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Characterizations of Gabor Systems via the Fourier Transform

Wojciech Czaja

Department of Mathematics, Washington University
Campus Box 1146, St. Louis MO 63130 USA
E-mail: wczaja@artsci.wustl.edu

Department of Mathematics, University of Wrocław
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

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Abstract

We give characterizations of orthogonal families, tight frames and orthonormal bases of Gabor systems. The conditions we propose are stated in terms of equations for the Fourier transforms of the Gabor system's generating functions.

1. Introduction

In many studies one seeks a particular function g that produces an orthonormal system or even a basis, as well as similar systems, when certain group actions are performed on g. This is well known in the case of wavelets, where the object is to find functions $\psi \in L^2(\mathbb{R})$, such that $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ forms an orthonormal basis. In this particular case we first apply the group action of translations, followed by dyadic dilations.

More than 50 years ago, Gabor proposed the study of systems defined by $g_{m,n}(x) = e^{2\pi imx}g(x-n), m,n \in \mathbb{Z}$. This, of course, is very natural from the point of view of Fourier analysis. A simple example of such a system is provided by $g = \chi_{[0,1]}$. In this case $\{g_{m,n} : m,n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. This example is typical for such systems since one cannot expect to have both smoothness and rapid decay at infinity for g, if we are seeking an orthonormal basis of this type. The well known result of Balian and Low states that in such a case either $\int x^2|g(x)|^2dx = \infty$ or $\int \xi^2|\hat{g}(\xi)|^2d\xi = \infty$.

One can consider somewhat more general systems, where translations and modulations have the form: translations by bn and modulations of the form $e^{2\pi iamx}$, where a and b are nonzero, real numbers.Again it was observed that there are limitations to such choices. In fact, Rieffel [13] has shown that if $ab > 1$, then such a system cannot be complete. If $ab = 1$, we are back to the situation of Gabor systems. If $ab < 1$ orthonormal bases are not possible; nevertheless one can obtain tight frames by forming these systems. For more information on this topic, we refer the reader to the book of Daubechies [8], or to the survey [1].

In the case of wavelet systems, simple equations have been found to completely characterize orthonormality, completeness and other important properties of these systems. Moreover, these equations are valid for various d-dimensional versions of wavelets. For references see the book of Hernández and Weiss [11], where the two equations of Wang [15] and Gripenberg, for wavelets with dyadic dilations appeared. For other dilations see [9], [6] or [4]. We also refer the reader to the works of Lemarié [12] and Daubechies [7], for similar considerations.

In this paper we shall consider the d-dimensional case as well, and we will present equations that characterize the orthogonal families, tight frames and orthonormal bases of Gabor systems. In a later paper we shall present the extension of the result of Rieffel and the special case of Daubechies, for multidimensional Gabor systems.

The characterizations of orthogonal families and tight frames, in case of one generating function of one variable, have been recently obtained by Casazza and Christensen, [5].

Before we proceed further let us present here the notation and several results that will be of use to us.

DEFINITION 1.1. Let $g^k \in L^2(\mathbb{R}^d)$, for $k = 1, \ldots, L$. By the Gabor system defined by the functions g^k, we mean the set of functions $\{g^k_{m,n} : m, n \in \mathbb{Z}^d, k = 1, \ldots, L\}$, where

$$g^k_{m,n}(x) = e^{2\pi iAm\cdot x}g^k(x - Bn),$$

for some nondegenerate linear maps A, B on $\mathbb{R}^d$.

We shall use the following notation: let a be the Jacobian of A and b be the Jacobian of B. Moreover, let B' denote $B^{t^{-1}}$. We use the Fourier transform defined by:

$$\hat{f}(\xi) = \int f(x)e^{-2\pi ix\cdot\xi}dx.$$

Proposition 1.2

The system $\{g_{0,n} : n \in \mathbb{Z}^d\}$ *is orthogonal if and only if*

$$\sum_{k\in\mathbb{Z}^d} |\hat{g}(\xi - B'k)|^2 = b\|g\|_2^2 \qquad a.e.\ \xi \in \mathbb{R}^d.$$

The proof presented in [11] (Proposition 2.1.11) can be adapted to this case.

Proposition 1.3 (see [11])

Let $\mathbb{H}$ *be a Hilbert space and* $\{e_j : j = 1, 2, \ldots\}$ *be a family of elements of* $\mathbb{H}$. *Then:*

$$\|f\|^2 = \sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2 \qquad holds\ for\ all\ f \in \mathbb{H}$$

if and only if

$$f = \sum_{j=1}^{\infty} \langle f, e_j\rangle e_j, \qquad with\ convergence\ in\ \mathbb{H},\ for\ all\ f \in \mathbb{H}.$$

A system $\{e_j : j = 1, 2, \ldots\}$ is called a *tight frame* for $\mathbb{H}$, with constant 1, if either of the two above conditions is satisfied.

Proposition 1.4 (see [11])

Suppose $\{e_j : j = 1, 2, \ldots\}$ *is a family of elements of Hilbert space* $\mathbb{H}$, *such that the first equality in Proposition 1.3 holds for all* f *in a dense subset of* $\mathbb{H}$. *Then this equality is valid for all* $f \in \mathbb{H}$.

Theorem 1.5 (see [11])

Suppose $\{e_j : j = 1, 2, \ldots\}$ *is a tight frame with constant 1. If* $\|e_j\| \geq 1$, *for all* j, *then* $\{e_j : j = 1, 2, \ldots\}$ *is an orthonormal basis for* $\mathbb{H}$.

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2. Orthogonal Gabor systems

Our first theorem will characterize the orthogonality of Gabor systems in terms of the Fourier transforms of the functions g^i.

Theorem 2.1

System $\{g^i_{m,n} : m, n \in \mathbb{Z}^d, i = 1, \ldots, L\}$ is orthogonal if and only if

$$\sum_{k \in \mathbb{Z}^d} |\widehat{g^i}(\xi - B'k)|^2 = b\|g^i\|_2^2 \quad a.e. \ \xi \in \mathbb{R}^d, i = 1, \ldots L, \tag{2.1}$$

$$\sum_{l \in \mathbb{Z}^d} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^i}(\xi + B'l)} = 0 \quad a.e. \ \xi \in \mathbb{R}^d, \tag{2.2}$$

for every $j \neq 0$, $i = 1, \ldots L$, and

$$\sum_{l \in \mathbb{Z}^d} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^k}(\xi + B'l)} = 0 \quad a.e. \ \xi \in \mathbb{R}^d, \tag{2.3}$$

for every j and $i \neq k$.

Corollary 2.2

System $\{g^i_{m,n} : m, n \in \mathbb{Z}^d, i = 1, \ldots, L\}$ is orthonormal if and only if

$$\sum_{k \in \mathbb{Z}^d} |\widehat{g^i}(\xi - B'k)|^2 = b \quad a.e. \ \xi \in \mathbb{R}^d, i = 1, \ldots L, \tag{2.1$'$}$$

and equations (2.2) and (2.3) are satisfied.

Remark. One can easily translate the above conditions, into conditions involving the functions g^i, rather than their Fourier transforms $\widehat{g^i}$. The reason we do it this way is that, in the future, we want to discuss similar problems for wave packets, which as in the case of wavelets, require this approach.

Proof. First let us assume that the system $\{g^i_{m,n} : m, n \in \mathbb{Z}^d, i = 1, \ldots, L\}$ is orthogonal. Proposition 1.2 implies then, that the first equation is satisfied, for every i. Moreover, using the Plancherel theorem we obtain

$$\begin{aligned}0 = \langle g^i_{j,k}, g^{i'} \rangle = \langle \widehat{g^i_{j,k}}, \widehat{g^{i'}} \rangle &= \int e^{2\pi i Bk\cdot Aj} e^{-2\pi i Bk\cdot\xi} \widehat{g^i}(\xi - Aj)\overline{\widehat{g^{i'}}(\xi)} d\xi \\ &= e^{2\pi i Bk\cdot Aj} \sum_{l\in\mathbb{Z}^d} \int_{Q_l} e^{-2\pi i Bk\cdot\xi} \widehat{g^i}(\xi - Aj)\overline{\widehat{g^{i'}}(\xi)} d\xi \\ &= e^{2\pi i Bk\cdot Aj} \sum_{l\in\mathbb{Z}^d} \int_{Q_0} e^{-2\pi i Bk\cdot\xi} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^{i'}}(\xi + B'l)} d\xi,\end{aligned}$$

where for $l = (l_1, \dots l_d)$, define $Q_l = B'(l + [0,1)^d)$. By Beppo-Levi's theorem, we can interchange the order of summation and integration in the integral $\int_{Q_0} \sum_{l\in\mathbb{Z}^d} e^{-2\pi i Bk\cdot\xi} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^{i'}}(\xi + B'l)}$, if the following series converges:

$$\sum_{l\in\mathbb{Z}^d} \int_{Q_0} |e^{-2\pi i Bk\cdot\xi} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^{i'}}(\xi + B'l)}|.$$

Observe, however, that

$$\begin{aligned}\sum_{l\in\mathbb{Z}^d} \int_{Q_0} \Big| e^{-2\pi i Bk\cdot\xi} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^{i'}}(\xi + B'l)} \Big| &= \int_{\mathbb{R}}^d \Big| \widehat{g^i}(\xi - Aj)\overline{\widehat{g^{i'}}(\xi)} \Big| d\xi \\ &\le \Big(\int_{\mathbb{R}}^d |\widehat{g^i}(\xi - Aj)|^2 d\xi \Big)^{1/2} \Big(\int_{\mathbb{R}}^d |\widehat{g^{i'}}(\xi)|^2 d\xi \Big)^{1/2} < \infty,\end{aligned}$$

since we assumed that $g^i \in L^2(\mathbb{R}^d)$. This shows that for $i = i'$, $j \neq 0$ and $k \in \mathbb{Z}^d$, or for $i \neq i'$ and $j, k \in \mathbb{Z}^d$ we have:

$$0 = \int_{Q_0} \Big(\sum_{l\in\mathbb{Z}^d} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^{i'}}(\xi + B'l)} \Big) e^{-2\pi i Bk\cdot\xi} d\xi.$$

Since $\{e^{2\pi i Bk\cdot} : k \in \mathbb{Z}^d\}$ is a basis for $L^2(Q_0)$ and $\sum_{l\in\mathbb{Z}^d} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^{i'}}(\xi + B'l)}$ is a Q_0-periodic function, we conclude that for $i = i'$ and $j \neq 0$, or for $i \neq i'$ and all j:

$$\sum_{l\in\mathbb{Z}^d} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^{i'}}(\xi + B'l)} = 0,$$

for a.e. $\xi \in \mathbb{R}^d$. This completes the proof of the first implication.

Now let us assume that all the three conditions are satisfied. It is easy to see that

$$\langle g^i_{j,k}, g^{i'}_{m,n}\rangle = e^{2\pi i A(j-m)\cdot Bn}\langle g^i_{j-m,k+n}, g^{i'}\rangle.$$

Moreover, Proposition 1.2 implies that the systems $\{g^i_{0,n} : n \in \mathbb{Z}\}$ are orthogonal for each i. Therefore to finish the proof, we can use Beppo-Levi's theorem, the Plancherel formula and the second and third conditions, to prove, as above, that

$$\langle g^i_{j,k}, g^{i'}\rangle = 0,$$

for $i = i'$, $j \neq 0$, $l \in \mathbb{Z}^d$ and for $i \neq i'$ and $j, k \in \mathbb{Z}^d$. □

In order to present the following results, we assume that B^tA maps $\mathbb{Z}^d$ into $\mathbb{Z}^d$. Let $W^i_j = \overline{\operatorname{span}\{g^i_{j,k} : k \in \mathbb{Z}^d\}}$ and suppose the system $\{g^i_{m,n} : m, n \in \mathbb{Z}^d, i = 1, \ldots, L\}$ is orthogonal. Let Q^i_j denote the orthogonal projection onto the space W^i_j. We have:

$$Q^i_j f = \sum_{k\in\mathbb{Z}^d} \langle f, g^i_{j,k}\rangle g^i_{j,k},$$

for every $f \in L^2(\mathbb{R}^d)$. Then:

$$\begin{aligned}\langle \hat{f}, \widehat{g^i_{j,k}}\rangle &= \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i Bk\cdot\xi - 2\pi i Aj\cdot Bk}\overline{\widehat{g^i}(\xi - Aj)}d\xi \\ &= \sum_{l\in\mathbb{Z}^d}\int_{Q_l} \hat{f}(\xi)e^{2\pi i Bk\cdot\xi}\overline{\widehat{g^i}(\xi - Aj)}d\xi \\ &= \sum_{l\in\mathbb{Z}^d}\int_{Q_0} \hat{f}(\xi + B'l)e^{2\pi i Bk\cdot(\xi+B'l)}\overline{\widehat{g^i}(\xi + B'l - Aj)}d\xi.\end{aligned}$$

Notice that:

$$\begin{aligned}\sum_{l\in\mathbb{Z}^d}\int_{Q_0}\left|\hat{f}(\xi + B'l)e^{2\pi i Bk\cdot\xi}\overline{\widehat{g^i}(\xi + B'l - Aj)}\right|d\xi &= \int_{\mathbb{R}^d}\left|\hat{f}(\xi)\overline{\widehat{g^i}(\xi - Aj)}\right|d\xi \\ \le \left(\int |\hat{f}(\xi)|^2 d\xi\right)^{1/2}\left(\int |\widehat{g^i}(\xi - Aj)|^2 d\xi\right)^{1/2} &= \|\hat{f}\|_2\|\hat{g}\|_2 < \infty.\end{aligned}$$

Thus, we can use Beppo-Levi's theorem, to obtain:

$$\langle \hat{f}, \widehat{g^i_{j,k}}\rangle = \int_{Q_0}\left(\sum_{l\in\mathbb{Z}^d}\hat{f}(\xi + B'l)\overline{\widehat{g^i}(\xi + B'l - Aj)}\right)e^{2\pi i Bk\cdot\xi}d\xi.$$

But these are evidently the Fourier coefficients of the Q_0 periodic function $\sum_{l\in\mathbb{Z}^d}\hat{f}(\xi + B'l)\overline{\widehat{g^i}(\xi + B'l - Aj)}$, so we can write:

$$\sum_{l\in\mathbb{Z}^d}\hat{f}(\xi + B'l)\overline{\widehat{g^i}(\xi + B'l - Aj)} = b\sum_{k\in\mathbb{Z}^d}\langle \hat{f}, \widehat{g^i_{j,k}}\rangle e^{-2\pi i Bk\cdot\xi}.$$

Multiplying both sides of this equation by $\widehat{g^i}(\xi - Aj)$, we obtain the following result:

Lemma 2.2

Let $B^tA : \mathbb{Z}^d \to \mathbb{Z}^d$. For the orthogonal projection operator Q_j^i we have

$$\sum_{l\in\mathbb{Z}^d} \hat{f}(\xi + B'l)\overline{\widehat{g^i}(\xi + B'l - Aj)}\widehat{g^i}(\xi - Aj)$$
$$= b \sum_{k\in\mathbb{Z}^d} \langle \hat{f}, \widehat{g^i_{j,k}}\rangle \widehat{g^i_{j,k}}(\xi) = \widehat{Q^i_j f}(\xi) \qquad a.e.\ \xi \in \mathbb{R}^d.$$

Having proved this result, we are ready to consider the completeness of Gabor systems when $A^{-1} = B^t$. (In particular this means that $ab = 1$.)

Theorem 2.3

Let $A = {B^t}^{-1} = B'$. Suppose that $g^i \in L^2(\mathbb{R}^d)$, $i = 1, \dots, L$ are such that the following equations hold:

$$\sum_{k\in\mathbb{Z}^d} |\widehat{g^i}(\xi - B'k)|^2 = b\|g^i\|_2^2 \qquad a.e.\ \xi \in \mathbb{R}^d, i = 1, \dots, L,$$

$$\sum_{l\in\mathbb{Z}^d} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^i}(\xi + B'l)} = 0 \qquad a.e.\ \xi \in \mathbb{R}^d,$$

for every $j \neq 0$, $i = 1, \dots, L$, and

$$\sum_{l\in\mathbb{Z}^d} \widehat{g^i}(\xi + B'l - Aj)\overline{\widehat{g^k}(\xi + B'l)} = 0 \qquad a.e.\ \xi \in \mathbb{R}^d,$$

for every j and $i \neq k$. Then the system $\{g^i_{m,n} : m, n \in \mathbb{Z}^d, i = 1, \dots, L\}$ is complete in $L^2(\mathbb{R}^d)$.

Proof. It is enough to show that

$$\sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} \widehat{Q^i_j f}(\xi) = \Big(\sum_{i=1}^{L} \|g^i\|_2^2\Big)\hat{f}(\xi) \qquad a.e.\ \xi \in \mathbb{R}^d,$$

and

$$\lim_{M\to\infty} \Big\| \sum_{i=1}^{L} \sum_{|j|\le M} \widehat{Q^i_j f}\Big\|_2 = \Big(\sum_{i=1}^{L} \|g^i\|_2^2\Big)\|\hat{f}\|_2,$$

for $f \in L^2(\mathbb{R}^d)$. But by Propositions 1.3 and 1.4, it suffices to prove the above equalities for f in some dense subset of $L^2(\mathbb{R}^d)$. Therefore, we consider only these functions f, such that $\mathrm{supp}(\hat{f})$ is compact.

Remark. To clarify the meaning of the series in the first of the equations, we point out that the convergence is to be taken as the convergence of the symmetric partial sums that are explicitly considered in the second equation. This applies repeatedly below.

By Theorem 2.1, we already know that system $\{g^i_{m,n} : m,n \in \mathbb{Z}^d, i = 1,\dots,L\}$ is orthogonal; hence, Lemma 2.2 applies to the projections Q^i_j, and we can write:

$$\begin{aligned}
\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\widehat{Q^i_j f}(\xi) &= \frac{1}{b}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\Big(\sum_{l\in\mathbb{Z}^d}\hat{f}(\xi+B'l)\overline{\widehat{g^i}(\xi+B'l-Aj)}\widehat{g^i}(\xi-Aj)\Big)\\
&= \frac{1}{b}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\Big(\hat{f}(\xi)|\widehat{g^i}(\xi-Aj)|^2+\sum_{l\neq 0}\hat{f}(\xi+B'l)\overline{\widehat{g^i}(\xi+B'l-Aj)}\widehat{g^i}(\xi-Aj)\Big)\\
&= \frac{1}{b}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\hat{f}(\xi)|\widehat{g^i}(\xi-Aj)|^2+\frac{1}{b}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{l\neq 0}\hat{f}(\xi+B'l)\\
&\quad\times\overline{\widehat{g^i}(\xi+B'l-Aj)}\widehat{g^i}(\xi-Aj)\\
&= \frac{1}{b}\hat{f}(\xi)\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}|\widehat{g^i}(\xi-Aj)|^2+\frac{1}{b}\sum_{i=1}^{L}\sum_{l\neq 0}\hat{f}(\xi+B'l)\\
&\quad\times\sum_{j\in\mathbb{Z}^d}\overline{\widehat{g^i}(\xi+B'l-Aj)}\widehat{g^i}(\xi-Aj)\\
&= \hat{f}(\xi)\sum_{i=1}^{L}\|g^i\|_2^2.
\end{aligned}$$

Here we have used our assumptions on the functions g^i and the fact that $A = B'$. The change in the order of summation is valid since $\hat{f}$ has compact support, which implies that the sum over $l \neq 0$ is finite.

For the second equality, since Q^i_j's are mutually orthogonal projections, we have:

$$\Big\|\sum_{i=1}^{L}\sum_{|j|\le M}\widehat{Q^i_j f}\Big\|_2 \le \Big(\sum_{i=1}^{L}\|g^i\|_2^2\Big)^{1/2}\|\hat{f}\|_2,$$

for every $M > 0$. The orthogonality of Q^i_j's implies that

$$\Big\|\sum_{i=1}^{L}\sum_{|j|\le M}\widehat{Q^i_j f}\Big\|_2 = \Big(\sum_{i=1}^{L}\sum_{|j|\le M}\|\widehat{Q^i_j f}\|_2^2\Big)^{1/2}$$

is an increasing sequence, bounded by $\left(\sum_{i=1}^{L}\|g^i\|_2^2\right)^{1/2}\|\hat{f}\|_2$. Therefore,

$$\lim_{M\to\infty}\Big\|\sum_{i=1}^{L}\sum_{|j|\le M}\widehat{Q_j^i f}\Big\|_2 \le \Big(\sum_{i=1}^{L}\|g^i\|_2^2\Big)^{1/2}\|\hat{f}\|_2.$$

The inequality

$$\lim_{M\to\infty}\Big\|\sum_{i=1}^{L}\sum_{|j|\le M}\widehat{Q_j^i f}\Big\|_2 \ge \Big(\sum_{i=1}^{L}\|g^i\|_2^2\Big)^{1/2}\|\hat{f}\|_2$$

follows from Fatou's lemma. □

Notice that this proof also yields the following corollary:

Corollary 2.4

Let $A = B^{t^{-1}} = B'$. Let $g^i \in L^2(\mathbb{R}^d), i = 1,\ldots,L$, be such that $\sum_{i=1}^{L}\|g^i\|_2^2 = 1$. Then with the assumptions of Theorem 2.3, the system $\{g^i_{m,n} : m,n \in \mathbb{Z}^d, i = 1,\ldots,L\}$ is a tight frame with constant 1.

Corollary 2.5

Let $A = B^{t^{-1}} = B'$. If the system $\{g^i_{m,n} : m,n \in \mathbb{Z}^d, i = 1,\ldots,L\}$ is an orthogonal system, then $L = 1$.

Proof. Notice that each of the systems $\{g^i_{m,n} : m,n \in \mathbb{Z}\}$, for $i = 1,\ldots,L$, satisfies the assumptions of the Theorem 2.3, and thus each of these systems is complete. It is then impossible for the whole system to orthogonal, if $L > 1$. (We would like to mention, that we use in the above the special case of Theorem 2.1, when $A = B'$.) □

Now, combining Corollary 2.4 with Theorem 1.5, we can obtain the following result for Gabor systems generated by a single function:

Theorem 2.6

Let $A = B^{t^{-1}} = B'$. Let $g \in L^2(\mathbb{R}^d)$, such that $\|g\|_2^2 = 1$. Then the system $\{g_{m,n} : m,n \in \mathbb{Z}^d\}$ is a tight frame with constant 1 if and only if the following equations hold:

$$\sum_{k\in\mathbb{Z}^d}|\hat{g}(\xi - B'k)|^2 = b \qquad a.e.\ \xi \in \mathbb{R}^d,$$

$$\sum_{l\in\mathbb{Z}^d}\hat{g}(\xi + B'l - Aj)\overline{\hat{g}(\xi + B'l)} = 0 \qquad a.e.\ \xi \in \mathbb{R}^d,$$

for every $j \neq 0$.

As an immediate application of Theorems 2.1 and 2.5, we obtain the following corollary:

Corollary 2.7

Let $A = B^{t^{-1}} = B'$. For any function $g \in L^2(\mathbb{R}^d)$, such that $\|g\|_2^2 = 1$, the system $\{g_{m,n} : m, n \in \mathbb{Z}^d\}$ is a tight frame with constant 1, if and only if it is an orthogonal system.

EXAMPLE. The above corollary allows us to easily verify whether a given Gabor system is an orthonormal basis since we only need to check the orthonormality of that system. For example, one can consider a Gabor system $\{g_{m,n} : m, n \in \mathbb{Z}^d\}$, with $g(x) = \chi_K(x)$, where K is a measurable subset of $\mathbb{R}^d$. It is an orthonormal basis for $L^2(\mathbb{R}^d)$, if and only if the set K is translation equivalent to $[0,1)^d$. (For the definition of the translation equivalence see [11].)

3. Tight frames of Gabor systems

We will now turn to the characterization of tight frames of Gabor systems, following the methods presented in [9].

Theorem 3.1

The Gabor system: $\{g^i_{m,n} : m, n \in \mathbb{Z}^d, i = 1, \ldots, L\}$ is a tight frame for $L^2(\mathbb{R}^d)$, if and only if it satisfies the two conditions:

$$\sum_{i=1}^{L} \sum_{j \in \mathbb{Z}^d} |\widehat{g^i}(\xi - Aj)|^2 = b \qquad a.e.\ \xi \in \mathbb{R}^d, \tag{3.1}$$

and

$$\sum_{i=1}^{L} \sum_{j \in \mathbb{Z}^d} \widehat{g^i}(\xi - Aj)\overline{\widehat{g^i}(\xi + B'l - Aj)} = 0 \qquad a.e.\ \xi \in \mathbb{R}^d, \tag{3.2}$$

for every $l \neq 0$.

Proof. By Proposition 1.4, it is enough to assume that the functions f, represented by this system, belong to some dense subset of $L^2(\mathbb{R}^d)$. Thus, let D be the set of all functions in $L^2(\mathbb{R}^d)$, such that their Fourier transforms have compact support and are bounded. Let $f \in D$. For fixed $i \in \{1, \dots, L\}$ and $j \in \mathbb{Z}^d$, define:

$$F(\xi) = F_j^i(\xi) = \hat{f}(\xi - Aj)\overline{\widehat{g^i}(\xi)}.$$

Then,

$$\begin{aligned}\hat{F}(Bk) &= \int_{\mathbb{R}^d} F(\xi)e^{-2\pi i Bk\cdot\xi}d\xi = \sum_{m\in\mathbb{Z}^d}\int_{Q_m} F(\xi)e^{-2\pi i Bk\cdot\xi}d\xi \\ &= \int_{Q_0} e^{-2\pi i Bk\cdot\xi}\Big(\sum_{m\in\mathbb{Z}^d} F(\xi + B'm)\Big)d\xi.\end{aligned}$$

The interchange of the order of summation and integration is valid, since we have assumed that function $\hat{f}$ is compactly supported, and, thus is function F, and, consequently the sum over m is finite. Therefore, we can write

$$\begin{aligned}b\sum_{k\in\mathbb{Z}^d}|\hat{F}(Bk)|^2 &= \Big\|\sum_{m\in\mathbb{Z}^d} F(\cdot + B'm)\Big\|^2_{L^2(Q_0)} \\ &= \int_{Q_0}\Big(\sum_{m\in\mathbb{Z}^d} F(\xi + B'm)\Big)\overline{\Big(\sum_{p\in\mathbb{Z}^d} F(\xi + B'p)\Big)}d\xi \\ &= \int_{\mathbb{R}^d}\Big(\sum_{m\in\mathbb{Z}^d} F(\xi + B'm)\Big)\overline{F(\xi)}d\xi.\end{aligned}$$

Therefore, from the definition of F, we obtained the following equality:

$$\begin{aligned}&b\sum_{k\in\mathbb{Z}^d}\left|\int_{\mathbb{R}^d}\hat{f}(\xi + Aj)\overline{\widehat{g^i}(\xi)}e^{-2\pi i Bk\cdot\xi}\right|^2 \\ &= \int_{\mathbb{R}^d}\overline{\hat{f}(\xi + Aj)}\widehat{g^i}(\xi)\Big(\sum_{m\in\mathbb{Z}^d}\hat{f}(\xi + B'm + j)\overline{\widehat{g^i}(\xi + B'm)}\Big)d\xi.\end{aligned}$$

Our task is to analyze the sum:

$$\begin{aligned}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|\langle f, g^i_{j,k}\rangle|^2 &= \sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|\langle \hat{f}, \widehat{g^i_{j,k}}\rangle|^2 \\ &= \sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\left|\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\widehat{g^i}(\xi - Aj)}e^{-2\pi i Bk\cdot\xi}d\xi\right|^2 \\ &= \sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\left|\int_{\mathbb{R}^d}\hat{f}(\xi + Aj)\overline{\widehat{g^i}(\xi)}e^{-2\pi i Bk\cdot\xi}d\xi\right|^2.\end{aligned}$$

Using the previous equality, we obtain:

$$\begin{aligned}
&\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|\langle f,g^i_{j,k}\rangle|^2\\
&=\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}\overline{\hat f(\xi+Aj)}\widehat{g^i}(\xi)\Big(\sum_{k\in\mathbb{Z}^d}\hat f(\xi+B'k+Aj)\overline{\widehat{g^i}(\xi+B'k)}\Big)d\xi\\
&=\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}|\hat f(\xi+Aj)|^2|\widehat{g^i}(\xi)|^2d\xi\\
&+\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}\overline{\hat f(\xi+Aj)}\widehat{g^i}(\xi)\Big(\sum_{0\neq k\in\mathbb{Z}^d}\hat f(\xi+B'k+Aj)\overline{\widehat{g^i}(\xi+B'k)}\Big)d\xi\\
&=\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}|\hat f(\xi)|^2|\widehat{g^i}(\xi-Aj)|^2d\xi\\
&+\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}\overline{\hat f(\xi)}\widehat{g^i}(\xi-Aj)\Big(\sum_{0\neq k\in\mathbb{Z}^d}\hat f(\xi+B'k)\overline{\widehat{g^i}(\xi+B'k-Aj)}\Big)d\xi.
\end{aligned}$$

For future reference let us introduce the following notation:

$$I(f)=\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|\langle f,g^i_{j,k}\rangle|^2,$$

$$I_0(f)=\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}|\hat f(\xi)|^2|\widehat{g^i}(\xi-Aj)|^2d\xi,$$

and

$$I_1(f)=\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}\overline{\hat f(\xi)}\widehat{g^i}(\xi-Aj)\Big(\sum_{0\neq k\in\mathbb{Z}^d}\hat f(\xi+B'k)\overline{\widehat{g^i}(\xi+B'k-Aj)}\Big)d\xi.$$

Thus, our decomposition can be written $I(f)=I_0(f)+I_1(f)$. To perform the next step, we will need the following result.

Fact 3.2

For $g^i\in L^2(\mathbb{R}^d)$, $i=1,\dots,L$, and $f\in D$, we have

$$\sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d}|\hat f(\xi+Aj)||\widehat{g^i}(\xi)|\Big(\sum_{0\neq k\in\mathbb{Z}^d}|\hat f(\xi+B'k+Aj)||\widehat{g^i}(\xi+B'k)|\Big)d\xi<\infty.$$

This is an easy consequence of the inequality:

$$|\widehat{g^i}(\xi)||\widehat{g^i}(\xi + B'k)| \leq |\widehat{g^i}(\xi)|^2 + |\widehat{g^i}(\xi + B'k)|^2,$$

and the fact, that f has compact support and is bounded.

Fact 3.2 allows us to change the orders of integration and summation in the expression for $\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|\langle f, g^i_{j,k}\rangle|^2$. Thus, using the equations (3.1) and (3.2), we obtain:

$$\begin{aligned}
\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|\langle f, g^i_{j,k}\rangle|^2 &= \sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}|\hat{f}(\xi)|^2|\widehat{g^i}(\xi - Aj)|^2 d\xi \\
&+ \sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\frac{1}{b}\int_{\mathbb{R}^d}\overline{\hat{f}(\xi)}\widehat{g^i}(\xi - Aj)\Big(\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}(\xi + B'k)\overline{\widehat{g^i}(\xi + B'k - Aj)}\Big)d\xi \\
&= \frac{1}{b}\int_{\mathbb{R}^d}|\hat{f}(\xi)|^2\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}|\widehat{g^i}(\xi - Aj)|^2 d\xi \\
&+ \frac{1}{b}\int_{\mathbb{R}^d}\overline{\hat{f}(\xi)}\sum_{i=1}^{L}\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}(\xi + B'k)\Big(\sum_{j\in\mathbb{Z}^d}\widehat{g^i}(\xi - Aj)\overline{\widehat{g^i}(\xi + B'k - Aj)}\Big)d\xi \\
&= \int_{\mathbb{R}^d}|\hat{f}(\xi)|^2 d\xi = \|f\|_2^2,
\end{aligned}$$

which completes the proof of the fact, that any Gabor system satisfying the two equations is a tight frame.

Now we will prove the converse implication. Namely, we assume that

$$\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}|\langle f, g^i_{j,k}\rangle|^2 = \|f\|_2^2,$$

for all $f \in L^2(\mathbb{R}^d)$, and we will derive from this the two equations (3.1) and (3.2). Clearly, our condition holds, in particular, for all functions $f \in D$. This leads to the following observation.

Remark. Recalling Fact 3.2, we see that for $f \in D$, $I(f) < \infty$ iff $I_0(f) < \infty$. Now taking $\hat{f} = \chi_C$, where $C \subset \mathbb{R}^d$ is any compact set, we see that $I_0(f) < \infty$, for all $f \in D$ iff $\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}|\widehat{g^i}(\xi - Aj)|^2$ is locally integrable in $\mathbb{R}^d$.

Thus by our assumption, the function:

$$\tau(\xi) = \frac{1}{b}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d} |\widehat{g^i}(\xi - Aj)|^2,$$

is locally integrable. Hence, almost every point in $\mathbb{R}^d$ is a point of differentiability of the integral of τ. This means, that if ξ_0 is such a point, then

$$\lim_{\delta\to 0^+} \frac{1}{|\mathbb{B}_\delta(\xi_0)|}\int_{\mathbb{B}_\delta(\xi_0)} \tau(\xi)d\xi = \tau(\xi_0).$$

Fix $\delta > 0$, and let f_δ be a function such that:

$$\hat{f}_\delta(\xi) = \frac{1}{\sqrt{|\mathbb{B}_\delta(\xi_0)|}}\chi_{\mathbb{B}_\delta(\xi_0)}(\xi).$$

Recall the decomposition, we have used in the proof of the first implication:

$$I(f_\delta) = I_0(f_\delta) + I_1(f_\delta).$$

Notice that

$$I(f_\delta) = \|f_\delta\|_2^2 = 1.$$

Thus, using the definition of I_0, we obtain:

$$1 = \frac{1}{|\mathbb{B}_\delta(\xi_0)|}\int_{\mathbb{B}_\delta(\xi_0)} \tau(\xi)d\xi + I_1(f_\delta).$$

If we can show that $\lim_{\delta\to 0^+} I_1(f_\delta) = 0$, then since ξ_0 is a point of differentiability of the integral of τ, we have the equality:

$$1 = \lim_{\delta\to 0^+} \frac{1}{|\mathbb{B}_\delta(\xi_0)|}\int_{\mathbb{B}_\delta(\xi_0)} \tau(\xi)d\xi + I_1(f_\delta) = \tau(\xi_0) + \lim_{\delta\to 0^+} I_1(f_\delta) = \tau(\xi_0).$$

Let us now look at $|I_1(f_\delta)|$. Arguing as in the proof of Fact 3.2, we see, that the above is bounded by the two following terms:

$$\frac{1}{b}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\sum_{0\neq k\in\mathbb{Z}^d} |\hat{f}_\delta(\xi)||\hat{f}_\delta(\xi + B'k)||\widehat{g^i}(\xi - Aj)|^2 d\xi$$

and

$$\frac{1}{b}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\sum_{0\neq k\in\mathbb{Z}^d}|\hat{f}_\delta(\xi)||\hat{f}_\delta(\xi+B'k)||\widehat{g^i}(\xi+B'k-Aj)|^2 d\xi.$$

But if δ is small enough, then:

$$|\hat{f}_\delta(\xi)||\hat{f}_\delta(\xi+B'k)|=0,$$

for every $k\neq 0$. Thus $I_1(f_\delta)=0$, for $\delta << 1$. This concludes the proof of the fact that $\tau(\xi)=1$, *a.e.* $\xi\in\mathbb{R}^d$. This also shows, that $I_1(f)=0$, for all $f\in D$, i.e.

$$\frac{1}{b}\int_{\mathbb{R}^d}\overline{\hat{f}(\xi)}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{0\neq k\in\mathbb{Z}^d}\hat{f}(\xi+B'k)\widehat{g^i}(\xi-Aj)\overline{\widehat{g^i}(\xi+B'k-Aj)}d\xi=0.$$

Moreover, using the polarization identity, we obtain that for any $f,h\in D$:

$$\frac{1}{b}\int_{\mathbb{R}^d}\overline{\hat{f}(\xi)}\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\sum_{0\neq k\in\mathbb{Z}^d}\hat{h}(\xi+B'k)\widehat{g^i}(\xi-Aj)\overline{\widehat{g^i}(\xi+B'k-Aj)}d\xi=0.$$

Remark. Using the inequality:

$$|\widehat{g^i}(\xi-Aj)||\widehat{g^i}(\xi+B'k-Aj)|\leq|\widehat{g^i}(\xi-Aj)|^2+|\widehat{g^i}(\xi+B'k-Aj)|^2,$$

we can argue as before, that the function:

$$\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\widehat{g^i}(\xi-Aj)\overline{\widehat{g^i}(\xi+B'k-Aj)}$$

is locally integrable, and thus almost every point in $\mathbb{R}^d$ is a point of differentiability of its integral.

Fix $k_0\neq 0$. Let ξ_0 be a point of differentiability of

$$\sum_{i=1}^{L}\sum_{j\in\mathbb{Z}^d}\widehat{g^i}(\xi-Aj)\overline{\widehat{g^i}(\xi+B'k_0-Aj)},$$

such that: $\xi_0\neq 0\neq\xi_0+k_0$. Let

$$\hat{f}_\delta(\xi)=\frac{1}{\sqrt{|\mathbb{B}_\delta|}}\chi_{\mathbb{B}_\delta(\xi_0)}(\xi),$$

and

$$\hat{h}_\delta(\xi) = \frac{1}{\sqrt{|\mathbb{B}_\delta|}} \chi_{\mathbb{B}_\delta(\xi_0 + B'k_0)}(\xi).$$

Then, in particular, we have

$$\hat{f}_\delta(\xi)\hat{h}_\delta(\xi + B'k_0) = \frac{1}{|\mathbb{B}_\delta|} \chi_{\mathbb{B}_\delta(\xi_0)}(\xi).$$

Let us now write the equality $I_1 = 0$, using the functions f_δ and h_δ:

$$\begin{aligned}
0 &= \frac{1}{b} \int_{\mathbb{R}^d} \overline{\hat{f}_\delta(\xi)} \sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} \sum_{0\neq k\in\mathbb{Z}^d} \hat{h}_\delta(\xi + B'k)\widehat{g^i}(\xi - Aj)\overline{\widehat{g^i}(\xi + B'k - Aj)} d\xi \\
&= \frac{1}{b} \int_{\mathbb{R}^d} \overline{\hat{f}_\delta(\xi)} \hat{h}_\delta(\xi + B'k_0) \sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} \widehat{g^i}(\xi - Aj)\overline{\widehat{g^i}(\xi + B'k_0 - Aj)} \\
&\quad + \frac{1}{b} \int_{\mathbb{R}^d} \overline{\hat{f}_\delta(\xi)} \sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} \sum_{0,k_0\neq k\in\mathbb{Z}^d} \hat{h}_\delta(\xi + B'k)\widehat{g^i}(\xi - Aj)\overline{\widehat{g^i}(\xi + B'k - Aj)} d\xi \\
&= \frac{1}{b} \frac{1}{|\mathbb{B}_\delta|} \int_{\mathbb{B}_\delta(\xi_0)} \sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} \widehat{g^i}(\xi - Aj)\overline{\widehat{g^i}(\xi + B'k_0 - Aj)} d\xi + J_\delta.
\end{aligned}$$

By examining the term J_δ we see, as before, that for δ small enough, it is equal to 0. Therefore, employing the fact that ξ_0 was chosen to be a point of differentiability of $\sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} \widehat{g^i}(\xi - Aj)\overline{\widehat{g^i}(\xi + B'k_0 - Aj)}$, we obtain the second equality:

$$\begin{aligned}
0 &= \lim_{\delta\to 0^+} \frac{1}{|\mathbb{B}_\delta|} \int_{\mathbb{B}_\delta(\xi_0)} \sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} \widehat{g^i}(\xi_0 - Aj)\overline{\widehat{g^i}(\xi_0 + B'k_0 - Aj)} d\xi \\
&= \sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} \widehat{g^i}(\xi_0 - Aj)\overline{\widehat{g^i}(\xi_0 + B'k_0 - Aj)},
\end{aligned}$$

for all $k_0 \neq 0$ and *a.e.* $\xi \in \mathbb{R}^d$. This completes the proof of Theorem 3.1. □

Remark. Notice that our characterization of tight frames of Gabor systems, gives a different approach to the general result obtained in [14], (Corollary 3.3.6, see also [2]).

We also would like to alert the reader to the fact that the roles of the matrices A and B are interchanged in Theorems 2.1 and 3.1. A short reflexion about this phenomenon is useful for avoiding some confusion about these matters.

4. Orthogonal bases of Gabor systems

In this section we will use previous results, to give characterizations of orthonormal bases of Gabor systems of $L^2(\mathbb{R}^n)$. In the first result we will make use of the equations we employed for the description of tight frames, in Section 3.

Theorem 4.1

The Gabor system: $\{g^i_{m,n} : m,n \in \mathbb{Z}^d, i = 1,\dots,L\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, if and only if it satisfies equations (3.1) and (3.2), and $\|g^i\|_2 \geq 1$, for $i = 1,\dots,L$.

Proof. If $\{g^i_{m,n} : m,n \in \mathbb{Z}^d, i = 1,\dots,L\}$ is an orthonormal basis, then it clearly is a tight frame. Then, by Theorem 3.1, equations (3.1) and (3.2) are satisfied. Orthonormality of the system implies also, that $\|g^i\|_2 \geq 1$, for $i = 1,\dots,L$.

Conversely, in view of Theorem 3.1, equations (3.1) and (3.2) imply that the system is a tight frame. Since $\|g^i\|_2 = \|g^i_{m,n}\|_2$, we see that by Theorem 1.5, the whole system is an orthonormal basis. □

The above result characterizes orthonormal bases using the description of tight frames. In the next theorem, we would like to consider a different set of conditions, based on the characterizing equations for orthonormality of a Gabor system.

Theorem 4.2

The Gabor system: $\{g^i_{m,n} : m,n \in \mathbb{Z}^d, i = 1,\dots,L\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$, if and only if it satisfies equations (2.1'), (2.2), (2.3) and (3.1).

Proof. An orthonormal basis is both, an orthonormal system (which implies equations (2.1'), (2.2), (2.3)) and a tight frame (which provides us with equation (3.1)).

The proof of the other implication follows closely a method from [3], adapted to the case of Gabor systems. Namely, notice that the calculations of the dual Gramian matrix $\tilde{G}(\xi)$ associated with the Gabor system $\{g^i_{m,n} : m,n \in \mathbb{Z}^d, i = 1,\dots,L\}$, defined by (2.15) in [2], give us that:

$$\langle \tilde{G}(\xi)e_k, e_k\rangle = \sum_{i=1}^{L} \sum_{j\in\mathbb{Z}^d} |\widehat{g^i}(\xi - Aj + B'k)|^2,$$

for $k \in \mathbb{Z}^d$ (for more about the Gramian matrices see, for instance, [2] or [14]). Thus, equation (3.1) implies that $\langle \tilde{G}(\xi)e_k, e_k\rangle = 1$. Since our system is an orthonormal basis, it follows from Theorem 2.5(i) in [2], that $\|\tilde{G}(\xi)\| \leq 1$. Therefore:

$$1 \geq \|\tilde{G}(\xi)e_k\|^2 = \sum_{l\in\mathbb{Z}^d} \left|\langle \tilde{G}(\xi)e_k, e_l\rangle\right|^2 = 1 + \sum_{l\in\mathbb{Z}^d, l\neq k} \left|\langle \tilde{G}(\xi)e_k, e_l\rangle\right|^2.$$

Therefore, $\langle \tilde{G}(\xi)e_k, e_l\rangle = 0$, for $l \neq k$, and $\tilde{G}(\xi)$ is an identity on $l^2(\mathbb{Z}^d)$ for a.e. ξ. Hence, by Theorem 2.5(ii) in [2], our Gabor system is a tight frame. Since it is an orthonormal system, $\|g^i_{m,n}\| \geq 1$, and Theorem 1.5 implies that we have in fact an orthonormal basis. □

The next two observations we make, result from the comparison between the results of Section 2 and Section 3. They reflect the very special, symmetric structure of Gabor systems.

Theorem 4.3

If $L = 1$, i.e. there is only one generating function g, the following are equivalent:

The system $\{g_{m,n}(x) = e^{2\pi i Am\cdot x} g^k(x - Bn) : m, n \in \mathbb{Z}^d\}$ is a tight frame with constant 1.

The system $\{g_{m,n}(x) = e^{2\pi i B'm\cdot x} g^k(x - A'n) : m, n \in \mathbb{Z}^d\}$ is an orthogonal system and $\|g\|_2^2 = ab$.

Equations (3.1) and (3.2) hold.

Theorem 4.4

If $L = 1$, and $A = B' = B^{t^{-1}}$, then equations (2.1'), (2.2) and (2.3) are the same as equations (3.1) and (3.2).

Especially the last theorem may be viewed as another approach to the results obtained at the end of Section 2 - Theorem 2.6 and Corollary 2.7.

5. Miscellaneous remarks

Now we may try to attempt proving some results about the Gabor systems using the characterizing equations. In [16], analogous characterizations for wavelets were used to study the wavelet multipliers. The following result is an immediate corollary of our characterizations, in case of Gabor multipliers.

Corollary 5.1

Assume that ν is a unimodular function on $\mathbb{R}$, such that $\nu(\cdot + 1)/\nu(\cdot)$ is a 1-periodic function. Then ν is a Gabor multiplier.

This provides us with an interesting example of a class of Gabor multipliers. Using a general method of [16], it is easy to show that all Gabor multipliers must be unimodular functions. Unfortunately to characterize them fully we would need a version of a multiresolution analysis of Gabor systems. That however, is impossible, as the following argument shows.

Assume that for every j, function $g(\cdot - j)$ is an element of the set $\{e^{2\pi i \cdot} g(\cdot - k), k \in \mathbb{Z}\}$. On the Fourier transform side it means, that for every j, there exists a 1-periodic function m_j, such that:

$$\hat{g}(\xi) = \hat{g}(\xi - j) m_j(\xi).$$

If we assume that the system $\{g(\cdot - j), j \in \mathbb{Z}\}$ is a Riesz basis for the closure of its span, then (as in [2] or [14]), we see that it satisfies:

$$0 < A \leq \sum_k |\hat{g}(\xi - k)|^2 \leq B < \infty,$$

for $a.e.$ $\xi \in \mathbb{R}$. In view of the previous equality, it gives us:

$$A \leq \sum_k |\hat{g}(\xi - k - j) m_j(\xi)|^2 = \Big(\sum_k |\hat{g}(\xi - k)|^2\Big) |m_j(\xi)|^2 \leq B,$$

or

$$A/B \leq |m_j(\xi)|^2 \leq B/A.$$

But now, if $|\hat{g}|$ is greater than some constant $C > 0$, on a set of positive measure, it will be greater than the constant AC/B, on all integer translates of this set. This contradicts the fact that $g \in L^2(\mathbb{R})$. For a different approach to the problem of nonexistence of Gabor multiresolution analyses, we refer the reader to [10].

Also, now it may be possible to combine the results of Sections 2 and 3, with those of [9], [11] and [15], to obtain similar characterizations of wave packets. As an introduction to this, let us have the following observation, which is a simple consequence of our results.

Fact 5.2

There are no orthonormal bases of wave packets of the form $\{g_{l,m,n}(x) = e^{2\pi i A m \cdot x} g^k(2^l x - A^{t^{-1}} n) : m, n \in \mathbb{Z}^d, l \in \mathbb{Z}\}$.

Further study of wave packets, especially of their completeness, will be the subject of our future work.

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