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A characterisation of the circle group

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Abstract

We show that if G is a compact connected Abelian group such that, for some $n \in \mathbb{N}$ and some closed subgroup H of $G_{(n)} = \{a \in G \mid na = 0\}$, the set $G \setminus H$ is disconnected, then G is topologically isomorphic with the circle group $\mathbb{T}$.

1. Introduction

Let G be a locally compact Abelian group with dual $\widehat{G}$. Given $n \in \mathbb{N}$, denote by $G^{(n)}$ and $G_{(n)}$ the image and kernel of the homomorphism $G \ni a \mapsto na \in G$, respectively. Given a subset $X \subset G$, let

$$-X = \left\{a \in G \mid -a \in X\right\}.$$

In agreement with the terminology introduced in [1], G is said to be *decomposable* if there exists an open subset $U \subset G$ such that $U \cup (-U) = G \setminus G_{(2)}$ and $U \cap (-U) = \emptyset$.

Let $\mathbb{T}$ be the circle group, this being the multiplicative group of complex numbers with unit modulus, endowed with the usual topology.

In [1] (see also [2]) the following theorem is established:

Theorem 1

Any decomposable compact connected Abelian group different from a singleton is topologically isomorphic with $\mathbb{T}$.

This result can be viewed as a characterisation of the circle group. The aim of this paper is to prove the following generalisation of Theorem 1:

Theorem 2

If G *is a compact connected Abelian group such that, for some* $n \in \mathbb{N}$ *and some closed subgroup* H *of* $G_{(n)}$, *the set* $G \setminus H$ *is disconnected, then* G *is topologically isomorphic with* $\mathbb{T}$.

Of course, the latter theorem can be regarded as yet another characterisation of the circle group.

2. An auxiliary result

This section is devoted to establishing an auxiliary result. We start by fixing notation and recalling some concepts from algebra and topology.

For a set A, denote by $\#A$ the cardinality of A, and by id_A the identity mapping of A onto itself.

For each $n \in \mathbb{N}$, let $\mathbb{Z}(n)$ be the cyclic group with n elements. Let $\mathbb{Q}$ be the additive group of rational numbers, equipped with the discrete topology.

If $\{G_i\}_{i\in I}$ is an indexed collection of Abelian groups, we write $\prod_{i\in I} G_i$ for the direct product of the G_i. If $I = \{1, \dots, n\}$, we also write $G_1 \times \cdots \times G_n$ in place of $\prod_{i\in I} G_i$. If $\mathfrak{m}$ is a cardinal number and if, for some fixed G, $G_i = G$ for each $i \in I$, where I is a set of cardinality equal to $\mathfrak{m}$, we write $G^{\mathfrak{m}}$ for $\prod_{i\in I} G_i$.

Let $(\cdot,\cdot)$ represent the pairing between elements of a locally compact Abelian group and elements of its dual.

For a subgroup H of a locally compact Abelian group G, denote by $H^\perp$ the *annihilator* of H in $\widehat{G}$, that is, the closed subgroup of $\widehat{G}$ defined as

$$H^\perp = \bigl\{\chi \in \widehat{G} \mid (a,\chi) = 1 \text{ for all } a \in H\bigr\}.$$

For a homomorphism f, designate by $\ker f$ the kernel of f.

Given locally compact Abelian groups G and H, and a continuous homomorphism $f\colon G \to H$, denote by $f^*\colon \widehat{H} \to \widehat{G}$ the *dual homomorphism* defined by

$$\big(a, f^*(\chi)\big) = \big(f(a), \chi\big) \qquad \big(a \in G,\ \chi \in \widehat{H}\big)\,.$$

An $\mathbb{N}$-indexed *projective* (or *inverse*) *system of groups* is a family $\{\Sigma_p, \pi_p^q\}$, where, for each $p \in \mathbb{N}$, Σ_p is a group, and, for all $p, q \in \mathbb{N}$ with $p \le q$, $\pi_p^q\colon \Sigma_q \to \Sigma_p$ is a homomorphism such that the following conditions hold:

(i) $\pi_p^p = \mathrm{id}_{\Sigma_p}$ for each $p \in \mathbb{N}$;

(ii) $\pi_p^q \pi_q^r = \pi_p^r$ for all $p, q, r \in \mathbb{N}$ with $p \le q \le r$.

If in addition each Σ_p is a topological group and each π_p^q is a continuous homomorphism, then $\{\Sigma_p, \pi_p^q\}$ is called a *topological projective system of groups*. The *projective limit* of such a system $\{\Sigma_p, \pi_p^q\}$ is the group $\varprojlim\{\Sigma_p, \pi_p^q\}$ defined as

$$\varprojlim\{\Sigma_p, \pi_p^q\} = \Big\{\{a_p\}_{p\in\mathbb{N}} \in \prod_{p\in\mathbb{N}} \Sigma_p \mid \pi_p^q(a_q) = a_p \text{ for all } p, q \in \mathbb{N} \text{ with } p \le q\Big\}.$$

For each $p \in \mathbb{N}$, let $\pi_p\colon \varprojlim\{\Sigma_p, \pi_p^q\} \to \Sigma_p$ be the homomorphism defined as the restriction to $\varprojlim\{\Sigma_p, \pi_p^q\}$ of the canonical projection of $\prod_{p\in\mathbb{N}} \Sigma_p$ onto Σ_p. The π_p are compatible with the π_p^q in the sense that $\pi_p^q \pi_q = \pi_p$ for all $p, q \in \mathbb{N}$ with $p \le q$. If $\{\Sigma_p, \pi_p^q\}$ is a topological projective system of groups, then $\varprojlim\{\Sigma_p, \pi_p^q\}$ can be given the weakest topology making all the projection maps π_p continuous. This topology is just the relativised topology from the direct product. Any family of sets of the form $\pi_p^{-1}(U_p)$, where p ranges over an arbitrarily fixed infinite subset of $\mathbb{N}$ and U_p is an open subset of Σ_p, is a base for the topology of $\varprojlim\{\Sigma_p, \pi_p^q\}$.

It is easy to check that projective limits satisfy the universal property that if Γ is another (topological) group with a family of (continuous) homomorphisms $\sigma_p\colon \Gamma \to \Sigma_p$ satisfying $\pi_p^q \sigma_q = \sigma_p$ for all $p, q \in \mathbb{N}$ with $p \le q$, then there is a unique (continuous) homomorphism $\sigma\colon \Gamma \to \varprojlim\{\Sigma_p, \pi_p^q\}$ satisfying $\pi_p \sigma = \sigma_p$ for all $p \in \mathbb{N}$.

Proposition 1

Let $\{\Sigma_p, \pi_p^q\}$ be an $\mathbb{N}$-indexed topological projective system of groups such that:

(i) there exists $l \in \mathbb{N}$ such that, for each $p \in \mathbb{N}$, Σ_p is topologically isomorphic with $\mathbb{T}^l$ by means of a homomorphism $\tau_p\colon \Sigma_p \to \mathbb{T}^l$;

(ii) for each $p, q \in \mathbb{N}$ with $p \le q$, π_p^q has the form

$$\pi_p^q = \tau_p^{-1} \tau_p^q \tau_q,$$

where $\tau_p^q\colon \mathbb{T}^l \to \mathbb{T}^l$ is the homomorphism

$$\tau_p^q\colon (t_1, \ldots, t_l) \mapsto \big(t_1^{n_p^q(1)}, \ldots, t_l^{n_p^q(l)}\big) \tag{1}$$

for some $n_p^q(1), \ldots, n_p^q(l) \in \mathbb{Z} \setminus \{0\}$;

(iii) if $l = 1$, then $\lim_{q\to\infty} |n_p^q(1)| = +\infty$ for each $p \in \mathbb{N}$.

Then $\Sigma = \varprojlim\{\Sigma_p, \pi_p^m\}$ has the property that, for each $n \in \mathbb{N}$ and each closed subgroup Γ of $\Sigma_{(n)}$, $\Sigma \setminus \Gamma$ is connected.

Proof. Without loss of generality, we may assume that $\Sigma_p = \mathbb{T}^l$ and $\tau_p = \mathrm{id}_{\Sigma_p}$ for all $p \in \mathbb{N}$, and $\pi_p^q = \tau_p^q$ for all $p, q \in \mathbb{N}$ with $p \le q$. That $\{\Sigma_p, \pi_p^q\}$ is an inverse system of groups now means that, for each $j = 1, \dots, l$, $n_p^p(j) = 1$ for each $p \in \mathbb{N}$, and $n_p^q(j) n_q^r(j) = n_p^r(j)$ for all $p, q, r \in \mathbb{N}$ with $p \le q \le r$. Suppose, on the contrary, that, for some $n \in \mathbb{N}$ and some closed subgroup Γ of $\Sigma_{(n)}$, $\Sigma \setminus \Gamma$ can be decomposed into two non-empty disjoint open (in $\Sigma \setminus \Gamma$) sets U_1 and U_2. Since Γ is a closed subgroup of $\Sigma_{(n)}$ and since $\Sigma_{(n)}$ is a closed subgroup of Σ, U_1 and U_2 are open subsets of Σ. For each $i = 1, 2$, choose $x_i \in U_i$ arbitrarily, and next select an open neighbourhood of x_i of the form $\pi_{p_i}^{-1}(V_i)$, contained in U_i, with $p_i \in \mathbb{N}$ and V_i an open subset of Σ_{p_i}. We claim that there exists $q \in \mathbb{N}$ with $q \ge p_i$ for each $i = 1, 2$ such that, for some $a_1, a_2 \in \Sigma_q$,

$$\pi_q^{-1}\big(\{a_i\}\big) \subset U_i \quad \text{for each } \ i = 1, 2 \tag{2}$$

and such that a_1 and a_2 can be joined by a closed arc I wholly contained in $\Sigma_q \setminus \pi_q(\Gamma)$.

First consider the case $l > 1$. Since, for each $p \in \mathbb{N}$, $\pi_p(\Gamma) \subset \pi_p(\Sigma_{(n)}) \subset (\Sigma_p)_{(n)}$ and since, clearly, $(\Sigma_p)_{(n)}$ is isomorphic with $\mathbb{Z}(n)^l$, we have

$$\#\pi_p(\Gamma) \le n^l. \tag{3}$$

Select $q \in \mathbb{N}$ so that $q \ge p_i$ for each $i = 1, 2$. In view of (3), $\pi_q(\Gamma)$ is finite. Hence, since Σ_q is an l-dimensional torus with $l > 1$, $\Sigma_q \setminus \pi_q(\Gamma)$ is arc-wise connected. For each $i = 1, 2$, set $a_i = \pi_q(x_i)$. Clearly, a_1 and a_2 can be linked by a closed arc I wholly contained in $\Sigma_q \setminus \pi_q(\Gamma)$. Furthermore

$$\pi_{p_i}^q(a_i) = \pi_{p_i}^q\big(\pi_q(x_i)\big) = \pi_{p_i}(x_i),$$

whence $\pi_q^{-1}(\{a_i\}) \subset \pi_{p_i}^{-1}(\{\pi_{p_i}(x_i)\})$. As $\pi_{p_i}(x_i) \in V_i$ and $\pi_{p_i}^{-1}(V_i) \subset U_i$, (2) is implied.

We now pass to the case $l = 1$. Using condition (iii) of the statement, choose $q \in \mathbb{N}$ so that $|n_{p_i}^q(1)| > n$ for each $i = 1, 2$. Being a subgroup of the cyclic group $(\Sigma_p)_{(n)}$, $\pi_q(\Gamma)$ is cyclic. Thus $\Sigma_q \setminus \pi_q(\Gamma)$ consists of open arcs, each of which has length equal to $2\pi/\#\pi_q(\Gamma)$. Let J be one of these arcs. In view of (3), the length of J is no smaller than $2\pi/n$. Now, for each $i = 1, 2$, $\ker \pi_{p_i}^q$ is the cyclic group of all roots of unity of order $|n_{p_i}^q(1)|$. Therefore, for each $i = 1, 2$, every coset of $\ker \pi_{p_i}^q$ consists of points evenly distributed around the circle, with the angular distance between any pair of two closest points being equal to $2\pi/|n_{p_i}^q(1)|$. As $2\pi/|n_{p_i}^q(1)| < 2\pi/n$, we see that, for each $i = 1, 2$, every coset of $\ker \pi_{p_i}^q$ has a point in common with J. For each $i = 1, 2$, pick $a_i \in (\ker \pi_{p_i}^q)\pi_q(x_i) \cap J$, where $(\ker \pi_{p_i}^q)\pi_q(x_i)$ denotes the coset of $\ker \pi^q_{p_i}$ containing $\pi_q(x_i)$. It is readily seen that $\pi^q_{p_i}(a_i) = \pi_{p_i}(x_i)$, and hence, as before, we obtain (2). Taking for I a subarc of J having the a_i for endpoints finally establishes the claim.

The inclusion $I \subset \Sigma_q \setminus \pi_q(\Gamma)$ now implies that $\pi_q^{-1}(I) \subset \Sigma \setminus \Gamma$. In view of the form of the bases for the topology of a projective limit, $\pi_q^{-1}(I)$ can be covered by open sets $\pi_r^{-1}(W_r)$, where $r \in \mathbb{N}$ satisfies $r \geq q$ and W_r is an open subset of Σ_r such that either $\pi_r^{-1}(W_r) \subset U_1$ or $\pi_r^{-1}(W_r) \subset U_2$ holds. Since $\pi_q^{-1}(I)$ is compact, we can choose a finite subcover $\pi_{r_1}^{-1}(W_{r_1}), \ldots, \pi_{r_d}^{-1}(W_{r_d})$. Fix $s \in \mathbb{N}$ so that $s \geq r_j$ for all $j = 1, \ldots, d$. For each $j = 1, \ldots, d$, set $W'_j = (\pi^s_{r_j})^{-1}(W_{r_j})$. Noting that $\pi_{r_j}^{-1}(W_{r_j}) = \pi_s^{-1}(W'_j)$, let, for each $i = 1, 2$, Z_i be the union of all those W'_j for which $\pi_s^{-1}(W'_j) \subset U_i$. Clearly, Z_1 and Z_2 are open disjoint subsets of Σ_s such that $(\pi^s_q)^{-1}(I) \subset Z_1 \cup Z_2$. Since π^s_q is a covering map, there is a continuous map $f \colon I \to \Sigma_s$ such that $\pi^s_q \circ f = \mathrm{id}_{\Sigma_q}$. Now, clearly, $f(I) \subset (\pi^s_q)^{-1}(I)$, and, since $f(I)$ is connected, we have either $f(I) \subset Z_1$ or $f(I) \subset Z_2$, and consequently either $\pi_s^{-1}(f(I)) \subset U_1$ or $\pi_s^{-1}(f(I)) \subset U_2$. But $\pi_s^{-1}(f(\{a_i\})) \subset \pi_q^{-1}(\{a_i\})$ and therefore either

$$\pi_q{}^{-1}\big(\{a_i\}\big) \cap U_1 \neq \emptyset \quad \text{for each} \;\; i = 1, 2$$

or

$$\pi_q{}^{-1}\big(\{a_i\}\big) \cap U_2 \neq \emptyset \quad \text{for each} \;\; i = 1, 2.$$

This, however, is incompatible with (2), as U_1 and U_2 are disjoint. The contradiction obtained establishes the result. □

3. Proof of the main result

This section is devoted to establishing Theorem 2.

Proof of Theorem 2. Let G be a compact connected Abelian group for which there exist $n \in \mathbb{N}$ and a closed subgroup $H \subset G_{(n)}$ such that $G \setminus H$ is disconnected. Then, necessarily, both G and $\widehat{G}$ are different from a singleton. By the connectedness of G, $\widehat{G}$ is torsion free (cf. [3, §24.25]). Let $\{\chi_\alpha\}_{\alpha \in A}$ be a maximal collection of independent elements of $\widehat{G}$. As is known, $\#A$ does not depend on the particular choice of a maximal family of independent members of $\widehat{G}$, and defines the (torsion-free) rank of $\widehat{G}$. By the maximality of $\{\chi_\alpha\}_{\alpha \in A}$, for each $\chi \in \widehat{G}$ there exist $n(\chi) \in \mathbb{Z}$ and an A-indexed family of integers $\{n_\alpha(\chi)\}_{\alpha \in A}$ such that: (i) $n_\alpha(\chi) \neq 0$ for only finitely many $\alpha \in A$; (ii) the equality $n(\chi)\chi = \sum_{\alpha \in A} n_\alpha(\chi)\chi_\alpha$ holds. By the independency of the χ_α, $n(\chi)$ can be taken to be non-zero so that – in particular –

for each $\alpha \in A$ the rational number $n_\alpha(\chi)/n(\chi)$ makes sense; moreover, this number depends only on χ. One verifies at once that, for each $\alpha \in A$, the function $\rho_\alpha : \chi \mapsto n_\alpha(\chi)/n(\chi)$ is a homomorphism from $\widehat{G}$ into $\mathbb{Q}$. Since $\widehat{G}$ is torsion free, we have

$$\bigcap_{\alpha \in A} \ker \rho_\alpha = \emptyset\,. \tag{4}$$

Let $\mathcal{P}_{\mathrm{fin}}(A)$ be the set of all finite subsets of A. For each $\mathbf{A} = \{\alpha_1, \dots, \alpha_k\}$ in $\mathcal{P}_{\mathrm{fin}}(A)$, define a homomorphism $\rho_{\mathbf{A}} : \widehat{G} \to \mathbb{Q}^k$ by setting

$$\rho_{\mathbf{A}} = (\rho_{\alpha_1}, \dots, \rho_{\alpha_k}).$$

Observe that, for each $\mathbf{A} \in \mathcal{P}_{\mathrm{fin}}(A)$, the dual of $\rho_{\mathbf{A}}(\widehat{G})$ is topologically isomorphic with $(\ker \rho_{\mathbf{A}})^\perp$, where the annihilator is taken in the dual of $\widehat{G}$ identified with G. In view of (4),

$$G = \bigcup_{\mathbf{A} \in \mathcal{P}_{\mathrm{fin}}(A)} (\ker \rho_{\mathbf{A}})^\perp,$$

and hence

$$G \setminus H = \bigcup_{\mathbf{A} \in \mathcal{P}_{\mathrm{fin}}(A)} \big(\ker \rho_{\mathbf{A}}\big)^\perp \setminus \big(H \cap (\ker \rho_{\mathbf{A}})^\perp\big).$$

Taking into account that $G \setminus H$ is disconnected, we immediately deduce from the last equality that for each $\mathbf{A} \in \mathcal{P}_{\mathrm{fin}}(A)$ there exists $\mathbf{B} \in \mathcal{P}_{\mathrm{fin}}(A)$ with $\mathbf{A} \subset \mathbf{B}$ such that $(\ker \rho_{\mathbf{B}})^\perp \setminus (H \cap (\ker \rho_{\mathbf{B}})^\perp)$ is disconnected.

Fix $\mathbf{A} \in \mathcal{P}_{\mathrm{fin}}(A)$ arbitrarily and choose $\mathbf{B} = \{\alpha_1, \dots, \alpha_l\}$ in $\mathcal{P}_{\mathrm{fin}}(A)$ so that $\mathbf{A} \subset \mathbf{B}$ and $(\ker \rho_{\mathbf{B}})^\perp \setminus (H \cap (\ker \rho_{\mathbf{B}})^\perp)$ is disconnected. For each $p \in \mathbb{N}$, let K_p be the cyclic subgroup of $\mathbb{Q}$ given by

$$K_p = \{m/p! \mid m \in \mathbb{Z}\}$$

and let $L_p = \rho_{\mathbf{B}}(\widehat{G}) \cap (K_p)^l$. It is clear that $L_p \subset L_{p+1}$ for each $p \in \mathbb{N}$ and that

$$\rho_{\mathbf{B}}(\widehat{G}) = \bigcup_{p=1}^{\infty} L_p. \tag{5}$$

Since, for each $p \in \mathbb{N}$, L_p is a subgroup of the direct product of l copies of the cyclic group K_p, it follows that L_p is a direct product of cyclic groups, and hence is up to isomorphism determined by its rank. It is apparent that

$$L_p \subset \big(\rho_{\alpha_1}(\widehat{G}) \cap K_p\big) \times \ldots \times \big(\rho_{\alpha_l}(\widehat{G}) \cap K_p\big). \tag{6}$$

From

$$\rho_{\mathbf{B}}(\chi_{\alpha_1}) = (1, 0, \dots, 0),$$
$$\rho_{\mathbf{B}}(\chi_{\alpha_2}) = (0, 1, \dots, 0),$$
$$\dots$$
$$\rho_{\mathbf{B}}(\chi_{\alpha_l}) = (0, 0, \dots, 1),$$

we infer that the rank of L_p is no smaller than l. On the other hand, for each $i = 1, \dots, l$, the group $\rho_{\alpha_i}(\widehat{G}) \cap K_p$ is cyclic, and so the rank of $(\rho_{\alpha_1}(\widehat{G}) \cap K_p) \times \dots \times (\rho_{\alpha_l}(\widehat{G}) \cap K_p)$ is equal to l. Coupling this with (6), we see that

$$L_p = (\rho_{\alpha_1}(\widehat{G}) \cap K_p) \times \dots \times (\rho_{\alpha_l}(\widehat{G}) \cap K_p).$$

For any $p, q \in \mathbb{N}$ with $p \le q$, let i_p^q be the canonical embedding of L_p into L_q. Clearly, $i_p^p = \mathrm{id}_{L_p}$ and hence $i_p^{p*} = \mathrm{id}_{\widehat{L_p}}$ for each $p \in \mathbb{N}$, and also $i_q^r i_p^q = i_p^r$ and hence $i_p^{q*} i_q^{r*} = i_p^{r*}$ for all $p, q, r \in \mathbb{N}$ with $p \le q \le r$. Thus $\{\widehat{L_p}, i_p^{q*}\}$ is a topological projective system of groups.

For each $p \in \mathbb{N}$, let i_p be the canonical embedding of L_p into $\rho_{\mathbf{B}}(\widehat{G})$. Clearly, $i_p = i_q i_p^q$ and so $i_p^* = i_p^{q*} i_q^*$ for any $p, q \in \mathbb{N}$ with $p \le q$. By the universal property of projective limits, there is a unique continuous homomorphism $\sigma\colon (\ker \rho_{\mathbf{B}})^{\perp} \to \varprojlim \{\widehat{L_p}, i_p^{q*}\}$ satisfying $\pi_p \sigma = i_p^*$ for all $p \in \mathbb{N}$; here, of course, π_p stands for the projection map from $\varprojlim \{\widehat{L_p}, i_p^{q*}\}$ onto $\widehat{L_p}$. From (5) we deduce that σ is injective. Since the i_p are injective, it follows that the σ_p are surjective, and consequently that σ is surjective. Thus σ is a continuous bijection, and as $(\ker \rho_{\mathbf{B}})^{\perp}$ is compact, σ is a homeomorphism and hence a topological isomorphism from $\varprojlim \{\widehat{L_p}, i_p^{q*}\}$ onto $(\ker \rho_{\mathbf{B}})^{\perp}$.

For each $p \in \mathbb{N}$ and each $j = 1, \dots, l$, let $\chi_{j,p} \in \widehat{G}$ be a generator of $K_p \cap \rho_{\alpha_j}(\widehat{G})$. For any $p, q \in \mathbb{N}$ with $p \le q \in \mathbb{N}$ and for each $j = 1, \dots, l$, let $n_p^q(j) \in \mathbb{Z} \setminus \{0\}$ be such that

$$\rho_{\alpha_j}(\chi_{j,p}) = n_p^q(j)\, \rho_{\alpha_j}(\chi_{j,q}).$$

For each j arbitrarily fixed, we can arrange all the $n_p^{p+1}(j)$ to be positive by replacing, if necessary, $\chi_{j,p}$ by $-\chi_{j,p}$ successively as p increases. Since

$$n_p^q(j) = \prod_{r=p}^{q-1} n_r^{r+1}(j), \tag{7}$$

all the $n_p^q(j)$ will then be positive too.

For each $p \in \mathbb{N}$, let $j_p\colon \mathbb{Z}^l \to L_p$ be the homomorphism

$$j_p\colon (a_1, \dots, a_l) \mapsto \big(a_1 \rho_{\alpha_1}(\chi_{1,p}), \dots, a_l \rho_{\alpha_l}(\chi_{l,p})\big).$$

Since j_p is bijective, the dual homomorphism $j_p^*\colon \widehat{L_p} \to \mathbb{T}^l$ is a topological isomorphism. Thus $\{\widehat{L_p}, i_p^{q*}\}$ satisfies condition (i) from Proposition 1.

For any $p, q \in \mathbb{N}$ with $p \leq q \in \mathbb{N}$, let $j_p^q\colon \mathbb{Z}^l \to \mathbb{Z}^l$ be the homomorphism

$$j_p^q\colon \big(a_1, \dots, a_l\big) \mapsto \big(n_p^q(1)a_1, \dots, n_p^q(l)a_l\big).$$

It is easily verified that $i_p^q j_p = j_q j_p^q$ and hence $j_p^* i_p^{q*} = j_p^{q*} j_q^*$. Since j_p^{q*} can be identified with τ_p^q given by (1), we see that $\{\widehat{L_p}, i_p^{q*}\}$ satisfies condition (ii) from Proposition 1.

Assume now that $\#\mathbf{A} \geq 2$. Then $\#\mathbf{B} \geq 2$ and since $\varprojlim\{\widehat{L_p}, i_p^{q*}\}$ is topologically isomorphic with $(\ker \rho_{\mathbf{B}})^{\perp}$ and

$$H \cap \big(\ker \rho_{\mathbf{B}}\big)^{\perp} \subset \big((\ker \rho_{\mathbf{B}})^{\perp}\big)_{(n)},$$

it follows from Proposition 1 that $(\ker \rho_{\mathbf{B}})^{\perp} \setminus (H \cap (\ker \rho_{\mathbf{B}})^{\perp})$ is connected, a contradiction. Therefore $\#\mathbf{A} = 1$ and consequently, in view of the arbitrariness of $\mathbf{A}$, A is a singleton. In particular, $\mathbf{A} = \mathbf{B} = A$, $l = 1$, and, by (4), $(\ker \rho_{\mathbf{B}})^{\perp} = G$.

Repeating the argument, we infer that $\{\widehat{L_p}, i_p^{q*}\}$ does not satisfy condition (iii) of Proposition 1. Now either there exists $p_0 \in \mathbb{N}$ such that $n_p^{p+1}(1) = 1$ for all $p \in \mathbb{N}$ with $p \geq p_0$, or there is a sequence $\{p_k\}_{k \in \mathbb{N}}$ in $\mathbb{N}$ diverging to infinity such that $n_{p_k}^{p_k+1}(1) = 1$ for all $k \in \mathbb{N}$. Using (7), it is easy to see that the first possibility holds precisely when condition (iii) of Proposition 1 is met. This implies that the second possibility holds, and now appealing to (7) again, we find that $n_p^q(1) = 1$ for all $p, q \in \mathbb{N}$ with $p, q \geq p_0$. Consequently, $\varprojlim\{\Sigma_p, \pi_p^q\}$ reduces to a group topologically isomorphic with $\mathbb{T}$. As $\varprojlim\{\Sigma_p, \pi_p^q\}$ is topologically isomorphic with $(\ker \rho_{\mathbf{B}})^{\perp}$ and as the latter group coincides with G, we finally conclude that G is topologically isomorphic with $\mathbb{T}$. $\square$

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