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Some geometric properties related to fixed point theory in Cesàro spaces

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Abstract

It is proved that for any $p\in(1,\infty)$ the Cesàro sequence space ces_p is $(kNUC)$ for any natural number k and it has the uniform Opial property. Moreover, weakly convergent sequence coefficient of those spaces is also calculated. It is also proved that for $1<p<\infty$ the spaces ces_p have property (L) and weak uniform normal structure. The packing rate of those spaces is also calculated.

1. Introduction

Our main aim is to calculate the weakly convergent sequence coefficient for Cesàro sequence space ces_p and to prove that for any $p \in (1,\infty)$, ces_p is $(kNUC)$ for any integer $k \geq 2$ and has the uniform Opial property and property (L). The

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weakly convergent sequence coefficient which is connected with normal structure is an important geometric constant. It was introduced by Bynum (see [4]).

In the following, X denotes a *Banach space* and $S(X)$ denotes the unit sphere of X.

For a sequence $\{x_n\} \subset X$, we consider

$$A(\{x_n\}) = \lim_{n\to\infty} \{\sup\{\|x_i - x_j\|: i,j \geq n, i \neq j\}\}$$

and

$$A_1(\{x_n\}) = \lim_{n\to\infty} \{\inf\{\|x_i - x_j\|: i,j \geq n, i \neq j\}\}.$$

The weakly convergent sequence coefficient of X, denoted by $WCS(X)$, is defined as follows:

$WCS(X) = \sup\{k > 0$: for each weakly convergent sequence $\{x_n\}$, there is $y \in \operatorname{conv}(\{x_n\})$ such that $k \cdot \limsup_{n\to\infty} \|x_n - y\| \leq A(\{x_n\})\}$, where $\operatorname{conv}(\{x_n\})$ denotes the convex hull of the elements of $\{x_n\}$ (see [4]).

The number $M(X) = 1/WCS(X)$ for a reflexive *Banach space* is called the *Maluta coefficient* and it is known that $M(X) = 1$ for every non-reflexive Banach space X (see [16]). It is also well known that a *Banach space* X with $WCS(X) > 1$ has weak normal structure (see [4]).

A sequence $\{x_n\}$ is said to be an asymptotic equidistant sequence if $A(\{x_n\}) = A_1(\{x_n\})$ (see [20]).

The formula $WCS(X) = \inf\{A(\{x_n\})\colon \{x_n\} \subset S(X)$ and $x_n \xrightarrow{w} 0\} = \inf\{A(\{x_n\})\colon \{x_n\}$ an asymptotic equidistant sequence in $S(X)$ and $x_n \xrightarrow{w} 0\}$ was obtained in [20].

A Banach space X is said to have weak uniform normal structure if $WCS(X) > 1$.

Recall that for a number $\varepsilon > 0$ a sequence $\{x_n\}$ is said to be an ε-separated sequence if

$$\operatorname{sep}(\{x_n\}) := \inf\{\|x_n - x_m\|: n \neq m\} > \varepsilon.$$

A Banach space X is said to have the uniform Kadec-Klee property ((UKK) for short) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if x is the weak limit of a normalized ε-separated sequence, then $\|x\| < 1 - \delta$ (see [10]).

The notion of nearly uniform convexity for *Banach spaces* was introduced in [10]. It is an infinite dimensional counterpart of the classical uniform convexity.

A Banach space X is said to be the nearly uniformly convex ((NUC) for short) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $\{x_n\} \subset B(X) := \{x \in X\colon \|x\| \leq 1\}$ with $\operatorname{sep}(\{x_n\}) > \varepsilon$, we have

$$\operatorname{conv}(\{x_n\}) \cap (1 - \delta)B(X) \neq \emptyset.$$

It is easy to see that every (NUC) space has the (UKK) property. Huff (see [10]) proved that X is (NUC) if and only if X is reflexive and X has the (UKK) property.

Let $k \geq 2$ be an integer. A Banach space X is said to be $(kNUC)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $\{x_n\} \subset B(X)$ with $\operatorname{sep}(x_n) > \varepsilon$ there are $n_1, n_2, \ldots, n_k \in N$ for which $\left\| \frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k} \right\| < 1 - \delta$ (see [13]). Of course a Banach space X is (NUC) whenever it is $(kNUC)$ for some integer $k \geq 2$.

A Banach space X is said to have the Opial property if every sequence $\{x_n\}$ weakly convergent to x_0 satisfies

$$\liminf_{n\to\infty} \|x_n - x_0\| \leq \liminf_{n\to\infty} \|x_n - x\|$$

for every $x \in X$ (see [17]).

Opial proved in [17] that l_p $(1 < p < \infty)$ satisfies this property but the spaces $L_p[0, 2\pi]$ $(p \neq 2,\ 1 < p < \infty)$ do not. Franchetti has shown in [8] that any infinite dimensional Banach space has an equivalent norm satisfying the Opial property.

A Banach space X is said to have the uniform Opial property if for every $\varepsilon > 0$ there exists $\tau > 0$ such that for each weakly null sequence $\{x_n\} \subset S(X)$ and $x \in X$ with $\|x\| \geq \varepsilon$, we have

$$1 + \tau \leq \liminf_{n\to\infty} \|x_n + x\|$$

(see [18]).

For a bounded subset $A \subset X$, the set-measure of noncompactness was defined in [12] by

$$\begin{aligned}\alpha(A) = \inf\{\varepsilon > 0\colon &\ A \text{ can be covered by finitely}\\ &\text{many sets of diameter } \leq \varepsilon\}.\end{aligned}$$

The ball-measure of noncompactness is defined by (see [9] and [11])

$$\begin{aligned}\beta(A) = \inf\{\varepsilon > 0\colon &\ A \text{ can be covered by finitely}\\ &\text{many balls of diameter } \leq \varepsilon\}.\end{aligned}$$

The functions α and β are called the Kuratowski measure of noncompactness and the Hausdorff measure of noncompactness in X, respectively. We can associate these functions with the notions of the set-contraction and the ball-contraction (see [6]). These notions are very useful tools to study nonlinear operator problems (see [6] and [18]).

The packing rate of a Banach space X is denoted by $\gamma(X)$ and it is defined by the formula

$$\gamma(X) = \delta(X)/\sigma(X),$$

where $\delta(X)$ and $\sigma(X)$ are defined as the supremum and the infimum, respectively, of the set

$$\Big\{\frac{\beta(A)}{\alpha(A)}\colon A\subset X,\quad A \text{ is } \alpha\text{-minimal},\ \ \alpha(A)>0\Big\}.$$

Recall that $A \subset X$ is said to be α-minimal if $\alpha(B) = \alpha(A)$ for any infinite subset B of A. For those definitions and for results concerning the existence of α-minimal and β-minimal sets we refer to [2, Chapter X].

For each $\varepsilon > 0$ define $\Delta(\varepsilon) = \inf\{1-\inf[\|x\|\colon x \in A]\colon A$ is a closed convex subset of $B(X)$ with $\beta(A) \geq \varepsilon\}$. The function Δ is called the modulus of noncompact convexity (see [9]).

A Banach space X is said to have property (L) if $\lim\limits_{\varepsilon\to 1-} \Delta(\varepsilon) = 1$. It has been proved in [18] that property (L) is a useful tool in the fixed point theory and that a Banach space X has property (L) if and only if it is reflexive and has the uniform Opial property.

For the definition of normal structure and weak uniform normal structure we refer to [2], [3] and [9].

The *Cesàro sequence space* was defined by J.S. Shue in 1970. It is useful in the theory of matrix operators and others. In this paper, we deal with the above geometric properties of Cesàro sequence spaces.

Let l^0 be the space of all real sequences. For $1 < p < \infty$, the *Cesàro sequence space* ces_p is defined by

$$\mathrm{ces}_p = \Big\{x \in l^0\colon \|x\| = \Big(\sum_{n=1}^{\infty}\Big[\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big]^p\Big)^{1/p} < \infty\Big\}.$$

2. Results

Theorem 1

If $1 < p < \infty$, then the space ces_p is $(kNUC)$ for any integer $k \geq 2$.

Proof. Let $\varepsilon > 0$ be given. For every sequence $\{x_n\} \subset B(X)$ with $\mathrm{sep}(\{x_n\}) > \varepsilon$, we put $x_n^m = (0, 0, \ldots, 0, x_n(m), x_n(m+1), \ldots)$. For each $i \in N$, the sequence $\{x_n(i)\}_{i=1}^{\infty}$ is bounded. Therefore, using the diagonal method one can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the sequence $\{x_{n_k}(i)\}$ converges for each $i \in N$.

Therefore, for any $m \in N$ there exists k_m such that $\operatorname{sep}(\{x^m_{n_k}\}_{k \geq k_m}) \geq \varepsilon$. Hence for each $m \in N$ there exists $n_m \in N$ such that

$$\|x^m_{n_m}\| \geq \frac{\varepsilon}{2}. \tag{0}$$

Write $I_p(x) = \sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{i=1}^{n} |x(i)|\Big)^p$ and put $\varepsilon_1 = \frac{k^{p-1}-1}{2k^p(k-1)}\Big(\frac{\varepsilon}{2}\Big)^p$. Then there exists $\delta > 0$ such that

$$|I_p(x+y) - I_p(x)| < \varepsilon_1 \tag{1}$$

whenever $I_p(x) \leq 1$ and $I_p(y) \leq \delta$ (see [5]).

There exists $m_1 \in \mathcal{N}$ such that $I_p(x_1^{m_1}) \leq \delta$. Next, there exists $m_2 > m_1$ such that $I_p(x_2^{m_2}) \leq \delta$. In such a way, there exists $m_2 < m_3 < \cdots < m_{k-1}$ such that $I_p(x_j^{m_j}) \leq \delta$ for all $j = 1, 2, \ldots, k-1$. Define $m_k = m_{k-1} + 1$. By condition (0), there exists $n_k \in \mathcal{N}$ such that $I_p(x^{m_k}_{n_k}) \geq (\frac{\varepsilon}{2})^p$. Put $n_i = i$ for $1 \leq i \leq k-1$. Then in virtue of (0), (1) and convexity of the function $f(u) = |u|^p$, we get

$$\begin{aligned}
I_p\Big(\frac{x_{n_1} + x_{n_2} + \cdots + x_{n_{k-1}} + x_{n_k}}{k}\Big) &= \sum_{n=1}^{m_1}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big|\frac{x_{n_1}(i) + \cdots + x_{n_k}(i)}{k}\Big|\Big)^p \\
&\quad + \sum_{n=m_1+1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big|\frac{x_{n_1}(i) + x_{n_2}(i) + \cdots + x_{n_{k-1}}(i) + x_{n_k}(i)}{k}\Big|\Big)^p \\
&\leq \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\Big(\frac{1}{n}\sum_{i=1}^{n}|x_{n_j}(i)|\Big)^p + \sum_{n=m_1+1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big|\frac{x_{n_2}(i) + \cdots + x_{n_k}(i)}{k}\Big|\Big)^p + \varepsilon_1 \\
&= \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\Big(\frac{1}{n}\sum_{i=1}^{n}|x_{n_j}(i)|\Big)^p + \sum_{n=m_1+1}^{m_2}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big|\frac{x_{n_2}(i) + \cdots + x_{n_k}(i)}{k}\Big|\Big)^p \\
&\quad + \sum_{n=m_2+1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big|\frac{x_{n_2}(i) + x_{n_3}(i) + \cdots + x_{n_{k-1}}(i) + x_{n_k}(i)}{k}\Big|\Big)^p + \varepsilon_1 \\
&\leq \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\Big(\frac{1}{n}\sum_{i=1}^{n}|x_{n_j}(i)|\Big)^p + \sum_{n=m_1+1}^{m_2}\frac{1}{k}\sum_{j=2}^{k}\Big(\frac{1}{n}\sum_{i=1}^{n}|x_{n_j}(i)|\Big)^p \\
&\quad + \sum_{n=m_3+1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}\Big|\frac{x_{n_3}(i) + x_{n_4}(i) + \cdots + x_{n_{k-1}}(i) + x_{n_k}(i)}{k}\Big|\Big)^p + 2\varepsilon_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\leq \frac{I_p(x_{n_1}) + \cdots + I_p(x_{n_{k-1}})}{k} + \frac{1}{k}\sum_{n=1}^{m_k-1}\Big(\frac{1}{n}\Big(\sum_{i=1}^{n}|x_{n_k}(i)|\Big)\Big)^p \\
&\quad + \sum_{n=m_{k-1}+1}^{\infty}\Big(\frac{1}{n}\Big(\sum_{i=1}^{n}\frac{|x_{n_k}(i)|}{k}\Big)\Big)^p + (k-1)\varepsilon_1
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{k-1}{k} + \frac{1}{k}\sum_{n=1}^{m_k-1}\Big(\frac{1}{n}\Big(\sum_{i=1}^{n}|x_{n_k}(i)|\Big)\Big)^p \\
&\quad + \frac{1}{k^p}\sum_{n=m_{k-1}+1}^{\infty}\Big(\frac{1}{n}\Big(\sum_{i=1}^{n}|x_{n_k}(i)|\Big)\Big)^p + (k-1)\varepsilon_1 \\
&= 1 - \frac{1}{k} + \frac{1}{k}\Big(1 - \sum_{n=m_{k-1}+1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|x_{n_k}(i)|\Big)\Big)^p \\
&\quad + \frac{1}{k^p}\sum_{n=m_{k-1}+1}^{\infty}\Big(\frac{1}{n}\Big(\sum_{i=1}^{n}|x_{n_k}(i)|\Big)\Big)^p + (k-1)\varepsilon_1 \\
&\le 1 + (k-1)\varepsilon_1 - \Big(\frac{k^{p-1}-1}{k^p}\Big)\sum_{n=m_{k-1}+1}^{\infty}\Big(\frac{1}{n}\Big(\sum_{i=1}^{n}|x_{n_k}(i)|\Big)\Big)^p \\
&\le 1 + (k-1)\varepsilon_1 - \Big(\frac{k^{p-1}-1}{k^p}\Big)(\frac{\varepsilon}{2})^p = 1 - \frac{1}{2}\Big(\frac{k^{p-1}-1}{k^p}\Big)(\frac{\varepsilon}{2})^p.
\end{aligned}
$$

Therefore, ces_p is $(kNUC)$ for any integer $k \ge 2$. □

Theorem 2

For any $1 < p < \infty$, the space ces_p has the uniform Opial property.

Proof. For any $\varepsilon > 0$ we can find a positive number $\varepsilon_0 \in (0, \varepsilon)$ such that

$$1 + \frac{\varepsilon^p}{2} > (1 + \varepsilon_0)^p.$$

Let $x \in X$ and $\|x\| \ge \varepsilon$. There exists $n_1 \in \mathcal{N}$ such that

$$\sum_{i=n_1+1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^p < (\frac{\varepsilon_0}{4})^p.$$

Hence we have

$$\Big\|\sum_{i=n_1+1}^{\infty} x(i)e_i\Big\| < \frac{\varepsilon_0}{4} < \frac{\varepsilon}{4},$$

where $e_i = (0, \ldots, \overset{ith}{1}, 0, 0, \ldots)$. Furthermore, we have

$$
\begin{aligned}
\varepsilon^p &\le \sum_{n=1}^{n_1}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^p + \sum_{n=n_1+1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^p \\
&< \sum_{n=1}^{n_1}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^p + (\frac{\varepsilon_0}{4})^p < \sum_{n=1}^{n_1}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^p + \frac{\varepsilon^p}{4},
\end{aligned}
$$

whence

$$\frac{3\varepsilon^p}{4} \leq \sum_{n=1}^{n_1} \Big(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \Big)^p .$$

For any weakly null sequence $\{x_m\} \subset S(X)$, in virtue of $x_m(i) \to 0$ for $i = 1, 2, \dots,$ there exists $m_0 \in N$ such that

$$\Big\| \sum_{n=1}^{n_1} x_m(i) e_i \Big\| < \frac{\varepsilon_0}{4}$$

when $m > m_0$. Therefore,

$$\begin{aligned} \|x_m + x\| &= \Big\| \sum_{i=1}^{n_1} (x_m(i) + x(i)) e_i + \sum_{i=n_1+1}^{\infty} (x_m(i) + x(i)) e_i \Big\| \\ &\geq \Big\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \Big\| - \Big\| \sum_{i=1}^{n_1} x_m(i) e_i \Big\| - \Big\| \sum_{i=n_1+1}^{\infty} x(i) e_i \Big\| \\ &\geq \Big\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \Big\| - \frac{\varepsilon_0}{2} \end{aligned}$$

when $m > m_0$. Moreover for $a := \sum_{i=1}^{n_1} |x(i)|$ there holds

$$\begin{aligned} &\Big\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \Big\|^p \\ &= \sum_{n=1}^{n_1} \Big(\frac{1}{n} \sum_{i=1}^{n} |x(i) e_i| \Big)^p + \sum_{n=n_1+1}^{\infty} \Big(\frac{1}{n} \sum_{i=1}^{n} (a + |x_m(i)|) \Big)^p \\ &\geq \sum_{n=1}^{n_1} \Big(\frac{1}{n} \sum_{i=1}^{n} |x(i) e_i| \Big)^p + \sum_{n=n_1+1}^{\infty} \Big(\frac{1}{n} \sum_{i=1}^{n} |x_m(i)| \Big)^p \\ &\geq \frac{3\varepsilon^p}{4} + \Big(1 - \frac{\varepsilon^p}{4} \Big) = 1 + \frac{\varepsilon^p}{2} > (1 + \varepsilon_0)^p . \end{aligned}$$

Therefore, combining this with the previous inequality, we get

$$\begin{aligned} \|x_m + x\| &\geq \Big\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \Big\| - \frac{\varepsilon_0}{2} \\ &\geq 1 + \varepsilon_0 - \frac{\varepsilon_0}{2} = 1 + \frac{\varepsilon_0}{2} . \end{aligned}$$

This means that ces_p has the uniform Opial property. □

By the facts presented in the introduction and the reflexity of ces_p for $1 < p < \infty$, we get the following

Corollary 1

For $1 < p < \infty$ the space ces_p has property (L) *and the fixed point property.*

We will now calculate the weakly convergence sequence coefficient of ces_p.

Theorem 3

There holds the equality $WCS(\mathrm{ces}_p) = 2^{1/p}$ whenever $1 < p < \infty$.

Proof. Take any $\varepsilon > 0$ and an asymptotic equidistant sequence $\{x_n\} \subset S(X)$ with $x_n \xrightarrow{w} 0$ and put $v_1 = x_1$. There exists $i_1 \in \mathcal{N}$ such that

$$\Big\| \sum_{i=i_1+1}^{\infty} v_1(i)e_i \Big\| < \varepsilon.$$

Since $x_n \to 0$ coordinatewise, there exists $n_2 \in \mathcal{N}$ such that

$$\Big\| \sum_{i=1}^{i_1} x_n(i)e_i \Big\| < \varepsilon$$

whenever $n \geq n_2$.

Take $v_2 = x_{n_2}$. Then there is $i_2 > i_1$ such that

$$\Big\| \sum_{i=i_2+1}^{\infty} v_2(i)e_i \Big\| < \varepsilon.$$

Since $x_n(i) \to 0$ coordinatewise, there exists $n_3 \in \mathcal{N}$ such that

$$\Big\| \sum_{i=1}^{i_2} x_n(i)e_i \Big\| < \varepsilon$$

whenever $n \geq n_3$.

Continuing this process in such a way by induction, we get a subsequence $\{v_n\}$ of $\{x_n\}$ such that

$$\Big\| \sum_{i=i_n+1}^{\infty} v_n(i)e_i \Big\| < \varepsilon$$

and

$$\Big\| \sum_{i=1}^{i_n} v_{n+1}(i)e_i \Big\| < \varepsilon.$$

Put

$$z_n = \sum_{i=i_{n-1}+1}^{i_n} v_n(i)e_i$$

for $n = 2, 3, \dots$. Then

$$\text{(2)} \qquad 1 \geq \|z_n\| = \Big\|\sum_{i=1}^{\infty} v_n(i)e_i - \sum_{i=1}^{i_{n-1}} v_n(i)e_i - \sum_{i=i_n+1}^{\infty} v_n(i)e_i\Big\|$$

$$\geq \Big\|\sum_{i=1}^{\infty} v_n(i)e_i\Big\| - \Big\|\sum_{i=1}^{i_{n-1}} v_n(i)e_i\Big\| - \Big\|\sum_{i=i_n+1}^{\infty} u_n(i)e_i\Big\| > 1 - 2\varepsilon.$$

Moreover, for any $n, m \in \mathcal{N}$ with $n \neq m$, we have

$$\text{(3)} \qquad \|v_n - v_m\| = \Big\|\sum_{i=1}^{\infty} u_n(i)e_i - \sum_{i=1}^{\infty} v_m(i)e_i\Big\|$$

$$\geq \Big\|\sum_{i=i_{n-1}+1}^{i_n} v_n(i)e_i - \sum_{i=i_{m-1}+1}^{i_m} v_m(i)e_i\Big\| - \Big\|\sum_{i=1}^{i_{n-1}} v_n(i)e_i\Big\|$$

$$- \Big\|\sum_{i=i_n+1}^{\infty} v_n(i)e_i\Big\| - \Big\|\sum_{i=1}^{i_{m-1}} v_m(i)e_i\Big\| - \Big\|\sum_{i=i_m+1}^{\infty} u_m(i)e_i\Big\|$$

$$\geq \|z_n - z_m\| - 4\varepsilon.$$

This means that $A(\{x_n\}) = A(\{v_n\}) \geq A(\{z_n\}) - 4\varepsilon$.

Put $u_n = \frac{z_n}{\|z_n\|}$ for $n = 2, 3, \dots$. Then

$$\text{(4)} \qquad u_n \in S(\mathrm{ces}_p);$$

$$\text{(5)} \qquad A(\{x_n\}) \geq 1 - \varepsilon A(\{u_n\}) - 4\varepsilon.$$

On the other hand

$$\|v_n - v_m\| \leq \|z_n - z_m\| + 4\varepsilon \leq \|u_n - u_m\| + 4\varepsilon$$

for every $m, n \in \mathcal{N},\ m \neq n$. Therefore

$$\text{(6)} \qquad A(\{u_n\}) \geq A(\{x_n\}) - 4\varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we have from (4), (5) and (6) that $WCS(\mathrm{ces}_p) = \inf\Big\{A(\{u_n\})\colon u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n(i)e_i \in S(\mathrm{ces}_p),\quad 0 = i_0 < i_1 < i_2 < \dots, u_n \xrightarrow{w} 0\Big\}$.

Using Lemma 2 in [20], we have $WCS(\text{ces}_p) = \inf\Big\{A(\{u_n\})\colon u_n = \sum_{i=i_{n-1}+1}^{i_n} u_n(i)e_i \in S(\text{ces}_p),\ 0 = i_0 < i_1 < i_2 < \cdots,\ u_n \xrightarrow{w} 0$ and $\{u_n\}$ is asymptotic equidistant$\Big\}$.

Take $m \in \mathcal{N}$ large enough such that

$$\sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{b}{k}\Big)^p < \varepsilon,$$

where $b := \sum_{i=i_{n-1}+1}^{i_n} |u_n(i)|$. We have for $n < m$

$$\begin{aligned}
&\|u_n - u_m\|^p \\
&= \sum_{k=i_{n-1}+1}^{i_{m-1}} \Big(\frac{1}{k}\sum_{i=1}^{k}|u_n(i)|\Big)^p + \sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{1}{k}\Big(b + \sum_{i=1}^{k}|u_m(i)|\Big)\Big)^p \\
&\geq \sum_{k=i_{n-1}+1}^{i_{m-1}} \Big(\frac{1}{k}\sum_{i=1}^{k}|u_n(i)|\Big)^p + \sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\Big)^p \\
&= \sum_{k=i_{n-1}+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k}|u_n(i)|\Big)^p - \sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{b}{k}\Big)^p + \sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\Big)^p \\
&> 1 - \varepsilon + 1 = 2 - \varepsilon, \quad \text{i.e. } A_1(\{u_1\}) \geq (2-\varepsilon)^{1/p}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\Big[\sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{1}{k}\Big(b + \sum_{i=1}^{k}|u_m(i)|\Big)\Big)^p\Big]^{1/p} = \Big[\sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{b}{k} + \frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\Big)^p\Big]^{1/p} \\
&\leq \Big[\sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{b}{k}\Big)^p\Big]^{1/p} + \Big[\sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\Big)^p\Big]^{1/p} < \varepsilon^{1/p} + 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|u_n - u_m\|^p \\
&= \sum_{k=i_{n-1}+1}^{i_{m-1}} \Big(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\Big)^p + \sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{1}{k}\Big(b + \sum_{i=1}^{k}|u_m(i)|\Big)\Big)^p \\
&\leq \sum_{k=i_{n-1}+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k}|u_m(i)|\Big)^p + \sum_{k=i_{m-1}+1}^{\infty} \Big(\frac{1}{k}\Big(b + \sum_{i=1}^{k}|u_m(i)|\Big)\Big)^p \\
&\leq 1 + (1 + \varepsilon^{1/p})^p
\end{aligned}$$

for any $n, m \in \mathcal{N},\ m \neq n$. This yields $A(\{u_n\}) \leq \left[1 + \left(1 + \varepsilon^{1/p}\right)^p\right]^{1/p}$ and, by the arbitrariness of $\varepsilon > 0$, we obtain $WCS(\text{ces}_p) = 2^{1/p}$.

Corollary 2

For $1 < p < \infty$, ces_p *has the weak uniform normal structure and normal structure.*

Corollary 3

For any $1 < p < \infty$ *there holds the equality* $\gamma(\text{ces}_p) = 2^{(p-1)/p}$.

Proof. By [1], if X is reflexive Banach space with the uniform Opial property, then $\gamma(X) = \frac{2}{WCS(X)}$. Since, by Theorem 1, ces_p is (NUC) for $1 < p < \infty$ and property (NUC) implies reflexivity, Theorem 3 yields $\gamma(\text{ces}_p) = 2^{(p-1)p}$. □

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