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On ℓ^∞ subspaces of Banach spaces

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Abstract

We obtain a refinement of a result of Partington on Banach spaces containing isomorphic copies of ℓ^∞. Motivated by this result, we prove that Banach spaces containing asymptotically isometric copies of ℓ^∞ must contain isometric copies of ℓ^∞.

A well-known result of R.C. James, dating back to the early sixties, says that if a Banach space contains an isomorphic copy of c_0 (respectively, ℓ^1), then it contains almost isometric copies of c_0 (respectively, ℓ^1) [10]. This result is often referred to as the *James Distortion Theorem*. A less well-known result due to Partington in the early eighties says that a similar result holds for Banach spaces containing isomorphic copies of ℓ^∞; that is, if a Banach space contains an isomorphic copy of ℓ^∞, then it contains almost isometric copies of ℓ^∞ [12].

In the past few years a number of authors have considered refinements of James's original result. In particular, the notions of a Banach space containing an asymptotically isometric copy of c_0 or ℓ^1 were directly motivated by James's result and have resulted in applications to fixed point theory [6, 7] and the isometric structure of Banach spaces [2, 5, 9]. In [8], the notion of a Banach space containing an asymptotically isometric copy of ℓ^∞ was introduced and an asymptotically isometric version of

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the classical Bessaga-Pełczyński Theorem [1, 4] was proved; namely, a dual Banach space, X^*, contains an asymptotically isometric copy of c_0 if and only if X^* contains an asymptotically isometric copy of ℓ^∞. This result was later improved to say that a dual Banach space, X^*, contains an asymptotically isometric copy of c_0 if and only if X^* contains an isometric copy of ℓ^∞ [9]. One consequence of these results is that dual Banach spaces contain an isometric copy of ℓ^∞ whenever they contain an asymptotically isometric copy of ℓ^∞. One of the aims of this short note is to prove that this result is true in general; that is, a Banach space contains an isometric copy of ℓ^∞ whenever it contains an asymptotically isometric copy of ℓ^∞. Although the proof of this result is not particularly difficult, we will see that the result is somewhat surprising when we consider refinements of Partington's result.

Before we get to the results of this paper, it should be pointed out that the proof of Theorem 6, though short, is not the shortest proof available. The proof of Theorem 6 follows quite easily using Definition 4. However, we have chosen to give a slightly longer proof, using Lemma 2, so as to highlight the similarities between the proofs of Theorem 3 and Theorem 6.

The results

We refer the reader to [4] and [11] for standard definitions and notation. We begin this section with Partington's result [12, Theorem 3].

Theorem 1

Let $\|\cdot\|$ *be an equivalent norm on* ℓ^∞ *and let* $\varepsilon > 0$. *Then there is a sequence* $u_1, u_2, \ldots$ *of disjoint vectors such that for each bounded sequence of real numbers* $a_1, a_2, \ldots$ *the norm of the formal sum* $\sum_{n=1}^{\infty} a_n u_n$ *lies between* $(1-\varepsilon)\sup|a_n|$ *and* $(1+\varepsilon)\sup|a_n|$.

The key ingredient Partington used in proving his result is the following lemma [12, Lemma 3], which will used twice in what follows.

Lemma 2

Let $\|\cdot\|$ *be an equivalent norm on* ℓ^∞ *and let* $\varepsilon \geq 0$. *If* $u_1, u_2, \ldots$ *is a bounded sequence of disjointly supported elements of* ℓ^∞ *and if for every* $n \geq 1$,

$$1 - \varepsilon/3 \leq \left\| u_n + \sum_{i=n+1}^{\infty} a_i u_i \right\| \leq 1 + \varepsilon/3,$$

provided that $\sup|a_i| \le 1$, *then*

$$1-\varepsilon \le \Big\|\sum_{i=1}^{\infty} a_i u_i\Big\| \le 1+\varepsilon/3$$

whenever $\sup|a_i| = 1$.

Our first result is a slight improvement of Partington's result. In the proof of this result we use the standard notation, $\operatorname{supp} u$, to denote the support of a vector u in ℓ^∞, $\|u\|_\infty$ is the usual sup norm of u, and $|A|$ denotes the cardinality of a subset A of $\mathbb{N}$.

Theorem 3

Let $\|\cdot\|$ be an equivalent norm on ℓ^∞ and let (ε_n) be a decreasing null sequence in $(0,1)$. Then there is a sequence $u_1, u_2, \dots$ of disjoint vectors such that for each bounded sequence of real numbers $a_1, a_2, \dots$ and for every $n \ge 1$, the norm of the formal sum $\sum_{i=n}^{\infty} a_i u_i$ satisfies

$$(*) \qquad (1-\varepsilon_n)\sup_{i\ge n}|a_i| \le \Big\|\sum_{i=n}^{\infty} a_i u_i\Big\| \le (1+\varepsilon_n)\sup_{i\ge n}|a_i|.$$

Proof. Let $T_0' = \mathbb{N}$. For $n \ge 0$ we inductively define

$$S_n = \big\{u \in \ell^\infty : \operatorname{supp} u \subseteq T_n' \text{ and } |T_n' \setminus \operatorname{supp} u| = \infty\big\}.$$

Given $v \in S_n$ with $\|v\|_\infty \le 1$, define

$$R_n(v) = \{w \in S_n : \|w\|_\infty = 1, \text{ and } w_i = v_i \text{ whenever } v_i \ne 0\}.$$

Note that if $w \in R_n(v)$, then $R_n(w) \subseteq R_n(v)$. Now define $M_n(v) = \sup\{\|w\| : w \in R_n(v)\}$ and $m_n(v) = \inf\{\|w\| : w \in R_n(v)\}$. Choose $v_n \in R_n(0)$ such that $M_n(0) < \|v_n\|(1+\varepsilon_n/6)$. Since $M_n(v_n) \le M_n(0)$, $M_n(v_n) < \|v_n\|(1+\varepsilon_n/6)$. Given $\delta > 0$, choose $y \in R_n(v_n)$ with $\|y\| < m_n(v_n) + \delta$. Then since $2v_n - y \in R_n(v_n)$, we have

$$2\|v_n\| \le \|2v_n - y\| + \|y\| \le M_n(v_n) + m_n(v_n) + \delta.$$

Consequently, since $\delta > 0$ was arbitrary, $2\|v_n\| \le M_n(v_n) + m_n(v_n)$ (this argument can also be found in [3; page 121]). Hence $m_n(v_n) > \|v_n\|(1-\varepsilon_n/6)$.

Now define $T_{n+1} = T'_n \setminus \operatorname{supp} v_n$ and $T'_{n+1} = T_{n+1} \setminus \{\min\{m : m \in T_{n+1}\}\}$. By construction we see that if $\sup |a_i| \le 1$, then the formal sum $v_n + \sum_{i=n+1}^{\infty} a_i v_i$ is both an element of S_n and the unit sphere of $(\ell^\infty, \|\cdot\|_\infty)$ and therefore satisfies

$$\|v_n\|(1-\varepsilon_n/6) < m_n(v_n) \le \left\| v_n + \sum_{i=n+1}^{\infty} a_i v_i \right\| \le M_n(v_n) < \|v_n\|(1+\varepsilon_n/6).$$

Since the $(\|v_n\|)$ is a bounded sequence of real numbers which is bounded away from 0, it has a subsequence which converges to a real number $M > 0$. Without loss of generality, we can assume that $(\|v_n\|)$ converges to M and for each $n \in \mathbb{N}$, $M(1-\varepsilon_n/7) \le \|v_n\| \le M(1+\varepsilon_n/7)$. Let $u_n = v_n/M$ and note that

$$(1-\varepsilon_n/3) \le \left\| u_n + \sum_{i=n+1}^{\infty} a_i u_i \right\| \le (1+\varepsilon_n/3),$$

whenever $\sup |a_i| \le 1$. The proof is now completed by an application of Lemma 2 and the fact that the sequence (ε_n) is decreasing. □

Remark. If one considers the expression $(*)$ in Theorem 3, one might ask if the expression can be improved by either removing the "$1-\varepsilon_n$" from the left hand side or removing the "$1+\varepsilon_n$" from the right hand side. For example, if we consider ℓ^∞ with the equivalent norm $|||\cdot|||$ defined by $|||(a_n)||| = \left[(\sup |a_n|)^2 + \sum_{n=1}^{\infty} 2^{-n}|a_n|^2\right]^{1/2}$, then we can in this case remove the "$1-\varepsilon_n$" from the left hand side. Specifically, if we define u_n to be the element of ℓ^∞ whose n-th coordinate is 1 and all other coordinates are 0, then for all bounded sequences (a_n) we have

$$\sup_{i\ge n} |a_i| \le \left|\left|\left| \sum_{i=n}^{\infty} a_i u_i \right|\right|\right| \le (1+2^{-n+1}) \sup_{i\ge n} |a_i|.$$

It is also worth noting that since $(\ell^\infty, |||\cdot|||)$ is strictly convex it does not contain an isometric copy of ℓ^∞.

We will see that removal of the "$1+\varepsilon_n$" from the right hand side leads us to asymptotically isometric copies of ℓ^∞, and concept which we now define.

DEFINITION 4. A Banach space X is said to contain an asymptotically isometric copy of ℓ^∞ if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a bounded linear operator $T : \ell^\infty \to X$ so that

$$\sup_n (1-\varepsilon_n)|a_n| \le \left\| T\left((a_n)_n\right) \right\| \le \sup_n |a_n| \ ,$$

for all $(a_n)_n \in \ell^\infty$.

The relationship between asymptotically isometric copies of ℓ^∞ and the last remark can be found in the following proposition, the proof of which is almost identical to the proof of Theorem 2 of [7], and is therefore omitted.

Proposition 5

A Banach space X contains an asymptotically isometric copy of ℓ^∞ if and only if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and an operator $T:\ell^\infty \to X$ so that

$$(1-\varepsilon_n)\sup_{i\geq n}|a_i| \leq \left\|T\left((a_i)_{i\geq n}\right)\right\| \leq \sup_{i\geq n}|a_i| \ ,$$

for all $(a_i)_i \in \ell^\infty$ and for all $n \in \mathbb{N}$.

We are now ready for our main result.

Theorem 6

Let X be a Banach space containing an asymptotically isometric copy of ℓ^∞. Then X contains an isometric copy of ℓ^∞.

Proof. Since X contains an asymptotically isometric copy of ℓ^∞, there is a null sequence (ε_n) in $(0,1)$ and a bounded linear operator $T:\ell^\infty \to X$ such that

$$\sup_n (1-\varepsilon_n)|t_n| \leq \left\|T\big((t_n)_n\big)\right\| \leq \sup_n |t_n| \ ,$$

for all $(t_n)_n \in \ell^\infty$.

Partition $\mathbb{N}$ into an infinite number of infinite (disjoint) subsets $(\mathbb{N}_j)$. For each $j \in \mathbb{N}$, let u_j be the element of ℓ^∞ whose n-th coordinate is 1 if $n \in \mathbb{N}_j$, and is 0 otherwise. Let (a_i) be a sequence with $\sup|a_i| \leq 1$ and for each $j \geq 1$ consider the expression

$$\Big\|T\big(u_j + \sum_{i>j} a_i u_i\big)\Big\|.$$

Since $\sup|a_i| \leq 1$ and the u_j's are disjointly supported norm 1 elements of ℓ^∞ we have

$$\Big\|T\big(u_j + \sum_{i>j} a_i u_i\big)\Big\| \leq 1.$$

On the other hand, since u_j has infinite support, we also have

$$\Big\|T\big(u_j + \sum_{i>j} a_i u_i\big)\Big\| \geq \sup_{n\in\mathbb{N}_j}(1-\varepsilon_n) = 1.$$

Therefore, by Lemma 2, we have

$$\Big\|T\big(\sum_{i=1}^{\infty} a_i u_i\big)\Big\| = 1, \text{ whenever } \sup|a_i| = 1.$$

This clearly implies that X contains an isometric copy of ℓ^∞. $\square$

Remark. As we mentioned earlier, the proof of Theorem 6 is not difficult but it is somewhat surprising because it says that if, in the expression $(*)$ of Theorem 3, we remove the "$1+\varepsilon_n$" from the right hand side, then we gain an isometric copy of ℓ^∞. However, removing the "$1-\varepsilon_n$" from the left hand side does not necessarily yield an isometric copy of ℓ^∞, as the example in the remark following Theorem 3 illustrates.

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