Collectanea Mathematica (electronic version): http://www.mat.ub.es/CM

Collect. Math. **51**, 2 (2000), 131–148

Duality and reflexivity in grand Lebesgue spaces

Alberto Fiorenza*

Dipartimento di Costruzioni e Metodi Matematici in Architettura,
via Monteoliveto, 3 I-80134 Napoli, Italy

E-mail: fiorenza@unina.it

Received February 19, 1999. Revised April 7, 1999

Abstract

The grand L^p space $L^{p)}(\Omega)$ $(1<p<+\infty)$ introduced by Iwaniec-Sbordone is defined as the *Banach Function Space* of the measurable functions f on Ω such that

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \frac{1}{|\,\Omega\,|} \int_\Omega |\, f \,|^{p-\varepsilon}\, dx \right)^{1/(p-\varepsilon)} < +\infty\,.$$

We introduce the *small* $L^{p'}$ *space* denoted by $L^{p)'}(\Omega)$ and we prove that the associate space of $L^{p)}(\Omega)$ is $L^{p)'}(\Omega)$. It turns out that $L^{p)'}(\Omega)$ is a *Banach Function Space* whose norm satisfy the Fatou property, and that it is the dual of the closure of $L^\infty(\Omega)$ in $L^{p)}(\Omega)$. Moreover, we give a characterization of $L^{p)}(\Omega)$ as dual space, and we prove that for any $1<p<+\infty$ the spaces $L^{p)}(\Omega)$ and $L^{p)'}(\Omega)$ are not reflexive.

* This work has been partially performed as a part of a National Research Project supported by M.U.R.S.T

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a set of Lebesgue measure $|\,\Omega\,| < +\infty$ and let $1 < p < +\infty$. The grand L^p space, that will be denoted by $L^{p)}(\Omega)$, introduced by Iwaniec-Sbordone in [8] is defined as the space of the measurable functions f on Ω such that

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \frac{1}{|\,\Omega\,|} \int_\Omega |\,f\,|^{p-\varepsilon}\,dx \right)^{1/(p-\varepsilon)} < +\infty\,.$$

Grand L^p spaces have been considered in various fields: in the theory of Partial Differential Equations (see e.g. [9], [10], [13], [14]), in the study of maximal operators and, more generally, quasilinear operators, and in interpolation theory (see e.g. [4], [2]). In particular, in the theory of Partial Differential Equations, it turns out that they are the right spaces in which some nonlinear equations have to be considered (see [7], [5]). Also, they have been studied in their own, various properties have been developed e.g. in [6], [3].

The aim of this paper is to find an explicit expression of the "best" functional $\mathcal{N}_{p'}$, usually called the *associate norm* of $\|\cdot\|_{p)}$, such that the following Hölder inequality holds

$$\fint_\Omega fg dx \leq \|f\|_{p)} \mathcal{N}_{p'}(g)$$

where the symbol $\fint_\Omega$ stands for $\frac{1}{|\Omega|}\int_\Omega$. Namely, the problem we solve is to find an expression of the associate norm free from the definition of the norm in grand L^p spaces. It turns out that the solution of this problem gives also a characterization of the dual of the closure of $L^\infty(\Omega)$ in $L^{p)}(\Omega)$.

Many concepts from the theory of Banach Function Spaces are used: we refer to the books by Zaanen ([15]) and Bennett-Sharpley ([1]) for the main results of this theory.

After introducing in Section 2 the auxiliary Banach space $L^{(p'}(\Omega)$, in Section 3 we study the *small* $L^{p'}$ *space* denoted by $L^{p)'}(\Omega)$ and we prove that the associate space of $L^{p)}(\Omega)$ is $L^{p)'}(\Omega)$. In particular, the following Hölder inequality holds

$$\fint_\Omega fg dx \leq \|f\|_{p)} \|g\|_{p)'} \qquad \forall f \in L^{p)}(\Omega), g \in L^{p)'}(\Omega)\,.$$

Finally, the fundamental function of the grand Lebesgue Space is estimated, and, as a consequence, the spaces $L^{p)}(\Omega)$ and $L^{p)'}(\Omega)$ are characterized as dual spaces. Moreover, we show that $L^{p)}(\Omega)$ and $L^{p)'}(\Omega)$ are not reflexive.

In order to have a simpler notation, unless differently specified, all the spaces considered in the sequel have to be intended as spaces of functions on Ω, therefore for instance we will write $L^{p)}$ instead of $L^{p)}(\Omega)$, L^∞ instead of $L^\infty(\Omega)$, etc.

2. The space $L^{(p'}$

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, $|\,\Omega\,| < +\infty$, and let $\mathcal{M}_0$ be the set of all measurable functions, whose values lie in $[-\infty, +\infty]$, finite a.e. in Ω. Also, let $\mathcal{M}_0^+$ be the class of functions in $\mathcal{M}_0$ whose values lie in $[0, +\infty]$.

Let us begin by proving the following

Lemma 2.1

If $f, g \in \mathcal{M}_0^+$ *and* $g \leq f = \sum_{k=1}^\infty f_k$ *with* $f_k \geq 0 \quad \forall k \in \mathbb{N}$ *then the functions defined in* Ω *by*

$$h_k = \left[f_k - \max\Big(g - \sum_{j=1}^{k-1} f_j, 0 \Big) \right] \chi_{\left\{ \sum_{j=1}^k f_j > g \right\}} \qquad \forall k \in \mathbb{N}$$

are such that

$$0 \leq h_k \leq f_k \qquad \forall k \in \mathbb{N} \tag{2.1}$$

and

$$g = \sum_{k=1}^\infty (f_k - h_k). \tag{2.2}$$

Proof. For a.e. $x \in \Omega$ such that $g(x) = f(x)$ we have $h_k(x) = 0 \quad \forall k \in \mathbb{N}$ and therefore (2.1) and (2.2) are obvious.

For a.e. $x \in \Omega$ such that $g(x) < f(x)$ let

$$\widehat{k} = \widehat{k}_x = \min \left\{ k : \sum_{j=1}^k f_j(x) > g(x) \right\}.$$

If $k < \widehat{k}$ we have $\sum_{j=1}^{k} f_j(x) \le g(x)$ and therefore $h_k(x) = 0$ from which (2.1) follows.

If $k = \widehat{k}$ we have $\sum_{j=1}^{\widehat{k}-1} f_j(x) \le g(x)$ and $\sum_{j=1}^{\widehat{k}} f_j(x) > g(x)$, therefore

$$h_{\widehat{k}}(x) = f_{\widehat{k}}(x) - \left(g(x) - \sum_{j=1}^{\widehat{k}-1} f_j(x) \right)$$

satisfies (2.1).

If $k > \widehat{k}$ then $h_k(x) = f_k(x)$ and therefore (2.1) is immediate.

Inequalities (2.1) are proved for any $k \in \mathbb{N}$.

On the other hand equality (2.2) holds because for a.e. $x \in \Omega$ such that $g(x) < f(x)$ we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \big(f_k(x) - h_k(x)\big) &= \sum_{k<\widehat{k}} \big(f_k(x) - h_k(x)\big) \\
&\quad + f_{\widehat{k}}(x) - h_{\widehat{k}}(x) + \sum_{k>\widehat{k}} \big(f_k(x) - h_k(x)\big) \\
&= \sum_{k<\widehat{k}} f_k(x) + f_{\widehat{k}}(x) - \left(f_{\widehat{k}}(x) - g(x) + \sum_{k<\widehat{k}} f_k(x) \right) \\
&\quad + \sum_{k>\widehat{k}} \big(f_k(x) - f_k(x)\big) = g(x)\,. \ \square
\end{aligned}$$

For each $g \in \mathcal{M}_0^+$ let us pose

$$\|g\|_{(p'} = \inf_{g=\sum_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega \mid g_k \mid^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} \right\}$$

where $1 < p < +\infty$, $p' = p/(p-1)$ and $g_k \in \mathcal{M}_0 \quad \forall k \in \mathbb{N}$.

Corollary 2.2

For each $g \in \mathcal{M}_0^+$ we have

$$\|g\|_{(p'} = \inf_{\substack{g=\sum_{k=1}^{\infty} g_k \\ g_k \ge 0}} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega g_k^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} \right\}.$$

Proof. For any sequence (g_k) in $\mathcal{M}_0$ put

$$\mathcal{S}\big((g_k)\big) = \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_{\Omega} |\, g_k \,|^{(p-\varepsilon)'} \, dx \right)^{1/((p-\varepsilon)')} . \tag{2.3}$$

We have to show that for any $g \in \mathcal{M}_0^+$ and for any decomposition $g = \sum_{k=1}^{\infty} g_k$ with $g_k \in \mathcal{M}_0$ there exists a decomposition $g = \sum_{k=1}^{\infty} \gamma_k$ with $\gamma_k \in \mathcal{M}_0^+ \quad \forall k \in \mathbb{N}$ such that $\mathcal{S}\big((g_k)\big) \geq \mathcal{S}\big((\gamma_k)\big)$.

Let us apply Lemma 2.1 replacing f_k by $g_k^+ = \max\{g_k, 0\} \quad \forall k \in \mathbb{N}$, and let us set $\gamma_k = g_k^+ - h_k \quad \forall k \in \mathbb{N}$. Since $|\, g_k^+ + g_k^- \,| = |\, g_k^+ + (-g_k^-) \,|$ where $g_k^- = \min\{g_k, 0\}$, we have

$$\begin{aligned} \mathcal{S}\big((g_k)\big) &= \mathcal{S}\big((g_k^+ + g_k^-)\big) = \mathcal{S}\big((g_k^+ + (-g_k^-))\big) \\ &\geq \mathcal{S}\big((g_k^+)\big) \geq \mathcal{S}\big((g_k^+ - h_k)\big) = \mathcal{S}\big((\gamma_k)\big) \end{aligned}$$

from which the assertion of Corollary 2.2 follows. □

Theorem 2.3

The space defined by

$$L^{(p'} = \left\{ g \in \mathcal{M}_0 : \|\, |\, g \,|\, \|_{(p'} < +\infty \right\}$$

is a Banach Function Space.

Proof. Most of the properties of $\|\, |\cdot|\, \|_{(p'}$ needed to show that $L^{(p'}$ is a Banach Function Space are trivial or easy to prove. Here we show only that the following properties hold for all f, g, $g^{(n)}$ $(n \in \mathbb{N})$, in $\mathcal{M}_0^+$:

(*i*) $\left\|\sum_{n=1}^{\infty} g^{(n)}\right\|_{(p'} \leq \sum_{n=1}^{\infty} \left\|g^{(n)}\right\|_{(p'}$.

(*ii*) If $g \leq f$ a.e. in Ω, then $\|g\|_{(p'} \leq \|f\|_{(p'}$.

Proof of (*i*). Let us assume that the functions $g_k^{(n)} \in \mathcal{M}_0^+$ are such that

$$\sum_{n=1}^{\infty} \left\| g^{(n)} \right\|_{(p'} < +\infty \quad \forall n \in \mathbb{N}\,,$$

otherwise the assertion is trivial. Let $\varepsilon > 0$ and let $g_k^{(n)} \in \mathcal{M}_0^+$ (the existence follows from Corollary 2.2) be such that

$$g^{(n)} = \sum_{k=1}^{\infty} g_k^{(n)} \qquad \forall n \in \mathbb{N}$$

and

$$\sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega (g_k^{(n)})^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} < \|g^{(n)}\|_{(p'} + \frac{\varepsilon}{2^n}\,.$$

We have

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} g^{(n)} \right\|_{(p'} &= \left\| \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} g_k^{(n)} \right\|_{(p'} = \left\| \sum_{n,k=1}^{\infty} g_k^{(n)} \right\|_{(p'} \\ &\le \sum_{n,k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega (g_k^{(n)})^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega (g_k^{(n)})^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} \\ &\le \sum_{n=1}^{\infty} \left\| g^{(n)} \right\|_{(p'} + \varepsilon \qquad \forall \varepsilon > 0\,. \end{aligned}$$

Proof of (ii). For any decomposition $f = \sum_{k=1}^{\infty} f_k$ with $f_k \in \mathcal{M}_0^+ \quad \forall k \in \mathbb{N}$ let (h_k) in $\mathcal{M}_0^+ \quad \forall k \in \mathbb{N}$ be given by Lemma 2.1, such that $g = \sum_{k=1}^{\infty}(f_k - h_k)$. By using Corollary 2.2 we have

$$\|f\|_{(p'} = \inf_{\substack{f=\sum_{k=1}^{\infty} f_k \\ f_k\ge 0}} \mathcal{S}\left((f_k)\right) \ge \inf_{\substack{f=\sum_{k=1}^{\infty} f_k \\ f_k\ge 0}} \mathcal{S}\left((f_k - h_k)\right) \ge \inf_{\substack{g=\sum_{k=1}^{\infty} g_k \\ g_k\ge 0}} \mathcal{S}\left((g_k)\right) = \|g\|_{(p'}$$

and the proof of (ii) is now complete. □

After Theorem 2.3 $L^{(p'}$ is a Banach space under the norm

$$\|g\|_{(p'} = \inf_{|g|=\sum_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega \mid g_k \mid^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} \right\}$$

where $g_k \in \mathcal{M}_0 \quad \forall k \in \mathbb{N}$. We remark that in the right hand side it is possible to replace $\mid g \mid$ simply by g, in fact we have

Proposition 2.4

For any $g \in L^{(p'}$

$$\|g\|_{(p'} = \inf_{g=\sum_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega \mid g_k \mid^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} \right\}$$

where $g_k \in \mathcal{M}_0 \quad \forall k \in \mathbb{N}$.

Proof. Let $| g |= \sum_{k=1}^{\infty} h_k$ be any decomposition of $| g |$ in $\mathcal{M}_0^+$. We have $g(x) = \operatorname{sgn}(g(x)) \mid g(x) \mid$ a.e. in Ω (where $\operatorname{sgn} = \chi_{]0,+\infty[} - \chi_{]-\infty,0[}$) and therefore

$$g(x) = \operatorname{sgn}\big(g(x)\big) \sum_{k=1}^{\infty} h_k(x) = \sum_{k=1}^{\infty} \operatorname{sgn}\big(g(x)\big) h_k(x) \qquad \text{a.e. in } \Omega$$

from which

$$\inf_{\substack{g=\sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0}} \mathcal{S}\left((g_k)\right) \leq \mathcal{S}\left((\operatorname{sgn}(g) h_k)\right) = \mathcal{S}\left((h_k)\right)$$

where $\mathcal{S}\left((g_k)\right)$ is given by (2.3). Therefore

$$\inf_{\substack{g=\sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0}} \mathcal{S}\left((g_k)\right) \leq \inf_{\substack{|g|=\sum_{k=1}^{\infty} h_k \\ h_k \in \mathcal{M}_0^+}} \mathcal{S}\left((h_k)\right) = \inf_{\substack{|g|=\sum_{k=1}^{\infty} h_k \\ h_k \in \mathcal{M}_0}} \mathcal{S}\left((h_k)\right).$$

On the other hand, let $g = \sum_{k=1}^{\infty} g_k$ be any decomposition of g in $\mathcal{M}_0$, and let

$$\gamma_{(g_k)} = \sum_{k=1}^{\infty} \mid g_k \mid .$$

We have $| g |\leq \gamma_{(g_k)}$ and therefore

$$\inf_{\substack{|g|=\sum_{k=1}^{\infty} h_k \\ h_k \in \mathcal{M}_0}} \mathcal{S}\left((h_k)\right) \leq \inf_{\substack{\gamma_{(g_k)}=\sum_{k=1}^{\infty} h_k \\ h_k \in \mathcal{M}_0}} \mathcal{S}\left((h_k)\right) \leq \mathcal{S}\left((g_k)\right)$$

from which

$$\inf_{\substack{|g|=\sum_{k=1}^{\infty} h_k \\ h_k \in \mathcal{M}_0}} \mathcal{S}\left((h_k)\right) \leq \inf_{\substack{g=\sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0}} \mathcal{S}\left((g_k)\right) .$$

Proposition 2.4 is therefore proved. □

We remark that the following inclusions are easy to prove

$$L^{p'+\varepsilon} \subset L^{(p'} \subset L^{p'} \qquad \forall \epsilon > 0 .$$

In particular, we have that $L^{\infty} \subset L^{(p'}$.

Next theorem is the main result of this section.

Theorem 2.5

The following Hölder-type inequality holds

$$\fint_\Omega fg dx \le \|f\|_{p)} \|g\|_{(p'} \qquad \forall f \in L^{p)}, g \in L^{(p'}.$$

Proof. Let $| g |= \sum_{k=1}^{\infty} g_k$ be any decomposition with $g_k \ge 0 \quad \forall k \in \mathbb{N}$ and let $f \in L^{p)}$. For each $k \in \mathbb{N}$ and for each $0 < \varepsilon < p-1$ we have

$$\begin{aligned}\fint_\Omega f g_k dx &\le \left(\fint_\Omega | f |^{p-\varepsilon} dx\right)^{1/(p-\varepsilon)} \left(\fint_\Omega | g_k |^{(p-\varepsilon)'} dx\right)^{1/((p-\varepsilon)')} \\ &= \left(\varepsilon \fint_\Omega | f |^{p-\varepsilon} dx\right)^{1/(p-\varepsilon)} \cdot \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega | g_k |^{(p-\varepsilon)'} dx\right)^{1/((p-\varepsilon)')} \\ &\le \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega | g_k |^{(p-\varepsilon)'} dx\right)^{1/((p-\varepsilon)')} \|f\|_{p)}\end{aligned}$$

and therefore

$$\fint_\Omega f g_k dx \le \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega | g_k |^{(p-\varepsilon)'} dx\right)^{1/((p-\varepsilon)')} \|f\|_{p)}$$

from which

$$\begin{aligned}\fint_\Omega fg dx &\le \fint_\Omega | f | \left|\sum_{k=1}^{\infty} g_k\right| dx \le \sum_{k=1}^{\infty} \fint_\Omega | f | g_k dx \\ &\le \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega | g_k |^{(p-\varepsilon)'} dx\right)^{1/((p-\varepsilon)')} \|f\|_{(p'} .\end{aligned}$$

Hence

$$\fint_\Omega fg dx \le \|g\|_{(p'} \|f\|_{p)} .$$

Theorem 2.5 is therefore proved. □

We will need in the following some other properties of the space $L^{(p'}$. Let us begin with the following

Lemma 2.6

Let $F_n \subset \Omega, \quad n \in \mathbb{N}$, *be such that* $\chi_{_{F_n}} \downarrow 0$ *a.e. in* Ω *and let* g *be any function in* $L^{(p'}$. *Then*

$$\|g\chi_{_{F_n}}\|_{(p'} \to 0\,.$$

Proof. Without loss of generality we can assume that g is nonnegative. Let $g = \sum_{k=1}^{\infty} g_k$ be a decomposition with $g_k \geq 0 \quad \forall k \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_{\Omega} g_k^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} < +\infty\,.$$

Setting

$$a_{k,n} = \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_{\Omega} g_k^{(p-\varepsilon)'} \chi_{_{F_n}} dx \right)^{1/((p-\varepsilon)')} \qquad \forall k \in \mathbb{N}, \forall n \in \mathbb{N}$$

we have

$$\|g\chi_{_{F_n}}\|_{(p'} \leq \sum_{k=1}^{\infty} a_{k,n} < +\infty \qquad \forall n \in \mathbb{N}\,.$$

Since $\sum_{k=1}^{\infty} a_{k,1} < +\infty$ and $a_{k,n} \downarrow 0 \quad \forall k \in \mathbb{N}$, the lemma is proved. □

Corollary 2.7

Let $E_n \subset \Omega, \quad n \in \mathbb{N}$, *be such that* $\chi_{_{E_n}} \uparrow \chi_{_\Omega}$ a.e. *in* Ω *and let* g *be any function in* $L^{(p'}$. *Then*

$$g\chi_{_{E_n}} \to g \quad \text{in} \quad L^{(p'}\,.$$

Proof. It suffices to apply Lemma 2.6 with $F_n = \Omega - E_n$. □

Corollary 2.8

Let $g \geq 0$ *be any function in* $L^{(p'}$ *and let* (g_n) *be an increasing sequence of nonnegative functions converging to* g *a.e. in* Ω. *Then* $\|g_n\|_{(p'} \uparrow \|g\|_{(p'}$.

Proof. Because of the order-preserving property of $\|\cdot\|_{(p'}$ it is clear that $\|g_n\|_{(p'}$ is an increasing sequence and $\lim_{n\to\infty}\|g_n\|_{(p'} \le \|g\|_{(p'}$. On the other hand, by Corollary 2.7, for any $\varepsilon > 0$ there exists M such that

$$\|g\|_{(p'} - \varepsilon \le \|\min\{M,g\}\|_{(p'} \le \|g\|_{(p'}.$$

Obviously $0 \le \min\{M,g_n\} \uparrow \min\{M,g\}$ a.e. in Ω and, since $\min\{M,g\} \in L^\infty$, we have also $\min\{M,g_n\} \to \min\{M,g\}$ in $L^{p'+1}$ therefore $\min\{M,g_n\} \to \min\{M,g\}$ in $L^{(p'}$. This means that there exists $\nu \in \mathbb{N}$ such that

$$\|\min\{M,g\}\|_{(p'} - \varepsilon \le \|\min\{M,g_\nu\}\|_{(p'} \le \|\min\{M,g\}\|_{(p'}.$$

Therefore for any $\varepsilon > 0$ we found $\nu \in \mathbb{N}$ such that

$$\|g\|_{(p'} - 2\varepsilon \le \|\min\{M,g_\nu\}\|_{(p'} \le \|g_\nu\|_{(p'} \le \|g\|_{(p'}. \ \square$$

Lemma 2.9

Let f be any function in L^∞. Then there exists $g \in L^\infty$ such that

$$\fint_\Omega fg\,dx = \|f\|_{p)}\|g\|_{p)'}.$$

Proof. If $f \in L^\infty$ then

$$\lim_{\varepsilon\to 0}\left(\varepsilon\fint_\Omega |f|^{p-\varepsilon}dx\right)^{1/(p-\varepsilon)} = 0$$

therefore

$$\sup_{0<\varepsilon<p-1}\left(\varepsilon\fint_\Omega |f|^{p-\varepsilon}dx\right)^{1/(p-\varepsilon)} = \left(\sigma\fint_\Omega |f|^{p-\sigma}dx\right)^{1/(p-\sigma)} = \|f\|_{p)}$$

where $\sigma = \sigma(f) \in]0,p-1[$.

Let $g \in L^\infty$ be such that

$$\fint_\Omega fg\,dx = \left(\fint_\Omega |f|^{p-\sigma}dx\right)^{1/(p-\sigma)}\left(\fint_\Omega |g|^{(p-\sigma)'}dx\right)^{1/((p-\sigma)')}$$

we have

$$\begin{aligned}
\fint_\Omega fg dx &\le \|f\|_{p)} \|g\|_{(p'} \le \|f\|_{p)} \inf_{0<\varepsilon<p-1} \varepsilon^{-1/(p-\varepsilon)} \left(\fint_\Omega \mid g \mid^{(p-\varepsilon)'} dx \right)^{1/((p-\varepsilon)')} \\
&\le \|f\|_{p)} \sigma^{-1/(p-\sigma)} \left(\fint_\Omega \mid g \mid^{(p-\sigma)'} dx \right)^{1/((p-\sigma)')} \\
&= \sigma^{1/(p-\sigma)} \left(\fint_\Omega \mid f \mid^{p-\sigma} dx \right)^{1/(p-\sigma)} \sigma^{-1/(p-\sigma)} \left(\fint_\Omega \mid g \mid^{(p-\sigma)'} dx \right)^{1/((p-\sigma)')} \\
&= \fint_\Omega fg dx \, .
\end{aligned}$$

Therefore the inequalities are in fact equalities, the first inequality is equality and the lemma is proved. □

In the following result we will consider the closure of L^∞ in $L^{p)}$, denoted by Σ^p in literature (see [6]). In order to use a notation closer to that one of the Banach Function Space theory, we will denote this space by $L_b^{p)}$. It is known that $L_b^{p)}$ is *strictly* contained in $L^{p)}$, therefore, the space $L_b^{p)}$ is not a Banach Function Space.

Corollary 2.10

Let f be any function in $L_b^{p)}$. Then

$$\|f\|_{p)} = \sup_{\substack{g \neq 0 \\ g \in L^{(p'}}} \frac{\fint_\Omega fg dx}{\|g\|_{(p'}} \, .$$

Proof. The inequality $\ge$ is true by virtue of the Hölder inequality for grand L^p spaces, already proved in Theorem 2.5. On the other hand, previous Lemma 2.9 gives the reversed inequality. □

3. The associate space of the grand Lebesgue space

For each $g \in \mathcal{M}_0$ let us set

$$\|g\|_{p)'} = \sup_{\substack{0 \le \psi \le |g| \\ \psi \in L^{(p'}}} \|\psi\|_{(p'} \, .$$

We remark that

$$\|g\|_{p)'} \le \|g\|_{(p'} \qquad \forall g \in \mathcal{M}_0$$

$$\|g\|_{p)'} = \|g\|_{(p'} < +\infty \qquad \forall g \in L^{(p'} \, .$$

Let us begin by proving the following

Proposition 3.1

The small $L^{p'}$ space defined by

$$L^{p)'} = \left\{ g \in \mathcal{M}_0 : \|g\|_{p)'} < +\infty \right\}$$

is a Banach Function Space.

Proof. All the properties of $\|g\|_{p)'}$ which have to be satisfied in order that $L^{p)'}$ be a Banach space are easy to prove, by using simply Theorem 2.3. It remains to prove only that the space $L^{p)'}$ satisfies the Fatou property

$$0 \le g_n \uparrow g \quad \text{a. e. in } \Omega \qquad \Rightarrow \qquad \|g_n\|_{p)'} \uparrow \|g\|_{p)'} \, .$$

Because of the order-preserving property of $\| \cdot \|_{p)'}$ it is clear that $\|g_n\|_{p)'}$ is an increasing sequence and

$$\lim_{n \to \infty} \|g_n\|_{p)'} \le \|g\|_{p)'} \, .$$

Now let us consider the following two possibilities: $g \in L^{p)'}, g \notin L^{p)'}$.

First case : $g \in L^{p)'}$.

Fix $\varepsilon > 0$, let $\psi \in L^{(p'}$ be such that $0 \le \psi \le |g|$ and

$$\|g\|_{p)'} - \varepsilon \le \|\psi\|_{(p'} \le \|g\|_{p)'} \, .$$

Obviously $0 \le \min\{\psi, g_n\} \uparrow \psi$ a.e. in Ω and therefore by Corollary 2.8 $\|\min\{\psi, g_n\}\|_{(p'} \uparrow \|\psi\|_{(p'}$. Let $\nu \in \mathbb{N}$ be such that

$$\|\psi\|_{(p'} - \varepsilon \le \|\min\{\psi, g_\nu\}\|_{(p'} \le \|\psi\|_{(p'}$$

then for any $\varepsilon > 0$ we find $\nu \in \mathbb{N}$ such that

$$\|g\|_{p)'} - 2\varepsilon \le \|\min\{\psi, g_\nu\}\|_{(p'} \le \|g_\nu\|_{p)'} \le \|g\|_{p)'} .$$

Second case : $g \notin L^{p)'}$.

Fix $M > 0$, let $\psi \in L^{(p'}$ be such that $0 \le \psi \le |g|$ and $\|\psi\|_{(p'} > M$. Obviously $0 \le \min\{\psi, g_n\} \uparrow \psi$ a.e. in Ω and therefore by Corollary 2.8 $\|\min\{\psi, g_n\}\|_{(p'} \uparrow \|\psi\|_{(p'}$. Let $\nu \in \mathbb{N}$ be such that $\|\min\{\psi, g_\nu\}\|_{(p'} > M$, then for any $\varepsilon > 0$ we find $\nu \in \mathbb{N}$ such that

$$\|g_\nu\|_{p)'} \ge \|\min\{\psi, g_\nu\}\|_{(p'} > M . \square$$

We prove now the Hölder inequality for Grand Lebesgue spaces.

Theorem 3.2

Let $1 < p < +\infty$ *and* $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$, $|\Omega| < +\infty$.
The following Hölder inequality holds

$$⨍_\Omega fg\,dx \le \|f\|_{p)} \|g\|_{p)'} \qquad \forall f \in L^{p)}, g \in L^{p)'} .$$

Proof. For any $f \in L^{p)}$ and for any $g \in \mathcal{M}_0$ we have, by Theorem 2.5,

$$⨍_\Omega |f||g|\,dx = \sup_{\substack{0 \le \psi \le |g| \\ \psi \in L^\infty}} ⨍_\Omega |f|\,\psi dx \le \sup_{\substack{0 \le \psi \le |g| \\ \psi \in L^{(p'}}} ⨍_\Omega |f|\,\psi dx$$

$$\le \sup_{\substack{0 \le \psi \le |g| \\ \psi \in L^{(p'}}} \|f\|_{p)} \|\psi\|_{(p'} = \|f\|_{p)} \|g\|_{p)'} .$$

Theorem 3.2 is proved. $\square$

The proof of next result is analogous to that one of Corollary 2.10.

Corollary 3.3

Let f be any function in $L_b^{p)}$. Then

$$\|f\|_{p)} = \sup_{\substack{g\neq 0\\ g\in L^{p)'}}} \frac{\displaystyle\fint_\Omega fg\,dx}{\|g\|_{p)'}} = \|f\|_{\left(L^{p)'}\right)'}.$$

Proposition 3.4

The spaces $L^{p)'}$ and $(L^{p)'})'$ are rearrangement-invariant spaces and $(L^{p)'})' = L^{p)}$.

Proof. Since the associate space of a rearrangement - invariant space is a rearrangement - invariant space, it is sufficient to prove that $(L^{p)'})'$ is rearrangement - invariant.

Let $f \in (L^{p)'})'$, and let us set $f_n = \min\{n, f\} \in L^\infty$.

We have

$$0 \le f_n \uparrow f \qquad \text{a.e. in } \Omega \tag{3.1}$$

and therefore $0 \le (f_n)^* \uparrow f^*$ a.e. in $[0, \Omega[$, where $(f_n)^*$ and f^* denote the decreasing rearrangements of f_n and f, respectively. Since the space $(L^{p)'})'$ satisfies the Fatou property, from (3.1) we get

$$\|f_n\|_{\left(L^{p)'}\right)'} \uparrow \|f\|_{\left(L^{p)'}\right)'}. \tag{3.2}$$

On the other hand we have also

$$\|(f_n)^*\|_{L^{p)}(0,|\Omega|)} \uparrow \|f^*\|_{L^{p)}(0,|\Omega|)}. \tag{3.3}$$

By Corollary 3.3

$$\|f_n\|_{\left(L^{p)'}\right)'} = \|f_n\|_{L^{p)}} = \|(f_n)^*\|_{L^{p)}(0,|\Omega|)}. \tag{3.4}$$

¿From (3.2), (3.3) and (3.4) we obtain $\|f\|_{\left(L^{p)'}\right)'} = \|f^*\|_{L^{p)}(0,|\Omega|)}$ and therefore the assertion is proved. □

Consequence of Proposition 3.4 and of the classical Lorentz-Luxemburg theorem (see [1], Theorem 2.7 p. 10) is the following

Theorem 3.5

The space $L^{p)}$ is associate to $L^{p)'}$ and vice versa.

Now we deal with the reflexivity problem for $L^{p)}$ and $L^{p)'}$. The space $L^{p)}$ is not reflexive, as established in the following simple

Proposition 3.6

The space $L^{p)}$ is not reflexive.

Proof. It suffices to construct a function which has not absolute continuous norm. Without loss of generality we can consider the space $L^{p)}(0,1)$.

It is easy to verify that the function $f(x) = x^{-1/p}$ has not absolute continuous norm, therefore the assertion follows.

In order to show that the same result is true for the space $L^{p)'}$, we need next result, due to J. Lang and L. Pick ([12]).

Theorem 3.7

Let φ be the fundamental function of the grand L^p space. Then we have

$$\varphi(t) \approx t^{1/p}\left[\log\left(\frac{1}{t}\right)\right]^{-1/p} \tag{3.5}$$

as $t \to 0$.

Proof. The fundamental function of the grand L^p space reads as

$$\varphi(t) = \sup_{0<\varepsilon<p-1} (\varepsilon t)^{1/(p-\varepsilon)}\,. \tag{3.6}$$

Defining $F(\varepsilon) = (\varepsilon t)^{1/(p-\varepsilon)}$, we get

$$F'(\varepsilon) = (\varepsilon t)^{1/(p-\varepsilon)}(p-\varepsilon)^{-2}\left(\frac{p}{\varepsilon} - 1 + \log\varepsilon + \log t\right).$$

Hence

$$F'(0+) = +\infty \qquad \text{and} \qquad F'(p-1) < 0 \tag{3.7}$$

if t is small enough.

The function

$$G(\varepsilon) = \frac{p}{\varepsilon} - 1 + \log\varepsilon + \log t$$

satisfies, for every fixed t, $G'(\varepsilon) = \varepsilon^{-1}(-p\varepsilon^{-1}+1) < 0$ for $\varepsilon < p$, hence G is strictly decreasing for every fixed t. Thus, for every fixed t, there exists an unique ε_t such that $G(\varepsilon_t) = 0$, hence $F'(\varepsilon_t) = 0$. By (3.7), ε_t is a point of maximum of F, the point where the supremum at (3.6) is attained, that is, $F(\varepsilon_t) = \varphi(t)$.

Now let β be a small number and let $t = \beta^{-1}\exp(1-p\beta^{-1})$. We have $G(\beta) = 0$, therefore, for such t, $\varepsilon_t = \beta$ and

$$\varphi(t) = (\beta t)^{1/(p-\beta)} = \exp\left(-\frac{1}{\beta}\right) .$$

Hence, asymptotically, we see that if β is small and

$$t = \frac{e}{p} \cdot \frac{p}{\beta} \exp\left(-\frac{p}{\beta}\right).$$

then $\varphi(t) = \exp\Big(-\frac{1}{\beta}\Big)$.

Now we claim that

$$\varphi(t) \approx t^{1/p}\left[\log\left(\frac{1}{t}\right)\right]^{-1/p} .$$

Indeed, for β small let t be given by $t = \beta^{-1}\exp(1-p\beta^{-1})$. Then

$$\begin{aligned} t^{1/p}\left[\log\left(\frac{1}{t}\right)\right]^{-1/p} &\approx \left(\frac{1}{\beta}\right)^{1/p}\left[\exp\left(-\frac{p}{\beta}\right)\right]^{1/p}\left[\log\left(\beta\exp(\frac{p}{\beta})\right)\right]^{-1/p} \\ &\approx \exp\Big(-\frac{1}{\beta}\Big) = \varphi(t) \end{aligned}$$

proving (3.5). □

Theorem 3.7 enables us to estimate the limit of the fundamental functions $\varphi = \varphi(t)$ of $L^{p)}$ and $L^{p)'}$. For both fundamental functions we have $\lim_{t\to 0}\varphi(t) = 0$, therefore we can state the following

Corollary 3.8

The dual of $L_b^{p)}$ is isometrically isomorphic to $L^{p)'}$ and the dual of $L_b^{p)'}$ is isometrically isomorphic to $L^{p)}$.

Remark 1. Corollary 3.8 shows that the spaces $L^{p)}$ and $L^{p)'}$ form a *complementary pair*. Complementary pairs of Banach spaces were introduced in papers by T.K. Donaldson, T.K. Donaldson and N.S. Trudinger, J.P. Gossez. For references on the subject we refer to [11].

Next proposition deals with the problem of the reflexivity of $L_b^{p)}$.

Proposition 3.9

The space $L_b^{p)}$ is not reflexive.

Proof. We have $(L^{p)'})' \subset (L^{p)'})^*$, therefore

$$L^{p)} \subset (L^{p)'})^* . \tag{3.8}$$

If the space $L_b^{p)}$ were reflexive we would have $(L_b^{p)})^{**} = L_b^{p)}$. On the other hand, by Corollary 3.8, $(L_b^{p)})^* = L^{p)'}$ and passing to the duals

$$(L^{p)'})^* = L_b^{p)} . \tag{3.9}$$

Equality (3.9) contradicts (3.8), therefore $L_b^{p)}$ is not reflexive. □

After Proposition 3.9 we can conclude that also the dual of $L_b^{p)}$ is not reflexive, therefore we have also

Corollary 3.10

The space $L^{p)'}$ is not reflexive.

Remark 2. After Theorem 3.5 we know that the associate space of $L^{p)}$ is $L^{p)'}$. We claim that the space $L^{p)'}$ is *not* isometrically isomorphic to the dual space of $L^{p)}$, since $L^{p)}$ is not of absolutely continuous norm (because, for instance, the function $f(x) = x^{-1/p}$ is in $L^{p)}(0,1)$ and has not absolute continuous norm).

Acknowledgments. The author wishes to thank Prof. David Cruz-Uribe, SFO for the stimulating discussions on the subject, and Dr. Jan Lang for the helpful remarks on the paper.

References

1. C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.
2. D. Cruz-Uribe, SFO and A. Fiorenza, in preparation.
3. M. Carozza and C. Sbordone, The distance to L^∞ in some function spaces and applications, *Differential Integral Equations*, **10** (1997), 599–607.
4. A. Fiorenza and M. Krbec, On the domain and range of the maximal operator, Preprint Academy of Sciences of the Czech Rep. **122**, 1997, to appear on *Nagoya Math. J.* (2000).
5. A. Fiorenza and C. Sbordone, Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1, *Studia Math.* **127**(3) (1998), 223–231.
6. L. Greco, A remark on the equality detDf = Det Df, *Differential Integral Equations* **6** (1993), 1089–1100.
7. L. Greco, T. Iwaniec and C. Sbordone, Inverting the p-harmonic operator, *Manuscripta Math.* **92** (1997), 249–258.
8. T. Iwaniec and C. Sbordone, On the integrability of the Jacobian under minimal hypotheses, *Arch. Rational Mech. Anal.* **119** (1992), 129–143.
9. T. Iwaniec and C. Sbordone, Weak minima of variational integrals, *J. Reine Angew. Math.* **454** (1994), 143–161.
10. T. Iwaniec and C. Sbordone, Riesz Transforms and elliptic pde's with VMO coefficients, *J. Analyse Math.* **74** (1998), 183–212.
11. A. Kufner, O. John and S. Fucik, *Function Spaces*, Noordhoff International Publishingr, Leyden, 1977.
12. J. Lang and L. Pick, personal communication.
13. C. Sbordone, Grand Sobolev spaces and their applications to variational problems, *Le Matematiche* **LI(2)** (1996), 335–347.
14. C. Sbordone, Nonlinear elliptic equations with right hand side in nonstandard spaces, *Rend. Sem. Mat. Fis. Modena*, Supplemento al **XLVI** (1998), 361–368.
15. A. C. Zaanen, *Integration*, North-Holland, 1967.