_{k-2l'}} = \delta_{l,l'}.$$

Thus, as the sequence $\langle f, \tau^j\Delta\rangle$ is square-summable, the sum over l in (7) also converges.

Now let us show that ρ satisfies (3):

$$\begin{aligned}\rho\tau f &= \sum_{l\in\mathbb{Z}} \langle \tau f, \tau^l\Delta\rangle \left(\sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta\right) = \sum_{l\in\mathbb{Z}} \langle f, \tau^{l-1}\Delta\rangle \left(\sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta\right)\\ &= \sum_{l\in\mathbb{Z}} \langle f, \tau^l\Delta\rangle \left(\sum_{k\in\mathbb{Z}} r_k\tau^{k+2(l+1)}\Delta\right) = \sum_{l\in\mathbb{Z}} \langle f, \tau^l\Delta\rangle \tau^2 \left(\sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta\right)\\ &= \tau^2 \left(\sum_{l\in\mathbb{Z}} \langle f, \tau^l\Delta\rangle \left(\sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta\right)\right) = \tau^2\rho f.\end{aligned}$$

Next we check that ρ preserves the norm of f as required by (4). As

$$\left\{\sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta\right\}_{l\in\mathbb{Z}}$$

is an orthonormal system, we have that

$$||\rho f||^2 = \left|\left|\sum_{l\in\mathbb{Z}} \langle f, \tau^l\Delta\rangle \left(\sum_{k\in\mathbb{Z}} r_k\tau^{k+2l}\Delta\right)\right|\right|^2 = \sum_{l\in\mathbb{Z}} \langle f, \tau^l\Delta\rangle \overline{\langle f, \tau^l\Delta\rangle} = ||f||^2.$$

Therefore, ρ is a (dyadic) dilation operator by definition. □

While these sequential properties of dyadic dilations are interesting and useful, we have found that many of the proofs that lay ahead are much easier to understand in the language of Fourier transforms. To begin with, let us consider a function $m \in L^2(\mathbb{T})$ whose Fourier coefficients are the elements of the sequence (r_k) defined in Proposition 3:

$$m(\xi) := \sum_{k\in\mathbb{Z}} r_k e^{ik\xi}, \tag{8}$$

where the convergence of the sum is taken in the $L^2([0, 2\pi))$-norm. We will denote the Fourier transform of a general sequence $(f_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ by $f^\vee$, that is,

$$f^\vee(\xi) = \sum_{k\in\mathbb{Z}} f_k e^{ik\xi}.$$

Proposition 5

For any $f \in \ell^2(\mathbb{Z})$,

$$(\rho f)^\vee(\xi) = m(\xi) f^\vee(2\xi) \quad a.e. \tag{9}$$

Proof. We saw earlier that $\{r_{k-2l}\}_{l\in\mathbb{Z}}$ is an orthonormal system in $\ell^2(\mathbb{Z})$. But

$$\begin{aligned}(r_{\cdot-2l})^\vee(\xi) &= \sum_{k\in\mathbb{Z}} r_{k-2l}e^{ik\xi} \\ &= \sum_{k\in\mathbb{Z}} r_k e^{i(k+2l)\xi} \\ &= e^{2il\xi}\sum_{k\in\mathbb{Z}} r_k e^{ik\xi} \\ &= e^{2il\xi}m(\xi),\end{aligned}$$

so $\{e^{2il\xi}m(\xi)\}_{l\in\mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{T})$. In particular,

$$g(\xi) := \sum_{l\in\mathbb{Z}} m(\xi)f_l e^{2il\xi}$$

is a convergent series in the L^2-norm. Let

$$S_N(\xi) = \sum_{l=-N}^{N} m(\xi)f_l e^{2il\xi}$$

and similarly take

$$R_N(\xi) = \sum_{|l|>N} m(\xi)f_l e^{2il\xi}.$$

We may then conclude

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} |m(\xi)f^\vee(2\xi) - g(\xi)|\ d\xi \\ &\quad= \frac{1}{2\pi}\int_0^{2\pi} |m(\xi)f^\vee(2\xi) - S_N(\xi) - R_N(\xi)|\ d\xi \\ &\quad\le \frac{1}{2\pi}\int_0^{2\pi} |m(\xi)f^\vee(2\xi) - S_N(\xi)|\ d\xi + \frac{1}{2\pi}\int_0^{2\pi} |R_N(\xi)|\ d\xi \\ &\quad\le ||m||\left|\left|f^\vee(2\cdot) - \sum_{l=-N}^{N} f_l e^{2il\cdot}\right|\right| + ||1||||R_N||,\end{aligned}$$

where the last line follows from the Cauchy-Schwartz inequality. It is clear, however, that both terms on the r.h.s. go to zero as $N\to\infty$, so we must have that the original integral is zero, which can happen only if the integrand is zero almost everywhere. In particular, we can conclude that $m(\xi)f^\vee(2\xi)\in L^2(\mathbb{T})$ and, even more, $\sum_{l\in\mathbb{Z}} m(\xi)f_l e^{2il\xi} = m(\xi)f^\vee(2\xi)$ in the L^2-norm. All that remains to be shown is that this sum is, in fact, equal to $(\rho f)^\vee$. As both ρ and the Fourier transform operator are continuous, we may appeal to sequential continuity and conclude that

$$\begin{aligned}(\rho f)^\vee &= \left(\rho\left(\sum_{l\in\mathbb{Z}} f_l \tau^l \Delta\right)\right)^\vee \\ &= \sum_{l\in\mathbb{Z}} f_l (\rho\tau^l\Delta)^\vee \\ &= \sum_{l\in\mathbb{Z}} f_l (\tau^{2l}\rho\Delta)^\vee \\ &= \sum_{l\in\mathbb{Z}} f_l e^{2il\cdot} m.\end{aligned}$$

This allows us to conclude that $(\rho f)^\vee(\xi) = m(\xi) f(2\xi)$ for almost every $\xi \in [0, 2\pi)$. □

Proposition 6

Given any $f^\vee \in L^2(\mathbb{T})$, $m \cdot f^\vee \in L^2(\mathbb{T})$ as well.

Proof. If it so happens that $f^\vee(\xi) = g^\vee(2\xi)$ for some $g^\vee \in L^2(\mathbb{T})$, we conclude that $(\rho g)^\vee = m \cdot f^\vee$ from the previous proposition, during which we showed that this function is square-integrable. But for arbitrary $f^\vee$, we have

$$f^\vee(\cdot) = \frac{1}{2}\left[f(\cdot) + f(\cdot + \pi)\right] + \frac{1}{2}\left[f(\cdot) - f(\cdot + \pi)\right].$$

The first term on the right-hand side is in fact π periodic, and the second is also π periodic if it is multiplied by $e^{i\cdot}$ (which will not change the fact that it is square-integrable). Thus,

$$f^\vee = (f_1)^\vee(2\cdot) + e^{-i\cdot}(f_2)^\vee(2\cdot), \quad \text{so} \quad m \cdot f^\vee = (\rho f_1)^\vee + e^{-i\cdot}(\rho f_2)^\vee$$

is square-integrable as well. □

As ρ is an isometry, it must have an adjoint operator ρ^* which is a left inverse to ρ and vanishes on the orthogonal complement of the image of ρ. For any sequence f, then, we will have $||\rho^* f|| \leq ||f||$. It is also easy to show that $(\rho^* f)_l = \sum_{k\in\mathbb{Z}} f_k \overline{r_{k-2l}}$, but we are not interested in this formula as much as we are in how ρ^* behaves in Fourier transform space.

Proposition 7

For any $f \in \ell^2(\mathbb{Z})$, given a dyadic dilation ρ, its adjoint ρ^* satisfies

$$(\rho^* f)^\vee(\xi) = \frac{1}{2}\left[f^\vee\left(\frac{\xi}{2}\right)\overline{m\left(\frac{\xi}{2}\right)} + f^\vee\left(\frac{\xi}{2}+\pi\right)\overline{m\left(\frac{x}{2}+\pi\right)}\right] \quad a.e. \tag{10}$$

where, as before, $m(\xi) = \Delta^\vee(\xi)$.

Proof. Let f, g be square-summable sequences.

$$\begin{aligned}(\rho^* f, g) &= (f, \rho g) \\ &= (f^\vee, (\rho g)^\vee)_{L^2(\mathbb{T})} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f^\vee(\xi)\overline{m(\xi)g^\vee(2\xi)}\, d\xi \\ &= \frac{1}{4\pi}\int_0^{4\pi} f^\vee\left(\frac{\xi}{2}\right)\overline{m\left(\frac{\xi}{2}\right)g^\vee(\xi)}\, d\xi \\ &= \frac{1}{4\pi}\int_0^{2\pi}\left[f^\vee\left(\frac{\xi}{2}\right)\overline{m\left(\frac{\xi}{2}\right)} + f^\vee\left(\frac{\xi}{2}+\pi\right)\overline{m\left(\frac{x}{2}+\pi\right)}\right]\overline{g^\vee(\xi)}\, d\xi.\end{aligned}$$

From the previous proposition, it is clear that the term in brackets is square-integrable, and it's also 2π-periodic. Thus, it is the Fourier transform of some square-summable sequence h. But by this calculation we can conclude that $(\rho^* f, g) - (h, g) = 0$ for any sequence g. Thus, we can take $g = \rho^* f - h$ and conclude that $||\rho^* f - h||^2 = 0$ which is only true when $h = \rho^* f$. But the Fourier transform of h is exactly the function given in (10). □

We are now positioned to characterize dyadic dilations once again, this time, in terms of Fourier transforms.

Theorem 1

A continuous linear operator ρ on $\ell^2(\mathbb{Z})$ is a dyadic dilation if and only if $\rho\tau = \tau^2\rho$ and $m = (\rho\Delta)^\vee$ satisfies

$$\frac{1}{2}\left[|m(\xi)|^2 + |m(\xi+\pi)|^2\right] = 1 \quad a.e. \tag{11}$$

Furthermore, for every $m \in L^2(\mathbb{T})$ with this property, there is a unique dyadic dilation ρ with $(\rho\Delta)^\vee = m$.

Proof. First we show that for any dyadic dilation, (11) is satisfied. This, however, is straightforward as

$$\begin{aligned}1 &= (\Delta)^\vee(\xi) \\ &= (\rho^*\rho\Delta)^\vee(\xi) \\ &= \frac{1}{2}\left[(\rho\Delta)^\vee\left(\frac{\xi}{2}\right)\overline{m\left(\frac{\xi}{2}\right)} + (\rho\Delta)^\vee\left(\frac{\xi}{2}+\pi\right)\overline{m\left(\frac{\xi}{2}+\pi\right)}\right] \\ &= \frac{1}{2}\left[m\left(\frac{\xi}{2}\right)\overline{m\left(\frac{\xi}{2}\right)} + m\left(\frac{\xi}{2}+\pi\right)\overline{m\left(\frac{\xi}{2}+\pi\right)}\right].\end{aligned}$$

So we replace $\xi/2$ with ξ and we conclude that $[|m(\xi)|^2 + |m(\xi+\pi)|^2]/2 = 1$ almost everywhere as desired.

Next we show that ρ with $\rho\tau = \tau^2\rho$ and $m = (\rho\Delta)^\vee$ satisfying (11) is actually a dyadic dilation. All we need to show is that ρ is an isometry. Notice that $\left\{e^{2il\cdot}m\right\}_{l\in\mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{T})$:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left(e^{2il\xi}m(\xi)\right)\overline{(e^{2il'\xi}m(\xi))}d\xi &= \frac{1}{2\pi}\int_0^{2\pi}e^{2i(l-l')\xi}|m(\xi)|^2d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi}e^{2i(l-l')(\xi+\pi)}|m(\xi+\pi)|^2d\xi\,.\end{aligned}$$

Averaging the last two lines, we conclude that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left(e^{2il\xi}m(\xi)\right)\overline{(e^{2il'\xi}m(\xi))}d\xi &= \frac{1}{2\pi}\int_0^{2\pi}e^{2i(l-l')\xi}\frac{1}{2}\left[|m(\xi)|^2+|m(\xi+\pi)|^2\right]d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi}e^{2i(l-l')\xi}d\xi \\ &= \delta_{l,l'}.\end{aligned}$$

As before, $(\rho\tau^l\Delta)^\vee = (\tau^{2l}\rho\Delta)^\vee = e^{2il\cdot}m$ so we conclude $(\rho f)^\vee = \sum_{l\in\mathbb{Z}} f_l e^{2il\cdot}m$. Clearly by this formula $||\rho f|| = ||f||$, so ρ is a dyadic dilation.

Finally, if $\rho\Delta = \rho'\Delta$ for some ρ, ρ' dyadic dilations, we will have by (9) that $\rho = \rho'$. □

Stepping back for a moment, we see that the problem of finding a dyadic dilation ρ is the same as the problem of finding a 2π-periodic function m which satisfies (11). Up to normalization, such functions have been extensively studied (as every MRA has an associated low-pass filter for which $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$ almost everywhere; see [2]). To give an idea of the many different types of dyadic dilations there may be, we present the following theorem, which says that in the case of m a trigonometric polynomial, we are free to choose half of its Fourier coefficients provided that they satisfy a given inequality. (For example, our chosen coefficients c_j necessarily must have $\sum |c_j|^2 \le 1$, but this is not sufficient. On the other hand, $\sum |c_j| < 1$ is sufficient but not necessary).

Theorem 2

Let c_j be a finitely supported sequence. There exists a sequence $\{r_k\}_{k\in\mathbb{Z}}$ which is finitely supported, satisfies (6) and has $r_{2k} = c_k$ for all k if and only if $|c^\vee(\xi)| = |\sum_{k\in\mathbb{Z}} c_k e^{ik\xi}| \le 1$ for all real ξ.

Proof. We prove the reverse direction first. Given any finitely supported sequence a_k (not necessarily satisfying the constraint given in the statement of the theorem), let us define the following polynomial:

$$p(z) := \sum_{j\in\mathbb{Z}} z^j \left(\sum_{k\in\mathbb{Z}} a_k\overline{a_{k-j}}\right).$$

Elementary algebra tells us that

$$p(z) = \left(\sum_{k\in\mathbb{Z}} a_k z^k\right)\left(\sum_{k\in\mathbb{Z}} \overline{a_k} z^{-k}\right).$$

When we take the sum $p(z)+p(-z)$, all terms in odd powers of z vanish. If, in addition, the a_k satisfy (6), we see that the coefficient of z^{2j} in $p(z)+p(-z)$ will equal $2\delta_{j,0}$, which means that $p(z)+p(-z)=2$. Knowing the factored form of p, we can conclude

$$\left(\sum_{k\in 2\mathbb{Z}} a_k z^k\right)\left(\sum_{k\in 2\mathbb{Z}} \overline{a_k} z^{-k}\right) + \left(\sum_{k\in 2\mathbb{Z}+1} a_k z^k\right)\left(\sum_{k\in 2\mathbb{Z}+1} \overline{a_k} z^{-k}\right) = 1.$$

This further reduces to

$$\left(\sum_{k\in\mathbb{Z}} a_{2k} z^{2k}\right)\left(\sum_{k\in\mathbb{Z}} \overline{a_{2k}} z^{-2k}\right) + \left(\sum_{k\in\mathbb{Z}} a_{2k+1} z^{2k}\right)\left(\sum_{k\in\mathbb{Z}} \overline{a_{2k+1}} z^{-2k}\right) = 1 \qquad (12)$$

because a factor of z cancels with a factor of z^{-1} in the sum over odd k. The goal, then, is to specify a_k for even k and use (12) as an equation to solve for a_k when k is odd. In particular, we look for a_{2k+1} such that the following equation is satisfied for all complex w:

$$\begin{aligned} p_0(w) &:= 1 - \left(\sum_{k\in\mathbb{Z}} c_k w^k\right)\left(\sum_{k\in\mathbb{Z}} \overline{c_k} w^{-k}\right) \\ &= \left(\sum_{k\in\mathbb{Z}} a_{2k+1} w^k\right)\left(\sum_{k\in\mathbb{Z}} \overline{a_{2k+1}} w^{-k}\right), \end{aligned}$$

where the c_k are the complex numbers described in the statement of the theorem (we assume that $c_k = 0$ if it is not explicitly given). Notice that, on the unit circle, $w^{-1} = \overline{w}$, so the polynomial p_0 takes on real values there, and we have

$$p_0(e^{i\theta}) = 1 - \left|\sum_{k\in\mathbb{Z}} c_k e^{ik\theta}\right|^2.$$

Suppose $p_0(e^{i\theta}) = 0$ for some θ (we will not consider the case p_0 identically zero, as this is easily shown to give rise to a dyadic dilation when we take $r_k = 0$ for odd k). As, by choice of the c_j, p_0 is non-negative on the unit circle, $e^{i\theta}$ must be a local minimum of p_0 (on the circle). This, however, can only be the case if the first non-vanishing derivative of p_0 (p_0', p_0'', ...) is an even order. Therefore, any roots of p_0 with modulus 1 have even multiplicity. Suppose that p_0 has a root whose modulus is strictly less than 1. As $p_0(1/\overline{z}) = \overline{p_0(z)}$, it must also have a corresponding root of magnitude greater than 1. (The reverse is true if p_0 has a root outside the unit disk). Therefore, we can define a (monic) polynomial b whose roots are precisely those roots of p_0 which (i) have magnitude is strictly less than one, or (ii) have magnitude 1, but in this case we take the

multiplicity of the roots to be exactly *half* the corresponding multiplicity for p_0. Thus, the polynomial $b(w)\overline{b(1/\overline{w})}$ has precisely the same roots and the same multiplicities as p_0, and so differ only by a multiplicative constant λ^2. For any w on the unit circle which is not a root of p_0, we must have $\lambda^2 = b(w)\overline{b(1/\overline{w})}/p_0(w) = |b(w)|^2/p_0(w)$. As $p_0(w)$ is real and positive, λ^2 is positive as well, so we may assume that λ itself is real and positive. The end result is that we have found b_k's (the coefficients of b) so that

$$\left(\sum_{k\in\mathbb{Z}} c_k w^k\right)\left(\sum_{k\in\mathbb{Z}} \overline{c_k} w^{-k}\right) + \left(\sum_{k\in\mathbb{Z}} \lambda b_k w^k\right)\left(\sum_{k\in\mathbb{Z}} \overline{\lambda b_k} w^{-k}\right) = 1.$$

Therefore, if we let $r_{2k} = c_k$ and $r_{2k+1} = \lambda b_k$, we can follow our initial algebra backwards and arrive at the conclusion that the sequence r_j must satisfy (6), and, by our construction, r_k is finitely supported.

For the converse, note that, if the c_k can be "interlaced" with another sequence so that the result satisfies (6), then p_0 is necessarily non-negative on the unit circle. Thus, if $|\sum_{k\in\mathbb{Z}} c_k e^{ik\xi}| > 1$ for any ξ, we have a contradiction. □

Note that our method to "fill in" the sequence r_k could have produced a different sequence if we had chosen different roots in the definition of b. (All we really needed was to make sure that the roots of b did not include both w and $\overline{w}^{-1}$). In other words, we stress that even though such a sequence r_j exists, it is not, in general, unique.

3. Wavelets in $\ell^2(\mathbb{Z})$

We now define what we mean by a wavelet.

DEFINITION 2. A sequence $\psi \in \ell^2(\mathbb{Z})$ is called a wavelet (with respect to the dyadic dilation ρ) if and only if $\{\rho^j \tau^{2k}\psi\}_{j,k\in\mathbb{Z},j\geq 0}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$.

To agree with usual wavelet conventions, we take translations first followed by dilations. Note that one could instead consider systems generated by dilating first and then translating. Such investigations on the real line lead to so-called quasi-affine systems [3]. We will not consider this case here.

It may seem curious that we have taken only translations by multiples of two in the definition of a wavelet. The developments of the following sections will lead us to the conclusion that this is the "natural" definition to use, and so, in some sense, the precise reasons must remain a mystery for the time being. We can, however, explain why the obvious choice (taking all translations) *cannot* be the best one.

Suppose s is a sequence (say with norm one), which is orthogonal to any of its (non-trivial) translates:

$$\langle s, \tau^j s\rangle = \delta_{j,0}.$$

As $(\tau^j s)^\vee(\xi) = e^{ij\xi} s^\vee(\xi)$, we have that

$$\begin{aligned}\delta_{j,0} &= \langle s, \tau^j s \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} s^\vee(\xi)\overline{s^\vee(\xi)} e^{-ij\xi}\, d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} |s^\vee(\xi)|^2 e^{-ij\xi}\, d\xi,\end{aligned}$$

therefore, $s^\vee(\xi)$ is almost everywhere unimodular. We will use this fact to show that the collection of the translates of s must, in fact, be a basis of $\ell^2(\mathbb{Z})$. Let f be a square-summable sequence. Then

$$\begin{aligned}\langle f, \tau^j s \rangle &= \frac{1}{2\pi} \int_0^{2\pi} f^\vee(\xi)\overline{s^\vee(\xi)} e^{-ij\xi}\, d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(f^\vee(\xi)\overline{s^\vee(\xi)} \right) e^{-ij\xi}\, d\xi,\end{aligned}$$

which is the j-th Fourier coefficient of $f^\vee(\xi)\overline{s^\vee(\xi)}$. Therefore, if we denote the projection of$f$ onto the closure of the span of $\{\tau^j s\}_{j\in\mathbb{Z}}$ by $P_s f$, we have

$$\begin{aligned}||P_s f||^2 &= \sum_{j\in\mathbb{Z}} |\langle f, \tau^j s \rangle|^2 \\ &= \sum_{j\in\mathbb{Z}} \left| \frac{1}{2\pi} \int_0^{2\pi} \left(f^\vee(\xi)\overline{s^\vee(\xi)} \right) e^{-ij\xi}\, d\xi \right|^2 \\ &= ||f^\vee(\xi)\overline{s^\vee(\xi)}||^2_{L^2(\mathbb{T})}.\end{aligned}$$

But multiplying a periodic function by a unimodular function does not change its norm:

$$\frac{1}{2\pi} \int_0^{2\pi} f^\vee(\xi)\overline{s^\vee(\xi)}\overline{f^\vee(\xi)\overline{s^\vee(\xi)}}\, d\xi = \frac{1}{2\pi} \int_0^{2\pi} f^\vee(\xi)\overline{f^\vee(\xi)}\, d\xi. \tag{13}$$

Therefore, $||P_s f|| = ||f^\vee||_{L^2(\mathbb{T})} = ||f||_{\ell^2(\mathbb{Z})}$, so the projection P_s must be the identity and the closure of the span of the translates of s must be $\ell^2(\mathbb{Z})$. We have shown the following:

Lemma 1

Suppose $s \in \ell^2(\mathbb{Z})$ is a norm one sequence which is orthogonal to all of its (non-trivial) translates. Then $\{\tau^j s\}_{j\in\mathbb{Z}}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$.

With this lemma, we can derive the following result:

Theorem 3

For any given dyadic dilation ρ, there does not exist $\psi \in \ell^2(\mathbb{Z})$ such that

$$\{\rho^j \tau^k \psi \mid j, k \in \mathbb{Z}, j \geq 0\}$$

is an orthonormal system.

Proof. Let ρ be a dyadic dilation and suppose that our conclusion is false for some non-zero ψ (normalized). In particular, ψ is orthogonal to its translates, so by the lemma, $\{\tau^k\psi\}_{k\in\mathbb{Z}}$ is an o.n.b. of $\ell^2(\mathbb{Z})$. As $\rho\psi$ is orthogonal to every translate of ψ, it must therefore be the zero sequence, and as ρ is injective, we must conclude that ψ is the zero sequence. But this is clearly a contradiction. □

Our chosen definition of a discrete wavelet, as shown by the theorem, is, then, the most natural choice from which we may be able to construct something analogous to a wavelet basis in $L^2(\mathbb{R})$. We will revisit this idea after we have developed the notion of a discrete MRA.

4. Decomposition of $\ell^2(\mathbb{Z})$

We can now use our operators ρ and ρ^* to decompose our space in a manner which is very "wavelet-like."

Theorem 4

Let V_{-k}, for $k \in \mathbb{N}$ be the image of $\ell^2(\mathbb{Z})$ under the operator ρ^k, and let U_{-k} be the null space of ${\rho^*}^k$. Then for any $k \in \mathbb{N}$, $l^2(\mathbb{Z}) = V_{-k} \oplus U_{-k}$.

Proof. We may express any sequence g in the following way: $g = \rho^k{\rho^*}^k g + (g - \rho^k{\rho^*}^k g)$. The first term in the sum is clearly a vector in V_{-k} and the term in parentheses must be in U_{-k}, for

$$\rho^{*k}(g - \rho^k{\rho^*}^k g) = {\rho^*}^k g - {\rho^*}^k g = 0.$$

We must therefore have that $\ell^2(\mathbb{Z}) = V_{-k} + U_{-k}$. Suppose that g has an alternate decomposition, i.e., for $a \in V_{-k}$ and $b \in U_{-k}$, $g = a + b$. We must then have that ${\rho^*}^k g = {\rho^*}^k a$ by definition of U_{-k}, so that $\rho^k{\rho^*}^k g = \rho^k{\rho^*}^k a$. Since $a \in V_{-k}$, $a = \rho^k c$ for some third sequence c, so that $\rho^k{\rho^*}^k a = \rho^k({\rho^*}^k\rho^k)c = \rho^k c = a$. Therefore $a = \rho^k{\rho^*}^k g$ and $b = g - \rho^k{\rho^*}^k g$. □

As suggested by the name, we will eventually see that the spaces V_k satisfy properties very similar to an MRA. Before we do this, though, we continue by decomposing the U_k.

Theorem 5

With V_{-k} and U_{-k} as defined above, let W_{-k} be the image of U_{-1} under the operator ρ^{k-1}. Then for all $k \in \mathbb{N}$, $U_{-k} = W_{-k} \oplus U_{-k+1}$.

Proof. Let g be a sequence in U_{-k}. Again we may say $g = \rho^{k-1}{(\rho^*)}^{k-1} g + (g - \rho^{k-1}{(\rho^*)}^{k-1} g)$. In this case, $\rho^{k-1}{(\rho^*)}^{k-1} g \in W_{-k}$ since $\rho^*({(\rho^*)}^{k-1} g) = 0$, and

$$(\rho^*)^{k-1}\big(g - \rho^{k-1}{(\rho^*)}^{k-1} g\big) = {(\rho^*)}^{k-1} g - {(\rho^*)}^{k-1} g = 0,$$

so $(g - \rho^{k-1}{(\rho^*)}^{k-1} \in U_{-k+1}$. Suppose now that $a \in W_{-k}$, $b \in U_{-k+1}$ satisfy $g = a + b$. Then ${(\rho^*)}^{k-1} g = {(\rho^*)}^{k-1} a$ (since this operation annihilates b). Again, since $a = \rho^{k-1} c$ for some c, we find that ${(\rho^*)}^{k-1} g = c$ and $\rho^{k-1}{(\rho^*)}^{k-1} g = \rho^{k-1} c = a$, which in turn shows that $b = (g - \rho^{k-1}{(\rho^*)}^{k-1})$, so our factors are unique. □

Corollary 1

For any negative integer k, $l^2(\mathbb{Z}) = V_k \oplus W_{-1} \oplus \cdots \oplus W_k$.

Proof. This is a straightforward induction argument. □

Proposition 8

For any $k \in \mathbb{N}$, V_{-k} and U_{-k} are orthogonal subspaces.

Proof. Let $\rho^k f \in V_{-k}$ and $g \in U_{-k}$. Then

$$\begin{aligned}\left\langle \rho^k f, g \right\rangle &= \left\langle f, \rho^{*k} g \right\rangle \\ &= \langle f, 0 \rangle = 0. \square\end{aligned}$$

Proposition 9

For any $k \in \mathbb{N}$, W_{-k} and U_{-k+1} are orthogonal subspaces.

Proof. Suppose $\rho^{k-1} f \in W_{-k}$ and $g \in U_{-k+1}$.

$$\begin{aligned}\left\langle \rho^{k-1} f, g \right\rangle &= \left\langle f, (\rho^*)^{k-1} g \right\rangle \\ &= \langle f, 0 \rangle = 0. \square\end{aligned}$$

Corollary 2

For any $j, k < 0$, $j \neq k$, W_j and W_k are orthogonal subspaces.

Proof. Without loss of generality, take $k > j$. Then W_k is a subspace of U_{j+1}, and hence is orthogonal to W_j. □

Proposition 10

For every $k \in \mathbb{N}$, V_{-k} is closed.

Proof. Suppose $\{\rho^k f_j\}$, $j \in \mathbb{N}$ is a convergent sequence of sequences in $\ell^2(\mathbb{Z})$ with limit sequence f.

$$\begin{aligned}||\rho^k \rho^{*k} f - \rho^k f_j|| &= ||\rho^k \rho^{*k} f - \rho^k (\rho^{*k} \rho^k) f_j|| \\ &= ||\rho^k (\rho^{*k} f - \rho^{*k} \rho^k f_j)|| \\ &= ||\rho^{*k} f - \rho^{*k} \rho^k f_j|| \\ &\leq ||f - \rho^k f_j|| \to 0.\end{aligned}$$

Thus, we have that $\rho^k f_j \to \rho^k \rho^{*k} f$. Since the limit is unique, $f = \rho^k \rho^{*k} f$, which implies that $f \in V_{-k}$, so V_{-k} is closed. □

Proposition 11

For every $k \in \mathbb{N}$, W_{-k} is closed.

Proof. Let $\{\rho^{k-1}f_j\}$, $j \in \mathbb{N}$ be a convergent sequence (of elements in $\ell^2(\mathbb{Z})$) with limit f such that $\rho^* f_j = 0$ for all j.

$$\begin{aligned} ||\rho^{k-1}(\rho^*)^{k-1}f - \rho^{k-1}f_j|| &= ||\rho^{k-1}(\rho^*)^{k-1}f - \rho^{k-1}(\rho^*)^{k-1}\rho^{k-1}f_j|| \\ &= ||(\rho^*)^{k-1}f - (\rho^*)^{k-1}\rho^{k-1}f_j|| \\ &\leq ||f - \rho^{k-1}f_j|| \to 0. \end{aligned}$$

As before, we can conclude that $f = \rho^{k-1}(\rho^*)^{k-1}f$, since the sequence converges to a unique limit. In addition,

$$\begin{aligned} \rho^*((\rho^*)^{k-1}f) &= \rho^{*k}f \\ &= \rho^{*k}\left(\lim_{j\to\infty}\rho^{k-1}f_j\right) \\ &= \lim_{j\to\infty}\rho^{*k}\rho^{k-1}f_j \\ &= 0. \end{aligned}$$

Therefore $(\rho^*)^{k-1}f$ is in the null space of ρ^* so $f = \rho^{k-1}((\rho^*)^{k-1}f)$ is in W_{-k}. Therefore W_{-k} is closed. □

Putting all of these rapid-fire propositions together, we have a very nice decomposition of $\ell^2(\mathbb{Z})$:

Corollary 3

For every $k \in \mathbb{Z}$, $k < 0$, $l^2(\mathbb{Z}) = V_k \oplus W_{-1} \oplus \cdots \oplus W_k$. Each summand is closed and orthogonal to every other summand, and $W_j = \rho^{-j+1}(W_{-1})$ for $j < -1$.

Proof. Only two parts of this result are left to show. First, W_j must be orthogonal to V_k for $j \geq k$ since it is a subspace of U_k which is orthogonal to V_k. Second, $W_j = \rho^{-j+1}(W_{-1})$ for $j < -1$. This, however, is clear from their definition. □

5. The Discrete MRA

Let us define, for convention, that $V_0 = \ell^2(\mathbb{Z})$.

Theorem 6

The collection of closed subspaces $\{V_j\}$, as defined in the previous section, satisfies, for $j = \ldots, -1, 0$:

$$V_j \subset V_{j+1} \tag{14}$$

$$\rho f \in V_j \Leftrightarrow f \in V_{j+1} \tag{15}$$

$$\bigcup_{j=-\infty}^{0} V_j = \ell^2(\mathbb{Z}) \tag{16}$$

$$\exists\ \varphi \in V_0 \text{ such that } \{\tau^k\varphi \mid k \in \mathbb{Z}\} \text{ is an orthonormal basis for } V_0. \tag{17}$$

Proof. $[V_j \subset V_{j+1}]$ As $V_j = \rho^{-j+1}(V_0)$, $V_j = \rho^{-j}(\rho V_0) \subset \rho^{-j}(V_0) = V_{j+1}$.
$[\rho f \in V_j \Rightarrow f \in V_{j+1}]$ Suppose $\rho f = \rho^{-j+1} g$ for some g (this is true if and only if $\rho f \in V_j$). Then $f = \rho^* \rho^{-j+1} g = \rho^{-j} g$, which is clearly in V_j.
$[\rho f \in V_j \Leftarrow f \in V_{j+1}]$ Suppose $f = \rho^{-j} g$ for some g. Then $\rho f = \rho^{-j+1} g$, and so is in V_j.
$[\bigcup_{j=-\infty}^{0} V_j = \ell^2(\mathbb{Z})]$ This is trivial because we already have containment and $V_0 = \ell^2(\mathbb{Z})$.
Similarly, the "scaling function" is trivial– take $\varphi = \Delta$, where Δ is an element of the standard basis. □

Suppose, conversely, that some collection of closed subspaces $\{V_j\}$ satisfies (14) - (17). The containment property (14) implies that the union of the sets V_j must be V_0. But, by (16), this means that $V_0 = \ell^2(\mathbb{Z})$. Finally, (15) implies that $V_j = \rho^{-j}(V_0) = \rho^{-j}(\ell^2(\mathbb{Z}))$. This is exactly how we constructed our particular V_j's. Therefore we can conclude:

Corollary 4

Given a dyadic dilation ρ, there exists a unique collection of subspaces $\{V_j\}$, $j \in \mathbb{Z} < 0$, which satisfy (14) - (17).

The conclusions of Theorem 6 seem so similar to the definition of an MRA on $L^2(\mathbb{R})$, that it seems natural to "finish out" the comparison and make the following definition:

DEFINITION 3. A Collection of closed subspaces $\{V_j\}$, together with a dyadic dilation ρ will be called a discrete MRA if and only if it satisfies (14) - (17) and the additional condition

$$\bigcap_{j=-\infty}^{0} V_j = \{0\}. \tag{18}$$

As before, there can be at most one collection of V_j's which produces a discrete MRA relative to a fixed dyadic dilation ρ.

Unfortunately, condition (18) is not always satisfied for a general dilation ρ and $\{V_{-j} := \rho^j(\ell^2(\mathbb{Z}))\}$. Consider the earlier example of the up-sampling dilation ρ_{ds}. The adjoint, ρ_u^*, does exactly the opposite, down-sampling,

$$\rho_u^*\big(\{\ldots, f_{-1}, f_0, f_1, \ldots\}\big) = \big\{\ldots, f_{-2}, f_0, f_2, \ldots\big\}. \tag{19}$$

Notice, though, that the sequence $\Delta \in V_j \ \ \forall j \geq 0$. Thus (18) is not true in this case. We can, however, show that (18) is equivalent to the condition that $||\rho^{*j} f|| \to 0$ as $j \to \infty$ for all sequences f.

To establish the desired conclusion, we need to construct a new operator P according to the following formula:

$$Pf := \sum_{j=0}^{\infty} \rho^j \big((\rho^*)^j f - \rho(\rho^*)^{j+1} f\big). \tag{20}$$

Each term in (20) is orthogonal to every other because

$$\rho^j\left((\rho^*)^j f - \rho(\rho^*)^{j+1} f\right) \in W_{j+1}.$$

In addition,

$$\begin{aligned}\sum_{j=0}^{N} \left|\left|\rho^j\left((\rho^*)^j f - \rho(\rho^*)^{j+1} f\right)\right|\right|^2 &= \left\| \sum_{j=0}^{N} \rho^j((\rho^*)^j f - \rho(\rho^*)^{j+1} f)\right\|^2 \\ &= ||f - \rho^{N+1}(\rho^*)^{N+1} f||^2 \\ &\leq \left(||f|| + ||\rho^{N+1}(\rho^*)^{N+1} f||\right)^2 \\ &\leq 4||f||^2,\end{aligned}$$

so the sum (20) converges. As the sum actually telescopes, we can conclude that

$$\lim_{j\to\infty} (\rho^j \rho^{*j} f) = f - Pf, \tag{21}$$

and hence, for fixed j',

$$\begin{aligned}\rho^{j'}\rho^{*j'}(f - Pf) &= \rho^{j'}\rho^{*j'}\left(\lim_{j\to\infty}(\rho^j\rho^{*j} f)\right) \\ &= \lim_{j\to\infty} \rho^{j'}\rho^{*j'}\left(\rho^j\rho^{*j} f\right) \\ &= \lim_{j\to\infty}(\rho^j\rho^{*j} f) = f - Pf.\end{aligned}$$

(We used the fact that for $j > j'$, $\rho^{*j'}\rho^j = \rho^{j-j'}$). We can hence conclude that P is a projection operator,

$$0 = \lim_{j\to\infty}\left(\rho^j\rho^{*j}(f - Pf)\right) - (f - Pf) = P(f - Pf). \tag{22}$$

Our plan is to show that the null space of P is precisely the intersection of the V_j's, and then investigate the conditions in which P has a trivial null space.

Proposition 12

The null space of P is exactly $\cap_j V_j$.

Proof. Suppose $g \in V_{-j} \quad \forall j \geq 0$. Then, for each j, there exists a sequence g_j such that $g = \rho^j g_j$, which implies that

$$\begin{aligned}\rho^j\rho^{*j} g &= \rho^j\rho^{*j}\rho^j g_j \\ &= \rho^j g_j = g.\end{aligned}$$

Therefore $g = \lim_{j\to\infty}\rho^j\rho^{*j} g = g - Pg$, so $Pg = 0$, and we have that $\cap_j V_j \subset null\ P$. Suppose instead that g is any sequence for which $Pg = 0$. Then for each j, $g = g - Pg = \rho^j\rho^{*j}(g - Pg) = \rho^j\rho^{*j} g$, so g must also be in V_{-j} for all j, and hence in the intersection as well. □

Corollary 5

The subspaces $\{V_j\}$ generated by the dilation operator ρ form a discrete MRA (with respect to ρ) if and only if

$$\lim_{j\to\infty} ||\rho^{*j} f|| = 0 \qquad \forall f \in \ell^2(\mathbb{Z}). \tag{23}$$

Proof. Condition (18) is true if and only if P has trivial null space which can occur if and only if P is the identity operator. But P is the identity if and only if $||f - Pf|| = 0 \ \ \forall f$. Finally, we have that

$$\begin{aligned} ||f - Pf|| &= \left|\left| \lim_{j\to\infty} \rho^j \rho^{*j} f \right|\right| \\ &= \lim_{j\to\infty} ||\rho^j \rho^{*j} f|| \\ &= \lim_{j\to\infty} ||\rho^{*j} f||, \end{aligned}$$

so our hypothesis is established. □

If ρ or ρ^* have any eigenvectors, these eigenvectors must be in $\cap V_j$. The next step is to show that any $f \in \cap V_j$ is itself an eigenvector of ρ, and that the eigenspace of ρ is at most dimension 1.

Lemma 2

Suppose $f, g \in L^2(\mathbb{T})$ have the property that, for any non-negative integer j, there exist functions s_j, f_j, g_j in $L^2(\mathbb{T})$ such that

$$f(x) = s_j(x) f_j(x) \ \ a.e. \tag{24}$$

$$g(x) = s_j(x) g_j(x) \ \ a.e. \tag{25}$$

$$f_j\Big(x + \frac{2\pi}{2^j}\Big) = f_j(x) \ \ a.e. \tag{26}$$

$$g_j\Big(x + \frac{2\pi}{2^j}\Big) = g_j(x) \ \ a.e. \tag{27}$$

Then f and g are linearly dependent, i.e., there exist $c_f, c_g \in \mathbb{C}$ not both zero such that $c_f f + c_g g = 0$ almost everywhere.

Proof. First of all, assume neither $||f||$ nor $||g||$ is zero, in which case the conclusion is obviously true. As the set of $x \in \mathbb{R}$ where (24) - (27) fail to be true for a particular j is a set of measure zero, the set where these equations fail to be true for *any particular* j is also a set of measure zero. Thus, on some set E containing almost every $x \in \mathbb{R}$ equations (24) - (27) hold for every x and every j. In particular, the set

$$E' = E \cap \bigcap_{j\geq 0, k\in\mathbb{Z}} \Big(E - \frac{2\pi k}{2^j}\Big)$$

is a set of full measure (as it is the countable intersection of sets of full measure) and has the additional property that $x \in E' \Rightarrow x + \frac{2\pi}{2^j} \in E'$ for all non-negative integers j. To see this last property, fix a non-negative j and let $x \in E'$. Consider any integer k and any non-negative integer j_0. Clearly

$$\frac{2\pi k}{2^{j_0}} - \frac{2\pi}{2^j} = \frac{2\pi(2^j k - 2^{j_0})}{2^{j+j_0}}, \quad \text{and} \quad x \in E - \frac{2\pi(2^j k - 2^{j_0})}{2^{j+j_0}}$$

by definition of E', so $x + \frac{2\pi}{2^j} \in E - \frac{2\pi k}{2^{j_0}}$ for any j_0, k. Thus $x + \frac{2\pi}{2^j} \in E'$. The translation invariance of E' will considerably simplify the rest of the argument.

We now define a new sequence of functions h_j:

$$h_j(x) = \begin{cases} \left| \frac{f_j(x)}{g_j(x)} \right| & g_j(x) \neq 0 \\ -1 & g_j(x) = 0. \end{cases} \tag{28}$$

Each h_j is measurable, so

$$h(x) = \liminf_{j \to \infty} \min \left\{ h_j(x), 2\frac{||f||}{||g||} \right\}$$

exists and is itself measurable. Let $x \in E'$ and fix a non-negative integer j. If $h(x) = -1$, then for infinitely many $N > j$ we have $g_N(x) = g_N(x + \frac{2\pi}{2^j}) = 0$, so $h(x + \frac{2\pi}{2^j}) = -1 = h(x)$. If $h(x) \neq -1$ then $g_N(x) \neq 0$ for all sufficiently large N, and as $f_N(x) = f_N(x + \frac{2\pi}{2^j})$ and $g_N(x) = g_N(x + \frac{2\pi}{2^j})$ for any $x \in E'$ and any $N > j$, we again conclude that $h(x) = h(x + \frac{2\pi}{2^j})$. Therefore, for almost every $x \in \mathbb{R}$, $h(x) = h(x + \frac{2\pi}{2^j})$ for all non-negative integers j. In particular, then, h is 2π-periodic and bounded, so $h \in L^2(\mathbb{T})$. This being the case, the fact that h has arbitrarily short period means that h must be almost everywhere equal to some constant c. This constant c cannot equal -1 as this would imply that $g = 0$ almost everywhere, contrary to assumption. Likewise c cannot equal $2||f||/||g||$, since, one can see that this would imply that $||f|| \geq 2||f||||g||/||g||$. Thus, $\liminf |f_j(x)/g_j(x)| = c$ for almost every $x \in \mathbb{R}$. By a similar construction, one can easily see that $\liminf \Re\{f_j(x)/g_j(x)\}$ and $\liminf \Re\{-if_j(x)/g_j(x)\}$ exist almost everywhere and are each almost everywhere constant, from which it easily follows that $f = c_g g$ almost everywhere for some $c_g \in \mathbb{C}$, so the hypothesis is true. □

Theorem 7

Given a dyadic dilation ρ and defining the spaces $V_j \subset \ell^2(\mathbb{Z})$ as before, we conclude that $f \in \bigcap_j V_j$ if and only if f is an eigenvector of ρ, and this space is at most one dimensional.

Proof. Given $f \in \bigcap_j V_j$, we have seen that $\rho^j \rho^{*j} f = f$ for all non-negative integers j. In terms of the Fourier transform of f,

$$f^\vee(\xi) = (\rho^{*j} f)^\vee(2^j \xi) \prod_{l=0}^{j-1} m(2^l \xi)$$

for all non-negative j and almost every $\xi \in [0, 2\pi)$. But we also have

$$(\rho f)^\vee(\xi) = (\rho^{*j} f)^\vee(2^{j+1}\xi)\ m(2^j\xi) \prod_{l=0}^{j-1} m(2^l\xi) \quad a.e.$$

again for all non-negative integers j (take the product to be 1 by convention if $j = 0$) . Thus, if we let $s_j(\xi) = \prod_{l=0}^{j-1} m(2^l\xi)$ and likewise take $f_j(\xi) = \left(\rho^{*j} f\right)^\vee (2^j\xi)$ and $g_j = (\rho^{*j} f)^\vee(2^{j+1}\xi)\ m(2^j\xi)$ ($s_j = (\rho^j \Delta)^\vee$, so it is square-integrable; f_j and g_j are clearly square-integrable), we may conclude from the lemma that $f^\vee$ and $(\rho f)^\vee$ are linearly dependent, so f and ρf must also be linearly dependent. Moreover, as $||f|| = ||\rho f||$, we conclude that $\rho f = \lambda f$ for some complex number λ of magnitude 1. Thus, every sequence in $\bigcap_j V_j$ is an eigenvector of ρ. The converse, however, is immediate, so we see that the eigenvectors of ρ are in fact, all the vectors in $\bigcap_j V_j$. Moreover, suppose f and g are any two non-zero eigenvectors of ρ. Then we have

$$f^\vee(\xi) = \left(\prod_{l=0}^{j-1} m(2^l\xi)\right) \lambda_f^{-j} f^\vee(2^j\xi) \quad a.e.$$

$$g^\vee(\xi) = \left(\prod_{l=0}^{j-1} m(2^l\xi)\right) \lambda_g^{-j} g^\vee(2^j\xi) \quad a.e.$$

Again we may apply the lemma and conclude that $f^\vee$ and $g^\vee$ are linearly dependent, thus f and g are linearly dependent as well. We can therefore conclude that the span of the eigenvectors of ρ is at most one-dimensional over $\mathbb{C}$ for any dyadic dilation ρ. □

Lemma 3

If f is any sequence in $\ell^2(\mathbb{Z})$ which is an eigenvector of a dyadic dilation ρ, then the Fourier transform of f is a bounded function.

Proof. Suppose ρ has a non-zero eigenvector f, otherwise we're done. This means that by (21), $Pf = 0$. As P is continuous and f is non-zero, there must be some finitely supported sequence g in a neighborhood of f for which $Pg \neq g$ (otherwise we could take a limit and conclude that $Pf = f \neq 0$). Combining (21) and the previous theorem, we conclude that

$$\lim_{j\to\infty} \rho^j \rho^{*j} g = g - Pg = cf$$

for some non-zero constant c since $g - Pg \neq 0$ and $P(g - Pg) = 0$, meaning that $g - Pg$ is in $\bigcap V_j$. As f is an eigenvector of both ρ and ρ^* (and the product of its eigenvalues is 1), we conclude that $g = cf + h$, where

$$\begin{aligned} \lim_{j\to\infty} ||\rho^{*j} h|| &= \lim_{j\to\infty} ||\rho^j \rho^{*j} h|| \\ &= || \lim_{j\to\infty} \rho^j \rho^{*j} (g - cf)|| \\ &= ||cf - cf|| = 0. \end{aligned}$$

Thus, if $\rho f = \lambda f$, $\lambda^j \rho^{*j} g = cf + \lambda^j \rho^{*j} h \to cf$ as $j \to \infty$. As g is finitely supported, its Fourier transform is a trigonometric polynomial and is a bounded function. But

$$\begin{aligned}
&\left| \left(\rho^{*j} g\right)^\vee (\xi) \right| \\
&= \frac{1}{2} \left| \left(\rho^{*j-1} g\right)^\vee (\xi/2) \overline{m(\xi/2)} + \left(\rho^{*j-1} g\right)^\vee (\xi/2 + \pi) \overline{m(\xi/2 + \pi)} \right| \\
&\leq \left[\frac{|m(\frac{1}{2}\xi)|^2 + |m(\frac{1}{2}\xi + \pi)|^2}{2} \; \frac{|\left(\rho^{*j-1} g\right)^\vee (\frac{1}{2}\xi)|^2 + |\left(\rho^{*j-1} g\right)^\vee (\frac{1}{2}\xi + \pi)|^2}{2} \right]^{1/2}
\end{aligned}$$

by the Cauchy-Schwartz inequality. But the first fraction on the right-hand side is one, and the second is bounded by the maximum of $|(\rho^{*j-1} g)^\vee|$. Since $|g^\vee(\xi)| < M$ for some M, we conclude by induction that $|(\rho^{*j} g)^\vee(\xi)| < M$. Thus $|\lambda^j (\rho^{*j} g)^\vee(\xi)| < M$ almost everywhere for all non-negative j. But this implies that $cf^\vee$, which is the L^2 limit of $\lambda^j (\rho^{*j} g)^\vee$, must be bounded as well. □

Theorem 8

If ρ fails to generate an MRA, i.e., if $\bigcap V_j$ is a one-dimensional subspace of $\ell^2(\mathbb{Z})$, then $m = (\rho\Delta)^\vee$ is unimodular. Equivalently, if m is not unimodular, then ρ generates an MRA.

Proof. Suppose that ρ possesses a non-zero eigenvector f; by the previous theorem, this is equivalent to our hypothesis. Then

$$|f^\vee(\xi)| = |m(\xi) f^\vee(2\xi)|$$

as the eigenvalue must have modulus one. We conclude that

$$\begin{aligned}
\frac{1}{2}\left[|f^\vee(\xi)|^2 + |f^\vee(\xi + \pi)|^2\right] &= \frac{1}{2}\left[|m(\xi)|^2 |f^\vee(2\xi)|^2 + |m(\xi + \pi)|^2 |f^\vee(2\xi + 2\pi)|^2\right] \\
&= |f^\vee(2\xi)|^2.
\end{aligned}$$

By the lemma, $|f^\vee(\xi)|$ is a bounded function, so $|f^\vee(\xi)|^4$ is also bounded. Therefore it is integrable and $g(\xi) = |f^\vee(\xi)|^2 \in L^2(\mathbb{T})$. The function g satisfies the equation $g(\xi) + g(\xi + \pi) = 2g(2\xi)$. If we take the inner product of both sides with $e^{2ik\xi}$, we find that the Fourier coefficients of g, denoted by g_k, satisfy $g_{2k} + g_{2k} = 2g_k$, or $g_{2k} = g_k$. Since $g \in L^2(\mathbb{T})$, this can only be true if $g_k = 0$ for $k \neq 0$, i.e., if g is almost everywhere constant. Thus, $|f^\vee(\xi)|$ must be almost everywhere constant as well and $|f^\vee(\xi)| = |\lambda||m(\xi)||f^\vee(2\xi)|$ implies (as $|\lambda| = 1$) that $|m(\xi)| = 1$ almost everywhere. □

In the case of MRAs on the real line, one can prove that $\bigcap_j V_j = \{0\}$ is in fact a consequence of the other properties of an MRA (see [2] for a proof). Though we have already seen that this is not the case for discrete wavelets, it is interesting that one is still able to make the next best possible conclusion.

In [1], the issue of deciding when a dilation generates an MRA is addressed by a theorem which shows that any ρ which can be extended to an operator on all bounded sequences, which sends the constant sequence to itself, and which has a C^∞ Fourier transform generates an MRA. These conditions insure that all dilations are "non-trivial" (i.e., dilations whose filters are not identically 1 in magnitude); finitely supported coefficients r_k as in (6) do, in fact, generate an MRA. They are, however, not necessary, as we have just seen.

So far we have been fairly silent about "discrete wavelets," which was the original focus of this project. Thankfully, though, we have developed almost all the machinery needed to say something meaningful on this subject. In particular, we will show that, if ρ is a dilation operator which generates a discrete MRA, there is an element $\psi \in W_{-1}$ which is a discrete wavelet.

6. The Discrete MRA wavelet

The goal of this section is to demonstrate that, associated with every discrete MRA which is generated by dyadic dilation ρ, there is a discrete wavelet in W_{-1}. First, we state precisely what we mean by a discrete MRA wavelet.

DEFINITION 4. A discrete wavelet ψ relative to the dyadic dilation ρ is called an MRA wavelet if and only if $\{\tau^{2k}\psi \mid k \in \mathbb{Z}\}$ is an orthonormal basis of W_{-1}.

Unlike wavelets on the real line, every discrete wavelet is an MRA wavelet as shown by the following theorem.

Theorem 9

Let $\psi \in \ell^2(\mathbb{Z})$ be a discrete wavelet with respect to the dyadic dilation ρ. Then ψ is an MRA wavelet.

Proof. By definition, $\{\rho^j\tau^{2k}\psi\}_{j,k\in\mathbb{Z},j\geq 0}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$. We can thus conclude:

$$\left\langle \rho^*\psi, \rho^j\tau^{2k}\psi \right\rangle = \left\langle \psi, \rho^{j+1}\tau^{2k}\psi \right\rangle = 0 \quad j \geq 0.$$

Therefore we must have $\rho^*\psi = 0$, meaning that, defining W_j and V_j as before (the "standard" MRA associated with ρ), we have $\psi \in W_{-1}$. As $\tau\rho^*\psi = \rho^*\tau^2\psi$, we get that $\tau^{2k}\psi \in W_{-1} \forall k \in \mathbb{Z}$. By hypothesis they are an orthonormal system. Now suppose $f \in W_{-1}$. It follows that

$$\left\langle f, \rho^j\tau^{2k}\psi \right\rangle = \left\langle \rho^* f, \rho^{j-1}\tau^{2k}\psi \right\rangle = 0$$

for $j \geq 0$. If $\left\langle f, \tau^{2k}\psi \right\rangle = 0 \ \forall k \in \mathbb{Z}$, then $f = 0$ by virtue of the fact that $\{\rho^j\tau^{2k}\psi\}_{j,k\in\mathbb{Z},j\geq 0}$ is an o.n.b. of $\ell^2(\mathbb{Z})$ and $\left\langle f, \rho^j\tau^{2k}\psi \right\rangle = 0$ for all j and k. Thus $\{\tau^{2k}\psi\}_{k\in\mathbb{Z}}$ is an o.n.b. of W_{-1} and so ψ must be a discrete MRA wavelet. □

Theorem 10

Suppose $\psi \in \ell^2(\mathbb{Z})$ is a wavelet relative to ρ_1 and ρ_2, both dyadic dilations. Then there is a unitary map $\mu : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ which commutes with translation such that $\rho_2 = \rho_1\mu$.

Proof. Observe that ψ is a wavelet relative to ρ_1 and ρ_2 if and only if the null space of ρ_1^* is equal to the null space of ρ_2^*. For any $f \in \ell^2(\mathbb{Z})$, we have

$$\rho_2 f = \rho_1 \rho_1^* \rho_2 f + (\rho_2 f - \rho_1 \rho_1^* \rho_2 f)$$

where the term in parentheses is in the null space of ρ_1^*, which in this case is equal to the null space of ρ_2^*. Thus, applying ρ_2^* to both sides,

$$f = \rho_2^* \rho_2 f = \rho_2^* \rho_1 \rho_1^* \rho_2 f.$$

Similarly, we can show

$$f = \rho_1^* \rho_2 \rho_2^* \rho_1 f.$$

Thus the operator $\mu = \rho_1^* \rho_2$ is invertible. It is also easy to check that μ commutes with translations. Also, as $\mu\mu^* f = \mu^* \mu f = f$, μ must be unitary.

Finally, we claim that $\rho_1 \mu = \rho_2$.

$$||\rho_1 \mu f - \rho_2 f||^2 = \langle \rho_1 \mu f - \rho_2 f, \rho_2 f \rangle + \langle \rho_1 \mu f - \rho_2 f, \rho_1 \mu f \rangle .$$

The first inner product is zero since $\langle \rho_1 \mu f - \rho_2 f, \rho_2 f \rangle = \langle \rho_2^* \rho_1 \mu f - f, f \rangle = \langle 0, f \rangle$. Thus,

$$||\rho_1 \mu f - \rho_2 f||^2 = \langle \rho_1 \mu f, \rho_1 \mu f \rangle - \langle \rho_2 f, \rho_1 \mu f \rangle = \langle f, f \rangle - \langle f, f \rangle = 0.$$

Therefore we may conclude that $\rho_2 = \rho_1 \mu$. □

Thus far, the results are quite interesting: every wavelet is an MRA wavelet, and up to the obvious equivalence suggested by the previous theorem, the dilation making ψ a wavelet is unique. Now let us actually construct a wavelet given a discrete MRA. We are going to define a *conjugate* linear operator F as follows:

$$(Ff)_j := (-1)^{1-j} \overline{f_{1-j}} \qquad \forall j \in \mathbb{Z}. \tag{29}$$

By conjugate linear, we mean that $F(\alpha f + \beta g) = \overline{\alpha} F f + \overline{\beta} F g$, which is clear from the definition. Some other immediate consequence are

$$F^2 f = -f \tag{30}$$

$$\langle Ff, Fg \rangle = \overline{\langle f, g \rangle} \quad = \quad \langle g, f \rangle \tag{31}$$

$$\langle Ff, g \rangle = - \langle Fg, f \rangle \tag{32}$$

where the last equality is taken by substituting $g = -F^2 g$ into (31). We will now proceed to show some more properties of F which are less apparent from the definition.

Proposition 13

$$\tau^{-1} F = -F\tau. \tag{33}$$

Proof. For any sequence f,

$$\begin{aligned}(F\tau f)_j &= (-1)^{1-j}\overline{(\tau f)_{1-j}} \\ &= (-1)^{1-j}\overline{f_{1-j-1}} \\ &= -\left((-1)^{-j}\overline{f_{-j}}\right) \\ &= -(Ff)_{j+1} = -(\tau^{-1}Ff)_j. \ \square\end{aligned}$$

Proposition 14

For all $k \in \mathbb{Z}$:

$$\left\langle Ff, \tau^{2k} f \right\rangle = 0. \tag{34}$$

Proof.

$$\begin{aligned}\left\langle Ff, \tau^{2k} f \right\rangle &= \sum_{l\in\mathbb{Z}} (Ff)_l \overline{(\tau^{2k} f)_l} \\ &= \sum_{l\in\mathbb{Z}} (-1)^{1-l}\overline{f_{1-l} f_{l-2k}} \\ &= \sum_{l\in\mathbb{Z}} (-1)^{l}\overline{f_{l} f_{1-l-2k}} \\ &= \sum_{l\in\mathbb{Z}} (-1)^{l-2k}\overline{f_{l-2k} f_{1-l}} = -\left\langle Ff, \tau^{2k} f \right\rangle,\end{aligned}$$

where in the last two lines we change the summation variable ($l \to 1-l$ and $l \to l+2k$). Since the inner product remains unchanged when multiplied by -1, it must be zero. $\square$

The usefulness of F comes from the next theorem, which says, as we shall see later, that for any dyadic dilation ρ, F is a bijection between V_1 and W_1. First, we need the following, rather unusual, property of F.

Proposition 15

For any dyadic dilation ρ and any sequence f (in $\ell^2(\mathbb{Z})$),

$$\left(\rho\rho^* - F\rho\rho^* F\right) f = f. \tag{35}$$

Proof. As we have done in the last few propositions, we will show that the two sides are equal when evaluated at any integer n. As in earlier sections, we make the definition $r_k := (\rho\Delta, \tau^k \Delta)$.

$$\begin{aligned}((\rho\rho^* - F\rho\rho^* F)f)_n &= \sum_{l\in\mathbb{Z}} (\rho^* f)_l r_{n-2l} - (-1)^{1-n}\overline{\sum_{l\in\mathbb{Z}} (\rho^* F f)_l r_{1-n-2l}} \\ &= \sum_{l\in\mathbb{Z}} \left[\left(\sum_{k\in\mathbb{Z}} f_k \overline{r_{k-2l}} \right) r_{n-2l} - (-1)^{1-n} \left(\sum_{k\in\mathbb{Z}} (Ff)_k \overline{r_{k-2l}} \right) r_{1-n-2l} \right] \\ &= \sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left[f_k \overline{r_{k-2l}} r_{n-2l} - (-1)^{1-n}(-1)^k f_k r_{1-k-2l} \overline{r_{1-n-2l}} \right] \\ &= \sum_{k\in\mathbb{Z}} f_k \sum_{l\in\mathbb{Z}} \left[\overline{r_{k-2l}} r_{n-2l} + (-1)^n (-1)^k r_{1-k-2l} \overline{r_{1-n-2l}} \right]. \end{aligned} \tag{36}$$

We know it is safe to change the order of summation because, as r and f are square-summable, their product is absolutely summable.

Suppose $n + k = 2m$ for some integer m. We can then change our variable of summation in (36) *only in the second term inside the sum over* l, by taking $l \to l - m$, in which case the sum over l becomes

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \left[\overline{r_{k-2l}} r_{n-2l} + r_{1-k-2(l-m)}\overline{r_{1-n-2(l-m)}}\right] &= \sum_{l\in\mathbb{Z}} \left[\overline{r_{k-2l}} r_{n-2l} + r_{n+1-2l}\overline{r_{k+1-2l}}\right] \\ &= \sum_{l\in\mathbb{Z}} \overline{r_{k-l}} r_{n-l},\end{aligned}$$

since the first half of the sum and the second half differed only by $2l$ in the first and $2l+1$ in the second. Since n and k differ by an even number, (6) tells us that the sum over l in (36) must equal $\delta_{n,k}$.

Suppose instead that $n + k = 2m + 1$ for some integer m. Performing the same trick we just used, the sum over l becomes:

$$\begin{aligned}\sum_{l\in\mathbb{Z}} \left[\overline{r_{k-2l}} r_{n-2l} - r_{1-k-2(l-m)}\overline{r_{1-n-2(l-m)}}\right] &= \sum_{l\in\mathbb{Z}} \left[\overline{r_{k-2l}} r_{n-2l} - r_{n-2l}\overline{r_{k-2l}}\right] \\ &= 0.\end{aligned}$$

Therefore

$$\left((\rho\rho^* - F\rho\rho^* F)f\right)_n = \sum_{k\in\mathbb{Z}} f_k \delta_{k,n} = f_n. \ \square$$

Corollary 6

F is a bijection from V_{-1} to W_{-1}.

Proof. Take $g \in V_{-1}$. We have already seen that $\rho\rho^* g = g$, so by the proposition, we must have $F\rho\rho^* Fg = 0$. Since F and ρ are both $1-1$, we must have $\rho^* Fg = 0$, meaning that F maps V_{-1} into W_{-1}. Similarly, take g now in W_{-1}. Now the proposition tells us that $g = -F\rho\rho^* Fg$ so we may conclude that $Fg = \rho\rho^* Fg$, meaning that $Fg \in V_{-1}$ by definition. Therefore, F is a bijection as described. $\square$

Theorem 11

For any dilation operator ρ, with V_{-j} and W_{-j} as generated by ρ, $\{\rho^j \tau^k \Delta\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_{-j} and $\{\rho^{j-1} F\rho\tau^k \Delta\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_{-j}.

Proof. $[V_{-j}]$ As ρ preserves inner products, and $\{\tau^k \Delta\}_{k\in\mathbb{Z}}$ is an o.n.b. of V_0, $\{\rho^j \tau^k \Delta\}_{k\in\mathbb{Z}}$ is a basis of $\rho^j V_0 = V_{-j}$.

$[W_{-j}]$ F is a bijection which (up to conjugation) preserves inner products. We know in particular that $F(V_{-1}) = W_{-1}$, so $\{F\rho\tau^k \Delta\}_{k\in\mathbb{Z}}$ must be an orthonormal basis of W_{-1}. But for $j > 1$, $W_{-j} = \rho^{j-1} W_{-1}$, so we must have that $\{\rho^{j-1} F\rho\tau^k \Delta\}_{k\in\mathbb{Z}}$ is an o.n.b. of W_{-j}. $\square$

Corollary 7

If ρ is a dyadic dilation which generates a discrete MRA, then there exists $\psi = F\rho\varphi$ which is a discrete orthonormal wavelet.

Proof. We already know that the dilates of the (2k)-translates of ψ generate $\overline{\bigoplus_j W_j}$ ($\rho^j\tau^{2k}F\rho\varphi = -\rho^j F\tau^{-2k}\rho\varphi = -\rho^j F\rho\tau^{-k}\varphi$. Collected over k, these sequences form a basis of W_{-j+1} by the above corollary). Since ρ generates a discrete MRA, we know that this space is equal to $\ell^2(\mathbb{Z})$. □

7. Connections to MRA wavelets in $L^2(\mathbb{R})$

Let $\psi \in L^2(\mathbb{R})$ be an MRA wavelet, and let φ be an associated scaling function with low-pass filter $m_0 \in L^2(\mathbb{T})$. As we have seen earlier, m_0 (when multiplied by $\sqrt{2}$) generates a dyadic dilation on $\ell^2(\mathbb{Z})$ in a very natural way, via the Fourier transform. Even more, we have an isometry mapping $\ell^2(\mathbb{Z})$ into $V_0 \subset L^2(\mathbb{R})$:

$$f \mapsto \sum_{k\in\mathbb{Z}} f_k\varphi(\xi - k). \tag{37}$$

Let us call this map Φ. As we are working with a real MRA, we have the two-scale equation

$$\frac{1}{2}\varphi\Big(\frac{\xi}{2}\Big) = \sum_{k\in\mathbb{Z}} \alpha_k\varphi(\xi + k), \tag{38}$$

where

$$m_0(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{ik\xi}. \tag{39}$$

To be explicit, the dyadic dilation on $\ell^2(\mathbb{Z})$ we choose has the property that $(\rho\Delta)_k = \alpha_{-k}$; the α_{-k} is accounted for by the fact that we chose to translate in one direction in the definition of Φ while the two-scale equation is typically written with translations in the opposite direction.

The first and most obvious property of Φ is that it respects translation by integers:

$$(\Phi\tau^l f)(\xi) = \sum_{k\in\mathbb{Z}} f_{k-l}\varphi(\xi - k) = \sum_{k\in\mathbb{Z}} f_k\varphi(\xi - k - l) = (\Phi f)(\xi - l).$$

If we consider the dyadic dilation of Φf, where in $\mathbb{R}$ this has an unambiguous meaning, we see that

$$\begin{aligned}
\frac{1}{2}(\Phi f)\Big(\frac{\xi}{2}\Big) &= \frac{1}{2}\sum_{k\in\mathbb{Z}} f_k\varphi\Big(\frac{\xi}{2} - k\Big) \\
&= \sum_{k\in\mathbb{Z}} f_k\frac{1}{2}\varphi\Big(\frac{\xi - 2k}{2}\Big) \\
&= \sum_{k\in\mathbb{Z}} f_k \sum_{k'\in\mathbb{Z}} \alpha_{k'}\varphi(\xi - 2k + k') \\
&= \sum_{k\in\mathbb{Z}} f_k \sum_{k'\in\mathbb{Z}} \alpha_{k'+2k}\varphi(\xi + k') \\
&= \sum_{k'\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} f_k\alpha_{-(k'-2k)}\varphi(\xi - k').
\end{aligned} \tag{40}$$

We needn't worry about interchanging the order of summation in the last line as we can exploit the properties of the isometry and approximate f by a finitely supported sequence, showing that the error goes to zero as our finite approximation goes to f. Notice that (40) is precisely $\frac{1}{\sqrt{2}}\Phi(\rho f)$. We can therefore conclude that

$$\frac{1}{\sqrt{2}}(\Phi f)\Big(\frac{\xi}{2}\Big) = [\Phi(\rho f)]\,(\xi). \tag{41}$$

Thus Φ respects dyadic dilation, and hence, the MRA structure, i.e., the discrete MRA generated by ρ is mapped via Φ into our MRA on $\mathbb{R}$ in such a way that

$$f \in V_j \text{ (the discrete MRA)} \Leftrightarrow \Phi f \in V_j \text{ (the real MRA)},$$

and

$$f \in W_j \text{ (the discrete MRA)} \Leftrightarrow \Phi f \in W_j \text{ (the real MRA)}.$$

Therefore, we can take $\frac{1}{\sqrt{2}}\psi(\frac{\xi}{2}) \in W_{-1}$ (remember that ψ is our real MRA wavelet), whose translates and dilates span V_0 in the limit, and construct for ourselves a discrete o.n. wavelet, namely, $\Phi^{-1}\frac{1}{\sqrt{2}}\psi(\frac{\cdot}{2})$.

Let us go back and explicitly show the correspondence just described above. First of all, we will need to adjust notation slightly to avoid confusion. When referring to the real MRA, we will use the standard notation. In the case of the discrete MRA generated by ρ (which is in turn generated by m_0), we will place a tilde over all the familiar symbols ($\widetilde{\varphi}$, $\widetilde{\psi}$, $\widetilde{V}_j$, $\widetilde{W}_j$, etc.).

First of all, we introduce the adjoint of Φ. As Φ^* is a left inverse of Φ, and Φ is a surjective map into V_0 (up to a.e.) we see that we must have

$$[\Phi^* y(\cdot)]_l = \ \langle y, \varphi(\cdot - l)\rangle$$

by the orthonormality of the translates of φ. Moreover, as the translates of φ taken together span V_0, we must have that Φ^* is also a right inverse of Φ (again, up to a.e.). Using these facts, we now show that Φ respects the MRA structures in $L^2(\mathbb{R})$ and $\ell^2(\mathbb{Z})$.

Proposition 16

For any $f \in \ell^2(\mathbb{Z})$,

$$f \in \widetilde{V}_j \Leftrightarrow \Phi f \in V_j.$$

Proof. Suppose $f \in \widetilde{V}_j$, $j \le 0$. By definition, $f = \rho^{-j} g$ for some second sequence g. Thus

$$(\Phi f)(\xi) = \frac{1}{\sqrt{2^{-j}}}(\Phi g)\Big(\frac{\xi}{2^{-j}}\Big) = 2^{j/2}(\Phi g)(2^j\xi).$$

As the image of Φ is contained in V_0, $\Phi f \in V_j$. For the converse, let us denote dyadic dilation on $\mathbb{R}$ by D, i.e.

$$Dy(\cdot) = \frac{1}{\sqrt{2}} y\Big(\frac{\cdot}{2}\Big).$$

We have shown that $D\Phi = \Phi\rho$. Therefore $\Phi^* D\Phi = \rho$ and so $\Phi^* D^* \Phi = \rho^*$. Suppose $\Phi f \in V_j$, $j \leq 0$. This means that $(D^*)^{-j}\Phi f \in V_0$, so $\Phi^*(D^*)^{-j}\Phi f = (\rho^*)^{-j} f \in \widetilde{V}_0$. But $\Phi^* = \Phi^{-1}$ (in the L^2-norm) so

$$\begin{aligned}\rho^{-j}(\rho^*)^{-j} f &= \Phi^*(D)^{-j}\Phi\Phi^*(D^*)^{-j}\Phi f \\ &= \Phi^*(D)^{-j}(D^*)^{-j}\Phi f \\ &= \Phi^*\Phi f = f.\end{aligned}$$

Therefore $f \in \widetilde{V}_j$. □

Corollary 8

For any $f \in \ell^2(\mathbb{Z})$,

$$f \in \widetilde{W}_j \Leftrightarrow \Phi f \in W_j.$$

Proof. In both the discrete MRA and the real MRA, the wavelet subspace W_j (or $\widetilde{W}_j$) is the orthogonal complement to V_j ($\widetilde{V}_j$) in V_{j+1} ($\widetilde{V}_{j+1}$). Since Φ is injective, surjective (up to a.e.), an isometry, and respects the V_j, we can use this characterization of W_j ($\widetilde{W}_j$) to see immediately that the conclusion must be true. □

One other important detail–we have yet to show that ρ, the dyadic dilation on $\ell^2(\mathbb{Z})$ actually generates an MRA. Since m_0 is a low-pass filter for a scaling function φ, m_0 cannot be constant, so we are clear. But we needn't even use this fact to prove that ρ generates an MRA.

Corollary 9

Given an MRA with low-pass filter m_0, the subspaces $\widetilde{V}_j$ generated by dyadic dilation ρ on $\ell^2(\mathbb{Z})$ (which is in turn generated by m_0) form a discrete MRA, i.e.,

$$\bigcap \widetilde{V}_j = \{0\}.$$

Proof. Let $f \in \bigcap \widetilde{V}_j$. We must then have $\Phi f \in \bigcap V_j$. As the V_j form a (real) MRA, this intersection is trivial, thus $f = 0$ as Φ is injective. □

As mentioned earlier, $\Phi\Phi^*[y(\cdot)] = y$ (up to a.e.), so if ψ is a wavelet associated with a given (real) MRA, then so is $\Phi\Phi^*\psi$. In particular, $\{\psi_{j,k}(\xi) = 2^{j/2}\psi(2^j\xi - k)\}_{j,k\in\mathbb{Z}}$ is an o.n.b. of $L^2(\mathbb{R})$, and $\{\psi_{j,k}\}_{j<0,k\in\mathbb{Z}}$ is an o.n.b. of V_0. The goal is to show that $\{\Phi^*\psi_{j,k}\}_{j<0,k\in\mathbb{Z}}$ is an o.n.b. of $V_0 = \ell^2(\mathbb{Z})$. To this end, it is sufficient to show that $\{\Phi^*\psi_{-1,k}\}_{k\in\mathbb{Z}}$ is an o.n.b. of $\widetilde{W}_{-1}$. Clearly this is an o.n. system as Φ is an isometry. If it does not span $\widetilde{W}_{-1}$, we can find an f in this subspace which is orthogonal to $\Phi^*\psi_{-1,k}$ for every k, which means that Φf is in W_{-1} and orthogonal to $\psi_{-1,k}$, for every k, meaning that Φf is almost everywhere zero and $f = 0$. Thus $\Phi^*\psi_{-1,k}$ is an o.n.b. of $\widetilde{W}_{-1}$, and, hence, its dyadic dilates span $\ell^2(\mathbb{Z})$. But

$$\Phi^*\big(\psi_{-1,k}(\cdot)\big) = \Phi^*\big(\psi_{-1,0}(\cdot - 2k)\big) = \tau^{2k}\Phi^*\psi_{-1,0},$$

so $\Phi^*\psi_{-1,0}$ is a discrete o.n. wavelet by definition (here we see where the $2k$ arises as well). We can therefore conclude:

Theorem 12

Let ψ be an MRA wavelet with scaling function φ and low-pass filter m_0. Then the sequence $\Phi^*\psi_{-1,0} = (\ \langle \psi_{-1,0}, \varphi(\cdot - l)\rangle\ )_{l\in\mathbb{Z}}$ is an o.n. wavelet in $\ell^2(\mathbb{Z})$ with respect to the dyadic dilation ρ generated by m_0.

8. Rational Dilations

We now consider the more general problem of "rational dilations" on $\ell^2(\mathbb{Z})$, i.e., those linear operators ρ for which

$$\tau^N \rho = \rho \tau^M$$

and

$$||\rho f|| = ||f|| \quad \forall f \in \ell^2(\mathbb{Z}).$$

These operators will correspond to N/M-dilations (more appropriately, contractions) in much the same way that the dyadic dilations on $\ell^2(\mathbb{Z})$ correspond to dyadic dilations on $L^2(\mathbb{R})$. Describing them as rational dilations, though, is slightly misleading–in our case a 6/4-dilation is not necessarily a 3/2-dilation. One could always demand that N and M be relatively prime, but we do not do so here because this extra hypothesis plays no role in the necessary proofs.

The first observation to make is that we may decompose $\ell^2(\mathbb{Z})$ into V_j and W_j subspaces as before, since the results of Section 4 rely only on the fact that ρ is an isometry of $\ell^2(\mathbb{Z})$ to the range of ρ. Let us define a family of projection operators on $\ell^2(\mathbb{Z})$ and their corresponding operators on $L^2(\mathbb{T})$.

DEFINITION 5. Given a positive integer n and a non-negative integer k less than n, the linear operator P_k^n on $\ell^2(\mathbb{Z})$ is defined as

$$(P_k^n f)_i = \begin{cases} f_i & (k-i) \in n\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that P_k^n is a projection operator and that, for a fixed n, the collection $\{P_k^n\}_{k=0,\ldots,n-1}$ is a commuting family of orthogonal projection operators with $\sum_{k=0}^{n-1} P_k^n$ equal to the identity operator. Another fact that is easily verified is that, on $L^2(\mathbb{T})$, the operators take the form:

$$(P_k^n f)^\vee(\xi) = \frac{e^{ik\xi}}{n} \sum_{l=0}^{n-1} e^{-2\pi i k l/n} f^\vee\left(\xi + \frac{2\pi l}{n}\right).$$

Back to the rational dilations. Let us define

$$m_0, \ldots, m_{M-1} \in L^2(\mathbb{T}) \text{ as } m_k(\xi) = (\rho \tau^k \Delta)^\vee(\xi).$$

It is immediate from our definition that for any

$$l \in \mathbb{Z}, (\rho \tau^{Ml+k} \Delta)^\vee(\xi) = (\tau^{Nl} \rho \tau^k \Delta)^\vee(\xi)$$

and this we conclude is equal to

$$e^{iNl\xi} m_k(\xi) = e^{-ikN\xi/M} e^{i(Ml+k)N/M\xi} m_k(\xi)$$

almost everywhere. By arguments similar to the one showing that $(\rho f)^\vee(\cdot) = m(\cdot)f^\vee(2\cdot)$ for dyadic dilations, we conclude that

$$(\rho P_k^M f)^\vee(\xi) = m_k(\xi)(P_k^M f)^\vee\left(\frac{N}{M}\xi\right) \quad a.e.$$

Therefore, by our previous observation that the sum of the P_k^n's is the identity,

$$(\rho f)^\vee(\xi) = \sum_{k=0}^{M-1} m_k(\xi)e^{-ikN/M\xi}(P_k^M f)^\vee\left(\frac{N}{M}\xi\right) \quad a.e. \tag{42}$$

Next we will derive the corresponding formula for ρ^*.

$$\begin{aligned}\langle g, \rho f\rangle &= \frac{1}{2\pi}\int_0^{2\pi} g^\vee(\xi)\sum_{k=0}^{M-1}\overline{m_k(\xi)}e^{ikN/M\xi}\overline{(P_k^M f)^\vee\left(\frac{N}{M}\xi\right)}\,d\xi \\ &= \frac{1}{2\pi}\sum_{k=0}^{M-1}\int_0^{2\pi} g^\vee(\xi)\overline{m_k(\xi)}e^{ikN/M\xi}\overline{(P_k^M f)^\vee\left(\frac{N}{M}\xi\right)}\,d\xi. \end{aligned}\tag{43}$$

We now employ two additional properties of the P_k^n (From now on, the symbol P_k^n will refer both operators on $\ell^2(\mathbb{Z})$ and the corresponding operators on $L^2(\mathbb{T})$ when no confusion will arise). First of all, if $g \in L^2(\mathbb{T})$ it is easy to check that $\int_0^{2\pi} g(\xi)\,d\xi = \int_0^{2\pi}(P_0^n g)(\xi)\,d\xi$ as a result of the periodicity of g. The other fact, easily verified, is that $f, g \in L^2(\mathbb{T})$, and if f has period $\frac{2\pi}{n}$, then $(P_0^n fg)(\xi) = f(\xi)(P_0^n g)(\xi)$. If we notice that $e^{ik\frac{N}{M}\xi}\overline{(P_k^M f)^\vee\left(\frac{N}{M}\xi\right)}$ has period $\frac{2\pi}{N}$, we may apply both these properties to (43) and obtain

$$\begin{aligned}\langle g, \rho f\rangle &= \frac{1}{2\pi}\sum_{k=0}^{M-1}\int_0^{2\pi}(P_0^N g^\vee\overline{m_k})(\xi)e^{ikN/M\xi}\overline{(P_k^M f)^\vee\left(\frac{N}{M}\xi\right)}\,d\xi \\ &= \frac{1}{2\pi}\sum_{k=0}^{M-1}\int_0^{2\pi}(P_0^N g^\vee\overline{m_k})\Big(\frac{M}{N}\xi\Big)e^{ik\xi}\overline{(P_k^M f)^\vee(\xi)}\,d\xi \\ &= \frac{1}{2\pi}\sum_{k=0}^{M-1}\int_0^{2\pi}(P_0^N g^\vee\overline{m_k})\Big(\frac{M}{N}\xi\Big)e^{ik\xi}(P_{-k}^M\overline{f}^\vee)(\xi)\,d\xi \\ &= \frac{1}{2\pi}\sum_{k=0}^{M-1}\int_0^{2\pi}(P_0^N g^\vee\overline{m_k})\Big(\frac{M}{N}\xi\Big)(P_0^M e^{ik\xi}\overline{f}^\vee)(\xi)\,d\xi \\ &= \frac{1}{2\pi}\sum_{k=0}^{M-1}\int_0^{2\pi}\left(P_0^M(P_0^N g^\vee\overline{m_k})\Big(\frac{M}{N}\xi\Big)e^{ik\xi}\overline{f}^\vee\right)(\xi)\,d\xi \\ &= \frac{1}{2\pi}\sum_{k=0}^{M-1}\int_0^{2\pi}(P_0^N g^\vee\overline{m_k})\Big(\frac{M}{N}\xi\Big)e^{ik\xi}\overline{f}^\vee(\xi)\,d\xi. \end{aligned}$$

As in the case of the dyadic dilation, this last integral permits us to conclude that

$$(\rho^* g)^\vee(\xi) = \sum_{k=0}^{M-1} e^{ik\xi}(P_0^N g^\vee\overline{m_k})\Big(\frac{M}{N}\xi\Big) \quad a.e. \tag{44}$$

We may now use this formula for ρ^* to find necessary (and sufficient) conditions on the m_k. Let $g = \rho\tau^l\Delta$, where l is non-negative and less than M. Then we can conclude that

$$e^{il\xi} = \sum_{k=0}^{M-1} e^{ik\xi}(P_0^N m_l\overline{m_k})\Big(\frac{M}{N}\xi\Big) \quad a.e.$$

If we now take any non-negative integer l' which is less than M we conclude that

$$\begin{aligned}
\delta_{l,l'} &= (P_0^M e^{i(l-l')\xi})(\xi) \\
&= \left(P_0^M \sum_{k=0}^{M-1} e^{i(k-l')\xi}(P_0^N m_l\overline{m_k})\Big(\frac{M}{N}\xi\Big)\right)(\xi) \\
&= \sum_{k=0}^{M-1} (P_0^N m_l\overline{m_k})\Big(\frac{M}{N}\xi\Big)(P_0^M e^{i(k-l')\xi})(\xi) \\
&= \sum_{k=0}^{M-1} (P_0^N m_l\overline{m_k})\Big(\frac{M}{N}\xi\Big)\delta_{k,l'} \\
&= (P_0^N m_l\overline{m_{l'}})\Big(\frac{M}{N}\xi\Big) \quad a.e.
\end{aligned}$$

The end result of all this calculation is that we can conclude

$$(P_0^N m_l\overline{m_{l'}})(\xi) = \delta_{l,l'} \quad a.e. \tag{45}$$

Not only is this condition necessary, but given m_k's satisfying it, we may define an operator ρ according to the formula (42) which satisfies our definition of an N/M-dilation.

Corollary 10

If $M > N$, then no such dilations exist.

Proof. Notice that on a pointwise level, $P_0^N m_l\overline{m_{l'}}$ is an inner product on $\mathbb{C}^N$. Thus, if $M > N$, (45) would imply that for almost every fixed ξ, the collection of M vectors given by $v_l = (m_l(\xi), \ldots, m_l(\xi + 2\pi(N-1)/N))$ would form an orthonormal system, which is impossible. □

Corollary 11

If $M < N$, then ρ is not surjective. In particular, W_{-1} is non-trivial.

Proof. If this were not the case, then ρ^* would be an M/N-dilation, which cannot exist. □

The proof of these corollaries rests on the fact that $P_0^N f\overline{g}$ is an inner product on $\mathbb{C}^N$ when interpreted in a pointwise sense. The problem of finding wavelets, then, reduces to completing the orthonormal system represented by (45). For the remainder

of this section, we will assume that there exist $m_M, \ldots, m_{N-1}$ such that, for any non-negative l, l' both less than N

$$\big(P_0^N m_l \overline{m_{l'}}\big)(\xi) = \delta_{l,l'} \quad a.e.$$

This condition implies that $\langle m_l, m_{l'} \rangle_{L^2(\mathbb{T})} = \delta_{l,l'}$ by earlier remarks. In particular, if $\psi_0, \ldots, \psi_{N-M-1}$ are sequences such that $(\psi_j)^\vee = m_{M+j}$, we are forced to conclude that $\psi_j \in W_{-1}$ and, even more, that $\tau^{Nk}\psi_j \in W_{-1}$ for $k \in \mathbb{Z}$ and all appropriate j.

Suppose $f \in W_{-1}$. Exploiting the inner-product formula in relation to our projection operators,

$$f^\vee = \sum_{k=0}^{N-1} m_k P_0^N \big(f^\vee \overline{m_k}\big).$$

By the formula for the adjoint operator ρ^* and orthogonality,

$$0 = \sum_{k=0}^{M-1} e^{ik\xi} P_0^N (f^\vee \overline{m_k}) \Big(\frac{M}{N}\xi\Big).$$

We can take this equation and apply P_l^M to both sides, where l is a non-negative integer less than M; the result is that

$$0 = \sum_{k=0}^{M-1} P_l^M \left(e^{ik\xi} P_0^N (f\overline{m_k}) \Big(\frac{M}{N}\xi\Big)\right) = e^{il\xi} P_0^N (f^\vee \overline{m_l}) \Big(\frac{M}{N}\xi\Big).$$

The ultimate result of these calculations is that $P_0^N(f^\vee \overline{m_l})(\xi) = 0$ for $l = 0, \ldots, M-1$, so we may conclude that, for $f \in W_{-1}$,

$$f^\vee = \sum_{k=M}^{N-1} m_k P_0^N \big(f^\vee \overline{m_k}\big).$$

But for a given k, $P_0^N(f^\vee \overline{m_k})$ is a $2\pi/N$-periodic function, so as noted earlier, $m_k P_0^N(f^\vee \overline{m_k})$ is a function in the span of $\{e^{iNl\cdot} m_k = (\tau^{Nl}\psi_{k-M})^\vee\}_{l\in\mathbb{Z}}$. Thus, we conclude that

$$\{\tau^{Nl}\psi_k\}_{l\in\mathbb{Z}, k=0,\cdots,N-M-1}$$

is an o.n.b. of W_{-1}, so the ψ_k are our wavelets.

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