Collectanea Mathematica (electronic version): http://www.mat.ub.es/CM

Collect. Math. **50**, 3 (1999), 311–321

Interpolation of bilinear operators between Banach function spaces

Mieczysław Mastyło* and Dariusz Staszak

Faculty of Mathematics and Computer Science, A. Mickiewicz University

Matejki 48/49, 60-769 Poznań, Poland.

E-mail: mastylo@math.amu.edu.pl

E-mail: staszak@math.amu.edu.pl

Received August 23, 1998. Revised January 15, 1999

Abstract

We study bilinear operators between couples of Banach function spaces with the second coordinate L_∞-space. We show an estimate in terms of the K-functional. This is used to prove a result on interpolation of bilinear operators between considered couples.

0. Introduction

It is well known that integral operators play an important role in the theory of operators between Banach function spaces (see [4], [5]). In the study of these operators a special bilinear operator plays a particular role. To see this recall that if $X = X(\mu_1)$ and $Y = Y(\mu_2)$ are Banach function spaces on $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, respectively, and $Z = Z(\mu_1 \times \mu_2)$ is a Banach function space on the product measure space $(\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2)$ then the following holds (see [9]):
For every $k \in Z'$ an integral operator

$$T_k x(t) := \int_{\Omega_1} k(s,t)x(s)d\mu_1 \quad \textit{for} \quad t \in \Omega_2$$

is bounded from X into Y' if and only if the bilinear tensor product operator $(x, y) \mapsto x \otimes y$ maps $X \times Y$ into Z, where $x \otimes y(s,t) = x(s)y(t)$ for $(s,t) \in \Omega_1 \times \Omega_2$.

---

* Research supported by KBN Grant 2 PO3A 050 09.

Here E' denotes the *Köthe dual space* of a Banach function space E on (Ω, μ), which can be identified with the space of all functionals possessing an integral representation, that is,

$$E' := \Big\{ y \in L^0(\mu);\ \|y\|_{E'} = \sup_{\|x\|_E \le 1} \int_\Omega |xy| d\mu < \infty \Big\}.$$

The proof of the above result is similar to the one given in [6] in the context of Orlicz spaces. For the study of the bilinear tensor product operator $B(x, y) := x \otimes y$ in various Banach function spaces we refer the reader to [9] and [1].

In the paper we study general bilinear bounded operators. The obtained results may be applied to the tensor product operators.

1. Bilinear operators between Banach function spaces

Throughout the paper if X_0 and X_1 are two Banach spaces both linearly and continuously embedded in a Hausdorff topological vector space $\mathcal{X}$, then (X_0, X_1) is said to be a *Banach couple* and is denoted by $\overline{X}$. For $x \in X_0 + X_1$, $t > 0$ the K and E functionals are defined as

$$K(t, x; \overline{X}) := \inf \Big\{ \|x_0\|_{X_0} + t\|x_1\|_{X_1}\,;\ x_0 \in X_0, x_1 \in X_1,\ x = x_0 + x_1 \Big\},$$

and respectively

$$E(t, x; \overline{X}) := \inf \Big\{ \|x - x_1\|_{X_0}\,;\ x - x_1 \in X_0, x_1 \in X_1,\ \|x_1\|_{X_1} \le t \Big\}.$$

By the definition of the K and E functionals we obviously have

$$K(t, x; \overline{X}) = \inf \Big\{ st + E(s, x; \overline{X})\,;\ s > 0 \Big\}.$$

Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ and $\overline{C} = (C_0, C_1)$ be Banach couples. In what follows we will write $T \in \mathcal{B}(\overline{A}, \overline{B}; \overline{C})$ or equivalently $T : \overline{A} \times \overline{B} \to \overline{C}$, whenever $T : (A_0 + A_1) \times (B_0 + B_1) \to C_0 + C_1$ is a bounded bilinear operator such that $T : A_j \times B_j \to C_j$ is bounded for $j = 0, 1$.

Clearly that if (Ω_1, μ_1) and (Ω_2, μ_2) are measure spaces and (Ω, μ) is a product of these spaces, then for the tensor product operator B defined by $B(x, y) := x \otimes y$ for $(x, y) \in L^0(\mu_1) \times L^0(\mu_2)$, we have

$$B : \Big(L_p(\mu_1), L_\infty(\mu_1)\Big) \times \Big(L_p(\mu_2), L_\infty(\mu_2)\Big) \to \Big(L_p(\mu), L_\infty(\mu)\Big)$$

for any $1 \le p \le \infty$.

In this section we are interested in more general case, where instead of L_p-spaces we have Banach function spaces and B is any bounded bilinear operator between considered couples. At first we give some more fundamental definitions and notation.

Let (Ω, μ) be a measure space with μ complete and σ-finite and let $L^0(\mu)$ denote the space of all equivalence classes of measurable functions on Ω with the topology convergence in measure relative to each set of finite measure.

The non-increasing *rearrangement* of $x \in L^0(\mu)$ is the function $x^* = x^*_\mu : (0, \infty) \to [0, \infty]$ defined by

$$x^*(t) := \inf \Big\{ \lambda > 0\,;\ \mu_x(\lambda) \le t \Big\} \ \text{ for } \ t > 0,$$

where $\mu_x(\lambda) := \mu\{\omega \in \Omega\,;\, |x(t)| > \lambda\}$ for $\lambda > 0$ and $\inf \emptyset = \infty$.

A Banach space $X \subset L^0(\mu)$ is called a *Banach function space* on (Ω, Σ, μ) if there exists $u \in X$ such that $u > 0$ a.e. and X satisfies the ideal property:

$$\Big(x \in L^0(\mu),\ y \in X,\ |x| \le |y| \ \text{ a.e.} \Big) \ \Rightarrow (x \in X \ \text{ and } \ \|x\| \le \|y\|).$$

If X is a Banach function space and $w \in L^0(\mu)$ with $w > 0$ a.e., we define the *weighted space* $X(w)$ by $\|x\|_{X(w)} = \|xw\|_X$.

Let X be a Banach function space. An element $x \in X$ is said to have an *order continuous* norm if $\|x_n\| \to 0$ whenever $x_n \le |x|$ and $x_n \downarrow 0$. The largest ideal consisting of all elements with order continuous norms will be denoted by X_a. Clearly that $X_a = \{x \in X\,;\, |x| \ge x_n \downarrow 0 \ \text{ implies } \ \|x_n\|_X \to 0\}$. The closure in X of the set of simple functions supported in sets of finite measure is denoted by X_b.

A Banach function space X on (Ω, μ) is said to be *symmetric* if whenever $x \in X$, $y \in L^0$, and $\mu_x(\lambda) = \mu_y(\lambda)$ for $\lambda > 0$, then $y \in X$ and $\|x\| = \|y\|$.

The *fundamental function* φ_X of a symmetric space X on (Ω, Σ, μ) is defined for each t belonging to the range of μ as $\varphi_X(t) = \|\chi_A\|_X$, for $A \in \Sigma$ with $\mu(A) = t$, where χ_A is the characteristic function of the set A.

We note that if X is a symmetric space on nonatomic measure space, then $X_a = X_b$ if and only if $\varphi_X(0+) = 0$ (see [2], Theorem 5.5). We refer the reader to [2] and [7] for a study of symmetric spaces.

In the sequel we will need the following result (see, e.g. [8]). For the sake of completeness we include a proof.

Lemma 1.1

Let X be a Banach function space on (Ω, Σ, μ). Then

$$E\Big(t, x; (X, L_\infty)\Big) = \|(|x| - t)_+\|_X$$

for any $x \in X + L_\infty$.

Proof. We begin to prove $\|(|x|-t)_+\|_X \le E(t,x) := E(t,x;(X,L_\infty))$. We may assume that $E(t,x)<\infty$ since it holds trivially otherwise. Take any $x_1 \in L_\infty$ with $\|x_1\|_{L_\infty} \le t$ such that $x - x_1 \in X$. This implies that

$$(|x|-t)_+ \le |x - x_1| \,.$$

By using the ideal property we obtain that

$$\|(|x|-t)_+\|_X \le \|x - x_1\|_X .$$

Since x_1 is arbitrary, we obtain the desired inequality.

In order to prove the converse inequality, we may assume that $(|x|-t)_+ \in X$ for $x \in X + L_\infty$ and $t>0$. We define a function $x^{(t)}$ by the formula:

$$x^{(t)}(s) = \min\{|x(s)|, t\} \operatorname{sign} x(s)$$

for $s \in \Omega$. Clearly that $x^{(t)} \in L_\infty$ with $\|x^{(t)}\|_{L_\infty} \le t$ and

$$x - x^{(t)} = (|x|-t)_+ \operatorname{sign} x \in X.$$

Combining the above we conclude that

$$E(t,x) \le \|x - x^{(t)}\|_X = \|(|x|-t)_+\|_X .$$

This completes the proof. □

Proposition 1.2

Let $X_j = X_j(\mu_j)$ be Banach function spaces on $(\Omega_j, \Sigma_j, \mu_j)$, $j=1,2,3$ and let $T:(X_1+L_\infty(\mu_1)) \times (X_2 + L_\infty(\mu_2)) \to X_3 + L_\infty(\mu_3)$ be a bilinear operator such that for $j=1,2$

$$\|T(x_1,x_2)\|_{X_3} \le \|x_1\|_{X_1}\|x_2\|_{X_2} \ \text{ for } \ x_j \in X_j,$$
$$\|T(x_1,x_2)\|_{L_\infty(\mu_3)} \le \|x_1\|_{L_\infty(\mu_1)}\|x_2\|_{L_\infty(\mu_2)} \ \text{ for } \ x_j \in L_\infty(\mu_j).$$

If X_2 is a symmetric space, then the following inequality holds:

$$K\Big(t, T(x,\chi_A); (X_3, L_\infty(\mu_3)\Big) \le \varphi_{X_2}(u) K\Big(t/\varphi_{X_2}(u), x; (X_1, L_\infty(\mu_1))\Big)$$

for any $x \in X_1 + L_\infty(\mu_1)$, $t>0$ and $A \in \Sigma_2$ such that $u = \mu_2(A) < \infty$.

Proof. Let $x \in X_1(\mu_1) + L_\infty(\mu_1)$. For any $s > 0$ let $x^{(s)}$ denotes the s-*truncation* of x, that is, the function

$$x^{(s)}(\omega) = \min\{|x(\omega)|, s\}\operatorname{sign} x(\omega) \quad \text{for} \quad \omega \in \Omega_1.$$

By the bilinearity of T, we have

$$\begin{aligned} T(x, \chi_A) &= T(x - x^{(s)}, \chi_A) + T(x^{(s)}, \chi_A) \\ &= f + g. \end{aligned}$$

Since $x - x^{(s)} = (|x| - s)_+ \operatorname{sign} x$, we obtain by Lemma 1.1

$$\begin{aligned} \|f\|_{X_3} &= \|T(x - x^{(s)}, \chi_A)\|_{X_3} \le \|x - x^{(s)}\|_{X_1} \|\chi_A\|_{X_2} \\ &\le \varphi_{X_2}\Big(\mu_2(A)\Big) \|(|x| - s)_+\|_{X_1} = \varphi_{X_2}(u) E\Big(s, x; (X_1, L_\infty)\Big). \end{aligned}$$

Moreover for g we get the following estimate

$$\begin{aligned} \|g\|_{L_\infty(\mu_3)} &= \|T(x^{(s)}, \chi_A)\|_{L_\infty(\mu_3)} \\ &\le \|x^{(s)}\|_{L_\infty(\mu_2)} \|\chi_A\|_{L_\infty(\mu_1)} \le s. \end{aligned}$$

Combining the above estimates, we obtain

$$\begin{aligned} K(t, T(x, \chi_A); (X_3, L_\infty)) &\le \|f\|_{X_3} + t\|g\|_{L_\infty} \\ &\le \varphi_{X_2}(u) E\Big(s, x; (X_1, L_\infty)\Big) + st \\ &\le \varphi_{X_2}(u) \Big(st/\varphi_{X_2}(u) + E\Big(s, x; (X_1, L_\infty)\Big)\Big). \end{aligned}$$

Taking the infimum over all $s > 0$, we obtain the desired inequality. □

Theorem 1.3

Assume that the assumptions of Proposition 1.2 *are satisfied and additionally*

(i) $(\Omega_3, \Sigma_3, \mu_3)$ *is nonatomic measure space and* $\varphi_{X_2}(0+) = 0$.

(ii) For every $x \in X_1(\mu_1) + L_\infty(\mu_1)$ *the operator* $T_x = T(x, \cdot)$ *is bounded from* $(X_2(\mu_2) + L_\infty(\mu_2))_b$ *into* $X_3(\mu_3) + L_\infty(\mu_3)$.

Then for any $x \in X_1(\mu_1) + L_\infty(\mu_1)$, $y \in (X_2(\mu_2) + L_\infty(\mu_2))_b$ *and* $t > 0$ *the following inequality holds:*

$$K\Big(t, T(x, y); (X_3(\mu_3), L_\infty(\mu_3))\Big) \le 2 \int_0^\infty K\Big(t/\varphi_{X_2}(s), x; (X_1, L_\infty)\Big) y^*(s) \varphi'_{X_2}(s)\, ds.$$

Proof. Let $\overline{X} = (X_1(\mu_1), L_\infty(\mu_1))$, $\overline{Y} = (X_2(\mu_2), L_\infty(\mu_2))$ and $\overline{Z} = (X_3(\mu_3), L_\infty(\mu_3))$. Suppose that $x \in X_0 + X_1$ and $y = c\chi_A$, where $A \in \Sigma_2$ with $\mu_2(A) < \infty$ and $c \in \mathbb{R}$. From Proposition 1.2 it follows that

$$K\Big(t, T(x,y); \overline{Z}\Big) \le |c|\varphi(u) K\Big(t/\varphi(u), x; \overline{X}\Big),$$

where $\varphi = \varphi_{X_2}$. Since $s \mapsto K(t/\varphi(s), x; \overline{X})$ is nonincreasing and $\varphi(0+) = 0$, we obtain

$$\begin{aligned}\int_0^u K\Big(t/\varphi(s), x; \overline{X}\Big)\varphi'(s)ds &\ge \int_0^u K\Big(t/\varphi(u), x; \overline{X}\Big)\varphi'(s)\,ds \\ &= \varphi(u) K\Big(t/\varphi(u), x; \overline{X}\Big).\end{aligned}$$

Since $y^* = y^*_{\mu_2} = |c|\chi_{(0,u)}$, we get

$$\begin{aligned}K\Big(t, T(x,y); \overline{Z}\Big) &\le |c| \int_0^u K\Big(t/\varphi(s), x; \overline{X}\Big)\varphi'(s)\,ds \\ &= \int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big) y^*(s)\varphi'(s)\,ds.\end{aligned}$$

Let $0 \le y = \Sigma_{k=1}^n y_k$ be a simple function, where $0 \le y_k = c_k\chi_{A_k}$, $A_k \in \Sigma_2$ $A_1 \subset ... \subset A_n$ with $\mu_2(A_k) < \infty$ for $k = 1, ..., n$. Then

$$y^* = y^*_{\mu_2} = \sum_{k=1}^n c_k \chi_{(0,\mu_2(A_k))} = \sum_{k=1}^n y_k^*.$$

By the bilinearity of T,

$$T(x,y) = \sum_{k=1}^n T(x, y_k).$$

Combining with the last inequality, we obtain

$$\begin{aligned}K\Big(t, T(x,y); \overline{Z}\Big) &\le \sum_{k=1}^n K\Big(t, T(x,y_k); \overline{Z}\Big) \\ &\le \sum_{k=1}^n \int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big) y_k^*(s)\varphi'(s)\,ds \\ &= \int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big)\Big(\sum_{k=1}^n y_k^*(s)\Big)\varphi'(s)\,ds \\ &= \int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big) y^*(s)\varphi'(s)\,ds.\end{aligned}$$

We show that the last inequality is true for any $y \in (Y_0+Y_1)_b$. Let $0 \le y \in (Y_0+Y_1)_b$. Take a sequence $\{y_n\}$ of simple functions that $0 \le y_n \uparrow y$. Since Y_0+Y_1 is a symmetric space and $\varphi_{Y_0+Y_1} = \min\{\varphi_{Y_0}, \varphi_{Y_1}\}$, we have $\varphi_{Y_0+Y_1}(0+) = 0$. It follows from Theorem 5.5 in [2] that $(Y_0+Y_1)_b = (Y_0+Y_1)_a$. In consequence $\|y_n - y\|_{Y_0+Y_1} \to 0$ by $0 \le y - y_n \downarrow 0$. Hence by the continuity of T_x, we obtain

$$\|T_x(y_n) - T_x(y)\|_{Z_0+Z_1} \to 0.$$

In consequence

$$K\Big(t, T_x(y_n); \overline{Z}\Big) \to K\Big(t, T_x(y); \overline{Z}\Big) \ \text{ for } \ t > 0.$$

We have proved that

$$\begin{aligned} K\Big(t, T_x(y_n); \overline{Z}\Big) &= K\Big(t, T(x, y_n); \overline{Z}\Big) \\ &\le \int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big) y_n^*(s)\varphi'(s)\, ds. \end{aligned}$$

Since $0 \le y_n \uparrow y$, $y_n^* \uparrow y^*$ (see [2], p. 41). Then using the monotone convergence theorem, we obtain

$$K\Big(t, T(x, y); \overline{Z}\Big) \le \int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big) y^*(s)\varphi'(s)\, ds$$

for $t > 0$. Now if $y \in (Y_0+Y_1)_b$ is an arbitrary element, then $y = y_+ - y_-$. Clearly that $y_+,\ y_- \in (Y_0+Y_1)_a$. Since T is bilinear operator, we obtain the following inequalities

$$\begin{aligned} K\Big(t, T(x, y); \overline{Z}\Big) &\le K\Big(t, T(x, y_+); \overline{Z}\Big) + K\Big(t, T(x, y_-); \overline{Z}\Big) \\ &\le \int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big)(y_+)^*(s)\varphi'(s)\, ds \\ &\quad + \int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big)(y_-)^*(s)\varphi'(s)\, ds \\ &\le 2\int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big) y^*(s)\varphi'(s)\, ds, \end{aligned}$$

which completes the proof. □

Since $K(t, f; (L_1(\mu), L_\infty(\mu)) = tf^{**}(t)$, where $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\, ds$ for $t > 0$ (see [2]), we obtain the following result (cf. [9]).

Corollary 1.4

If $T : (L_1(\mu_1), L_\infty(\mu_1)) \times (L_1(\mu_2), L_\infty(\mu_2)) \to (L_1(\mu_3), L_\infty(\mu_3))$, then there exists a constant $C > 0$ such that

$$T(x,y)^{**}(t) \le C \int_0^\infty x^{**}(t/s) y^*(s) \frac{ds}{s}$$

for any $x \in L_1(\mu_1) + L_\infty(\mu_1)$, $y \in (L_1(\mu_2) + L_\infty(\mu_2))_a$ and $t > 0$.

2. Applications

In this section we present an application of the obtained results to interpolation of bilinear operators. We recall that in every symmetric space X on $\mathbb{R}_+$, *dilation operators* D_s $(0 < s < \infty)$ defined by $D_s f(t) = f(t/s)$ for $f \in X$ are bounded (see [7]).

We now need a technical result. The proof is a modification of the proof of Lemma II 4.7 in [7].

Lemma 2.1

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a quasi-concave function, $w \in L^0(\mathbb{R}_+, m)$ and let E be a symmetric space on $\mathbb{R}_+$. Then for any $x \in E$ the following inequality holds:

$$\Big\| \int_0^\infty (D_{\varphi(s)} x^*) w(s)\, ds \Big\|_E \le \int_0^\infty \|D_{\varphi(s)} x\|_E \, w(s)\, ds.$$

Proof. Let $\lambda > 1$. Since for any $x \in E$ the function $s \mapsto D_{\varphi(s)} x^*$ is nondecreasing, we obtain for $t > 0$

$$0 \le \int_0^\infty D_{\varphi(s)} x^*(t) w(s)\, ds = \sum_{k=-\infty}^{\infty} \int_{\lambda^k}^{\lambda^{k+1}} D_{\varphi(s)} x^*(t) w(s)\, ds$$
$$\le \sum_{k=-\infty}^{\infty} D_{\varphi(\lambda^{k+1})} x^*(t) \int_{\lambda^k}^{\lambda^{k+1}} w(s)\, ds.$$

Since for any $s > 0$, $\|D_s\|_{E \to E} \le \max\{1, s\}$ (see [7], p. 98), we obtain

$$\Big\| \int_0^\infty D_{\varphi(s)} x^*(t) w(s)\, ds \Big\| \le \sum_{k=-\infty}^{\infty} \|D_{\varphi(\lambda^{k+1})} x^*\|_E \int_{\lambda^k}^{\lambda^{k+1}} w(s)\, ds$$
$$\le \sum_{k=-\infty}^{\infty} \|D_{\varphi(\lambda^k)\overline{\varphi}(\lambda)} x^*\|_E \int_{\lambda^k}^{\lambda^{k+1}} w(s)\, ds$$
$$\le \overline{\varphi}(\lambda) \sum_{k=-\infty}^{\infty} \|D_{\varphi(\lambda^k)} x^*\|_E \int_{\lambda^k}^{\lambda^{k+1}} w(s)\, ds,$$

where $\overline{\varphi}(\lambda) := \sup\{\varphi(u\lambda)/\varphi(u) \,:\, u > 0\}$.

On the other hand, since the function $s \mapsto \|D_{\varphi(s)}x^*\|_E$ is nondecreasing, we have

$$\int_0^\infty \|D_{\varphi(s)}x^*\|_E\, w(s)\, ds = \sum_{k=-\infty}^{\infty} \int_{\lambda^k}^{\lambda^{k+1}} \|D_{\varphi(s)}x^*\|_E\, w(s)\, ds$$
$$\geq \sum_{k=-\infty}^{\infty} \|D_{\varphi(\lambda^k)}x^*\|_E \int_{\lambda^k}^{\lambda^{k+1}} w(s)\, ds.$$

Combining with the previous inequality, we obtain

$$\Big\| \int_0^\infty D_{\varphi(s)}x^*(t)w(s)\, ds \Big\| \leq \overline{\varphi}(\lambda) \sum_{k=-\infty}^{\infty} \int_{\lambda^k}^{\lambda^{k+1}} \|D_{\varphi(s)}x^*\|_E\, w(s)\, ds$$
$$= \overline{\varphi}(\lambda) \int_0^\infty \|D_{\varphi(s)}x\|_E\, w(s)\, ds.$$

Since the fundamental function of any symmetric space is quasi-concave, $\overline{\varphi}$ is also a quasi-concave. Thus $\overline{\varphi}$ is continuous. In consequence, we obtain the required inequality, by $\lim_{\lambda\to 1+} \overline{\varphi}(\lambda) = 1$. □

Let $\Phi \subset L^0(\mathbb{R}_+, m)$ be a Banach lattice such that $\min\{1, t\} \in \Phi$ and let $\overline{X} = (X_0, X_1)$ be a Banach couple. The K-*method space* defined by

$$\overline{X}_\Phi := \{x \in X_0 + X_1 \,;\, K(\cdot, x; \overline{X}) \in \Phi\}.$$

is a Banach space equipped with the norm $\|x\| = \|K(\cdot, x; \overline{X})\|_\Phi$, which is an exact interpolation space with respect to $\overline{X}$ (see [3]).

In what follows, if (Ω, Σ, μ) is a measure space and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a quasi-concave function, then $\Lambda(\psi)$ denotes a symmetric *Lorentz space* defined by

$$\Lambda(\psi) := \Big\{ x \in L^0(\mu) \,;\, \|x\|_{\Lambda(\psi)} = \int_0^\infty x^*(s)\psi(s)\, \frac{ds}{s} < \infty \Big\}.$$

We are now in a position to prove the main result of this section.

Theorem 2.2

Let E be a symmetric space on $\mathbb{R}_+$ such that $\min\{1, 1/t\} \in E$ and let $X_j = X_j(\mu_j)$, $j = 1, 2, 3$ be Banach lattices with X_2 a symmetric space on a nonatomic measure space such that $\varphi_{X_2}(0+) = 0$. If $T : (X_1, L_\infty) \times (X_2, L_\infty) \to (X_3, L_\infty)$, then T is a bounded bilinear operator from $(X_1, L_\infty)_\Phi \times \Lambda(\psi)_a$ into $(X_3, L_\infty)_\Phi$, where $\Phi = E(1/t)$, $\psi(s) = \|D_{\varphi(s)}\|_{E\to E}$ and $\varphi(s) = \varphi_{X_2}(s)$ for $s > 0$.

Proof. Let $\overline{X} = (X_1(\mu_1), L_\infty(\mu_1))$, $\overline{Y} = (X_2(\mu_2), L_\infty(\mu_2))$ and $\overline{Z} = (X_3(\mu_3), L_\infty(\mu_3))$. Without loss of generality we may assume that

$$\|T(x,y)\|_{Z_j} \leq \|x\|_{X_j}\|y\|_{Y_j} \text{ for } (x,y) \in X_j \times Y_j,\ j = 0,1.$$

Since $T : (X_1, L_\infty) \times (X_2, L_\infty) \to (X_3, L_\infty)$, all the assumptions of Theorem 1.3 are satisfied. Observe that $\min\{1,t\} \in \Phi$, by $\min\{1,1/t\} \in E$.

Let $x \in \overline{X}_\Phi$ and $y \in L_1(\mu_2) \cap L_\infty(\mu_2)$. Then by Theorem 1.3 and the inequality $\varphi'(s) \leq \varphi(s)/s$ for a.e. $s > 0$, we obtain

$$\begin{aligned}\frac{K(t, T(x,y); \overline{Z})}{t} &\leq \frac{2}{t}\int_0^\infty K\Big(t/\varphi(s), x; \overline{X}\Big) y^*(s)\varphi'(s)\, ds \\ &\leq 2\int_0^\infty K_*\Big(t/\varphi(s), x; \overline{X}\Big) y^*(s)\, \frac{ds}{s},\end{aligned}$$

where $f_*(u) := f(u)/u$ for $u > 0$. This implies that

$$\frac{K(t, T(x,y); \overline{Z})}{t} \leq 2\int_0^\infty D_{\varphi(s)} K_*(t, x; \overline{X}) y^*(s)\, \frac{ds}{s}.$$

Since $t \mapsto K_*(t, x; \overline{X})$ for $t > 0$ is nonincreasing function, we have from Lemma 2.1

$$\begin{aligned}\Big\| \int_0^\infty D_{\varphi(s)} K_*(\cdot, x; \overline{X}) y^*(s) \frac{ds}{s}\Big\|_E &\leq \int_0^\infty \|D_{\varphi(s)} K_*(\cdot, x; \overline{X})\|_E\, y^*(s)\, \frac{ds}{s} \\ &\leq \Big(\int_0^\infty y^*(s)\, \|D_{\varphi(s)}\|_{E\to E} \frac{ds}{s}\Big) \|K_*(\cdot, x; \overline{X})\|_E \\ &= \|x\|_{\overline{X}_\Phi} \|y\|_{\Lambda(\psi)}.\end{aligned}$$

Combining the above we conclude

$$\|T(x,y)\|_{\overline{Z}_\Phi} = \|K_*(\cdot, T(x,y); \overline{Z})\|_E \leq 2\|x\|_{\overline{X}_\Phi}\|y\|_{\Lambda(\psi)}.$$

Since $T \in \mathcal{B}(\overline{X}, \overline{Y}; \overline{Z})$, $T_x = T(x, \cdot)$ is continuous from $Y_0 + Y_1$ into $Z_0 + Z_1$. Thus the proof is complete by the density of $L_1(\mu_2) \cap L_\infty(\mu_2)$ in $\Lambda(\psi)_a$. $\square$

References

1. S.V. Astashkin, Tensor products in symmetric function spaces, *Collect. Math.* **48** (1997), 375–391.
2. C. Bennett and B. Sharpley, *Interpolation spaces. An Introduction*, Academic Press Inc., New York, 1988.
3. Yu.A. Brudnyiĭ, and N.Ya. Krugljak, *Interpolation Functors and Interpolation Spaces*, I, North Holland, Amsterdam, 1991.
4. C. Corduneanu, *Integral equations and stability of Feedback systems*, Academic Press, New York, 1973.
5. B. Korotkov, *Integral operators*, Nauka, Novosibirsk, 1983.
6. M.A. Krasnosel'skiĭ and Ja. Rutyckiĭ, *Convex Functions and Orlicz Spaces*, Groningen, 1961.
7. S.G. Krein, Yu.I. Petunin and E.M. Semenov, *Interpolation of Linear Operators*, Nauka, Moscow, 1978 (in Russian); English transl.: Amer. Math. Soc., Providence, 1982.
8. L. Maligranda and L.E. Persson, The E-functional for some pairs of groups, *Results Mat.* **20** (1991), 538–553.
9. M. Milman, Some new function spaces and their tensor products, *Notas Mat.* **20** (1978), 1–128.