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A support theorem for the complex Radon transform

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Abstract

Let $C_c(\mathbb{C}^n)$ be the space of compactly supported continuous functions on $\mathbb{C}^n$. For $f \in C_c(\mathbb{C}^n)$, $\hat{f}$ denotes the complex Radon transform of f. We say that for a compact set K the support theorem holds if every function $\varphi \in C_c(\mathbb{C}^n)$ with $\operatorname{supp}(\hat{\varphi}) \subset \hat{K}$ vanishes outside K. The goal of this paper is to establish conditions on K under which the support theorem holds for K.

If f is a function defined on $\mathbb{R}^n(\mathbb{C}^n)$, the classical Radon transform of f is a function $\hat{f}$ defined on hyperplanes; the value of $\hat{f}$ at a given hyperplane is the integral of f over that hyperplane. The Radon transform was studied by J. Radon [9], F. John [5, 6], S. Helgason [3, 4], I.M. Gel′fand, M.I. Graev, and N. Ya. Vilenkin [1], D. Ludwig [7]. One of the basic results on the classical Radon transform is Helgason′s support theorem [3]: A rapidly decreasing function must vanish outside a ball if its real Radon transform does. This theorem holds for every convex compact set in $\mathbb{R}^n$ [4], [7]. There are many applications of the support theorem. For instance, some of applications in complex analysis are contained in [2], [10]. In the present paper we investigate the relation between the supports of f and $\hat{f}$ in the case of the complex Radon transform.

Notation. For $z, w \in \mathbb{C}^n$ we set $\langle z, w\rangle = \sum z_j w_j$; the standard Lebesgue measure in $\mathbb{C}^n$ is $d\omega_{2n}$, and $S^{2n-1} = \{z \in \mathbb{C}^n |\ |z| = 1\}$, $B^n(z, R) = \{w \in \mathbb{C}^n |\ |w - z| < R\}$.

$\mathcal{D}(\mathbb{C}^n)$ denotes the space consisting of all infinitely differentiable functions with compact support, and $C_c(\mathbb{C}^n)$ is used to denote the space of compactly supported continuous functions on $\mathbb{C}^n$. If $\varphi \in C_c(\mathbb{C}^n)$, the standard complex Radon transform of φ (denoted by $\hat{\varphi}$) is defined by

$$\hat{\varphi}(\xi, s) = \frac{1}{|\xi|^2} \int\limits_{\langle z,\xi\rangle = s} \varphi(z)\, d\lambda(z), \tag{1}$$

where $(\xi, s) \in (\mathbb{C}^n \setminus 0) \times \mathbb{C}$, and $d\lambda(z)$ is the area element on the hyperplane $\{z : \langle z, \xi\rangle = s\}$. For a set $A \subset \mathbb{C}^n$, we denote by $\hat{A}$ the set of all $(\xi, s) \in (\mathbb{C}^n \setminus 0) \times \mathbb{C}$ such that the hyperplane $\{z : \langle z, \xi\rangle = s\}$ meets A.

A set $A \subset \mathbb{C}^n$ is called linearly convex if, for every $w \notin A$, there is a complex hyperplane $\{z : \langle z, \xi\rangle = s\}$ which contains w and does not meet A (see Martineau [8]).

Let $K \subset \mathbb{C}^n$ be a linearly convex compact set. We say that for K the support theorem holds if every function $\varphi \in C_c(\mathbb{C}^n)$ with $\mathrm{supp}(\hat{\varphi}) \subset \hat{K}$ vanishes outside K.

We consider the problem of description of linearly convex compact sets for which the support theorem holds. The following theorem yields the sufficient condition.

Theorem 1

Let $K \subset \mathbb{C}^n$ be a linearly convex compact set. Suppose that for every $z \notin K$ there exists a hyperplane $P = \{\lambda : \langle \lambda, \xi_0\rangle = s_0\}$ satisfying the following conditions:
(i) P contains z.
(ii) P does not meet K.
(iii) The set $\mathbb{C} \setminus K_{\xi_0}$ is connected, where $K_{\xi_0} = \{\langle \lambda, \xi_0\rangle\}_{\lambda \in K}$ is the projection of K on ξ_0.

Then the support theorem holds for K.

Corollary

Let $K_j \subset \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K_j$ is connected, $j = 1, \ldots, n$. Then the support theorem holds for

$$K = K_1 \times K_2 \times \ldots K_n.$$

The proof of Theorem 1 is based on the following result:

Theorem 2

Let $\varphi(z) \in C_c(\mathbb{C}^n)$ and let $(\xi_0, s_0) \notin \mathrm{supp}\,\hat{\varphi}$, where $\hat{\varphi}(\xi, s)$ is the Radon transform of φ. Suppose that there exists a connected unbounded open set $D \subset \mathbb{C}$ such that $s_0 \in D$ and $(\{\xi_0\} \times D) \cap \mathrm{supp}\,\hat{\varphi} = \emptyset$. Then the function φ vanishes on the hyperplane $\{z : \langle z, \xi_0\rangle = s_0\}$.

Proof of Theorem 2. Let $\psi \in \mathcal{D}(\mathbb{C}^n)$ and let $P(z,\bar{z})$ be a homogeneous polynomial of bidegree (k,m). Denote by $\hat{\psi}(\xi,s)$ and $\widehat{P\psi}(\xi,s)$ respectively the Radon transforms of $\psi(z)$ and $P(z,\bar{z})\psi(z)$. Then

$$\frac{\partial^{k+m}\widehat{P\psi}(\xi,s)}{\partial s^k \partial \bar{s}^m} = (-1)^{k+m} P\Big(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \bar{\xi}}\Big)\hat{\psi}(\xi,s). \tag{2}$$

An analogue of this formula for the real Radon transform is proved in [6, Chapter I]. The same proof is valid in the case of the complex Radon transform.

Suppose that $K \subset \mathbb{C}^n$, $\varphi \in C_c(\mathbb{C}^n)$, $(\xi_0, s_0) \in (\mathbb{C}^n \setminus 0) \times \mathbb{C}$ and $D \subset \mathbb{C}$ satisfy the hypotheses of the theorem. We assume at first that $\varphi \in \mathcal{D}(\mathbb{C}^n)$. For every $(\xi,s) \notin \operatorname{supp}\hat{\varphi}$ the function $\hat{\varphi}$ vanishes on some neighborhood of (ξ,s). In particular, we have

$$\frac{\partial^{|\alpha|+|\beta|}\hat{\varphi}(\xi_0,s)}{\partial \xi^\alpha \partial \bar{\xi}^\beta} = 0, \quad s \in D \tag{3}$$

for all multi-indices α and β.

Let α and β be multi-indices with $|\alpha| + |\beta| > 1$. Assume, without loss of generality, that $|\alpha| > 0$. Let $\varphi_{\alpha\beta}(z) = z^\alpha \bar{z}^\beta \varphi(z)$. It follows from (2) and (3) that

$$\frac{\partial^{|\alpha|+|\beta|}\hat{\varphi}_{\alpha\beta}(\xi_0,s)}{\partial s^{|\alpha|} \partial \bar{s}^{|\beta|}} = 0, \quad s \in D. \tag{4}$$

Denote by $f(s)$ the function

$$\frac{\partial^{|\alpha|-1+|\beta|}\hat{\varphi}_{\alpha\beta}(\xi_0,s)}{\partial s^{|\alpha|-1} \partial \bar{s}^{|\beta|}}.$$

By (4) the function $\bar{f}(s) = \operatorname{Re}(f(s)) - i\operatorname{Im}(f(s))$ is holomorphic on D. Since $\varphi \in \mathcal{D}(\mathbb{C}^n)$, there is $R > 0$ such that $\hat{\varphi}_{\alpha\beta}(\xi,s)$ vanishes on $\{(\xi,s) : |s| \geq R|\xi|\}$. Thus $\bar{f}(s) = 0$ on $\{s : |s| \geq R|\xi_0|\}$. Since the domain D is unbounded, and $\bar{f}(s)$ is holomorphic on D, it follows from the uniqueness theorem that $\bar{f}(s)$ vanishes on D. Therefore

$$\frac{\partial^{|\alpha|-1+|\beta|}\hat{\varphi}_{\alpha\beta}(\xi_0,s)}{\partial s^{|\alpha|-1} \partial \bar{s}^{|\beta|}} = 0, \quad s \in D.$$

It is easy to see that an induction on $|\alpha|$ gives

$$\frac{\partial^{|\beta|}\hat{\varphi}_{\alpha\beta}(\xi_0,s)}{\partial \bar{s}^{|\beta|}} = 0, \quad s \in D. \tag{5}$$

Similarly, from (5) we obtain

$$\hat{\varphi}_{\alpha\beta}(\xi_0, s) = 0, \quad s \in D.$$

In particular, $\hat{\varphi}_{\alpha\beta}(\xi_0, s_0) = 0$. Thus, for every polynomial $P(z, \bar{z})$, we have

$$\int\limits_{\langle \xi_0, z\rangle = s_0} P(z, \bar{z})\varphi(z)\, d\lambda(z) = 0. \tag{6}$$

Let $\varphi_{\xi_0, s_0}(\mu) \in \mathcal{D}(\mathbb{C}^{n-1})$ be the restriction of φ to the hyperplane $\{z : \langle z, \xi_0\rangle = s_0\}$. Then (6) implies

$$\int\limits_{\mathbb{C}^{n-1}} \varphi_{\xi_0, s_0}(\mu) Q(\mu, \bar{\mu})\, d\omega_{2n-2}(\mu) = 0$$

for every polynomial $Q(\mu, \bar{\mu})$. Therefore $\varphi_{\xi_0, s_0}(\mu) \equiv 0$, i.e., $\varphi(z)$ vanishes on $\{z : \langle z, \xi_0\rangle = s_0\}$.

Now suppose, without the assumption of smoothness, that $\varphi(z) \in C_c(\mathbb{C}^n)$ satisfies the hypotheses of the theorem. Let $\{\chi_m(z)\}_{m=1}^{\infty}$ be a sequence of smooth functions on $\mathbb{C}^n$ with $\operatorname{supp}\chi_m \subset \{z : |z| \le 1/m\}$ that converges in the space of measures to the delta function at the origin. Then the convolution

$$\varphi_m(z) = (\varphi * \chi_m)(z) = \int\limits_{\mathbb{C}^n} \varphi(w)\chi_m(z - w)\, d\omega_{2n}(w)$$

is a smooth function with compact support on $\mathbb{C}^n$, and $\varphi_m(z) \to \varphi(z)$ for every $z \in \mathbb{C}^n$. Let $\hat{\varphi}_m(\xi, s)$ be the Radon transform of $\varphi_m(z)$. Then [6, Chapter II]

$$\hat{\varphi}_m(\xi, s) = \int\limits_{\mathbb{C}} \hat{\varphi}(\xi, \lambda)\hat{\chi}_m(\xi, s - \lambda) d\omega_2(\lambda).$$

We shall show that for $m \ge m_0$ the functions φ_m satisfy the hypotheses of the theorem. We have $\operatorname{supp}\varphi \subset B^n(0, R)$ for some $R > 1$. Then (1) implies that $\hat{\varphi}(\xi, s) = 0$ for $|s|/|\xi| \ge R$. Since D is unbounded, there exists $s_1 \in D$ such that $|s_1|/|\xi_0| > 4R$. Let $\Gamma \subset D$ be a broken line joining s_0 to s_1. Denote by D_Γ^δ the set

$$\bigcup_{s \in \Gamma} B^2(s, \delta).$$

Since $(\{\xi_0\} \times \Gamma) \cap \operatorname{supp}\hat{\varphi} = \emptyset$, and since $\{\xi_0\} \times \Gamma$ is a compact subset of $(\mathbb{C}^n \setminus 0) \times \mathbb{C}$, there exists δ $(0 < \delta \le |\xi_0|/2)$ such that $\hat{\varphi}(\xi, s) = 0$ on $B^n(\xi_0, \delta) \times D_\Gamma^\delta$. By (1),

$\hat{\chi}_m(\xi, s-\lambda) = 0$ for $|s-\lambda|/|\xi| \geq 1/m$. Therefore, for $\xi \in B^n(\xi_0, \delta)$, we have (since $|\xi| \leq 2|\xi_0|$ if $\xi \in B^n(\xi_0, \delta)$)

$$\begin{aligned} |\hat{\varphi}_m(\xi, s)| &\leq \int\limits_{|s-\lambda|\leq|\xi|/m} |\hat{\varphi}(\xi,\lambda)||\hat{\chi}_m(\xi, s-\lambda)|\, d\omega_2(\lambda) \\ &\leq \int\limits_{|s-\lambda|\leq 2|\xi_0|/m} |\hat{\varphi}(\xi,\lambda)||\hat{\chi}_m(\xi, s-\lambda)| d\omega_2(\lambda). \end{aligned} \tag{7}$$

Under the conditions $(\xi, s) \in B^n(\xi_0, \delta) \times D_\Gamma^{\delta/2}$, $|s-\lambda| \leq 2|\xi_0|/m$, $2|\xi_0|/m < \delta/2$, the point (ξ, λ) lies in $B^n(\xi_0, \delta) \times D_\Gamma^{\delta}$,i.e., $\hat{\varphi}(\xi, \lambda) = 0$. Then (7) implies that $\hat{\varphi}_m(\xi, s)$ vanishes on $B^n(\xi_0, \delta) \times D_\Gamma^{\delta/2}$ for $m > 4|\xi_0|/\delta$. Since $\operatorname{supp}\varphi_m \subset B^n(0, R+1/m)$, we have $\hat{\varphi}_m(\xi, s) = 0$ for $|s|/|\xi| \geq R + 1/m$. From this it follows that $\hat{\varphi}_m(\xi, s) = 0$ for $(\xi, s) \in B^n(\xi_0, \delta) \times \{s : |s| > 4R|\xi_0|\}$. We set

$$\tilde{D} = D_\Gamma^{\delta/2} \cup \{s : |s| > 4R|\xi_0|\}.$$

Since $s_1 \in D_\Gamma^{\delta/2} \cap \{s : |s| > 4R|\xi_0|\}$, $\tilde{D}$ is a connected unbounded open set. For $m \geq m_0$ the functions $\hat{\varphi}_m(\xi, s)$ vanish on $B^n(\xi_0, \delta) \times \tilde{D}$. By what has been proved, for $m \geq m_0$ the functions φ_m vanish on the hyperplane $P = \{z : \langle z, \xi_0\rangle = s_0\}$. Since the sequence $\{\varphi_m(z)\}$ converges to $\varphi(z)$, the function $\varphi(z)$ vanishes on P. The theorem is proved. $\square$

Proof of Theorem 1. Let $\hat{\varphi}(\xi, s)$ be the Radon transform of $\varphi(z) \in C_c(\mathbb{C}^n)$. Suppose that $\hat{\varphi}(\xi, s) = 0$ for $(\xi, s) \notin \hat{K}$. Fix $z_0 \notin K$. Then there exists a point $(\xi_0, s_0) \in (\mathbb{C}^n \setminus 0) \times \mathbb{C}$ such that $\{z : \langle z, \xi_0\rangle = s_0\} \cap K = \emptyset$, $\langle z_0, \xi_0\rangle = s_0$ and the set $\mathbb{C} \setminus \{\langle z, \xi_0\rangle\}_{z\in K}$ is connected. The set $K_{\xi_0} = \{\langle z, \xi_0\rangle\}_{z\in K}$ is compact because $K_{\xi_0} = f_{\xi_0}(K)$, where $f_{\xi_0}(z) = \langle \xi_0, z\rangle$. Thus $\mathbb{C} \setminus K_{\xi_0}$ is a connected unbounded open set. It is easy to see that $\hat{K}$ is (relatively) closed in $(\mathbb{C}^n \setminus 0) \times \mathbb{C}$. Then $\operatorname{supp}\hat{\varphi} \subset \hat{K}$. Since $(\{\xi_0\} \times (\mathbb{C} \setminus K_{\xi_0})) \cap \hat{K} = \emptyset$, we have $(\{\xi_0\} \times (\mathbb{C} \setminus K_{\xi_0})) \cap \operatorname{supp}\hat{\varphi} = \emptyset$. Then by Theorem 2, $\varphi(z)$ vanishes on the hyperplane $P = \{z : \langle z, \xi_0\rangle = s_0\}$ that contains the point z_0. Theorem 1 is proved.

We shall show the need for hypothesis (iii) in Theorem 1. Let $K = \{(z_1, z_2) \in \mathbb{C}^2 |\ 1 \leq |z_1| \leq 2, |z_2| \leq 2\}$ and let $\operatorname{conv}(K)$ be the convex hull of K. We have $\operatorname{conv}(K) = \{(z_1, z_2) \in \mathbb{C}^2 |\ |z_1| \leq 2, |z_2| \leq 2\}$. It is easy to see that the hyperplane $P_{\xi,s} = \{z : \langle z, \xi\rangle = s\}$ meets $\operatorname{conv}(K)$ and does not meet K if and only if

$$|s| < |\xi_1| - 2|\xi_2|. \tag{8}$$

In particular, P_{ξ^0,s_0} contains the origin and does not meet K if and only if $s_0 = 0$ and $0 < |\xi_1^0| - 2|\xi_2^0|$. For such a hyperplane we have

$$K_{\xi^0} = \{\langle z, \xi^0 \rangle\}_{z \in K} = \{s \in \mathbb{C} \mid |\xi_1^0| - 2|\xi_2^0| \leq |s| \leq 2|\xi_1^0| + 2|\xi_2^0|\},$$

i.e., $\mathbb{C} \setminus K_{\xi^0}$ is not connected. Therefore K is a linearly convex compact set which does not satisfy the condition (iii) of Theorem 1. Fix $0 < \delta < 1$. Let $h(x) \in \mathcal{D}(\mathbb{R})$ be such that $\varphi_1(z_1) = h(|z_1|^2) = 1$ for $|z_1| \leq 2 - 2\delta$ and $\varphi_1(z_1) = 0$ for $|z_1| \geq 2$. Let the function $g(\lambda) \in \mathcal{D}(\mathbb{C})$ be such that $\operatorname{supp} g \subset \{\lambda : |\lambda| \leq 1/2\}$ and $g\prime(0) = \partial g(0)/\partial\lambda = 1$. We set

$$\varphi(z) = \varphi_1(z_1) g\prime(z_2), \qquad (z_1, z_2) \in \mathbb{C}^2,$$

where $g\prime(\lambda) = \partial g(\lambda)/\partial\lambda$. Denote by $\hat{\varphi}(\xi, s)$ the Radon transform of φ. For every hyperplane $P_{\xi,s} = \{z : \langle z, \xi \rangle = s\}$ with $P_{\xi,s} \cap \operatorname{conv}(K) = \emptyset$ we have $\hat{\varphi}(\xi, s) = 0$ because $\operatorname{supp} \varphi \subset \operatorname{conv}(K)$. Let $P_{\xi,s}$ be a hyperplane such that $P_{\xi,s} \cap \operatorname{conv}(K) \neq \emptyset$ and $P_{\xi,s} \cap K = \emptyset$. Then ξ and s satisfy (8). We have

$$\begin{aligned}
\hat{\varphi}(\xi, s) &= \frac{1}{|\xi|^2} \int_{\mathbb{C}} \varphi_1\left(\frac{s\bar{\xi}_1}{|\xi|^2} + \mu\frac{\xi_2}{|\xi|}\right) g'\left(\frac{s\bar{\xi}_2}{|\xi|^2} - \mu\frac{\xi_1}{|\xi|}\right) d\omega_2(\mu) \\
&= \frac{1}{|\xi|^2} \int\limits_{\left|\frac{s\bar{\xi}_2}{|\xi|^2} - \mu\frac{\xi_1}{|\xi|}\right| \leq c\frac{1}{2}} \varphi_1\left(\frac{s\bar{\xi}_1}{|\xi|^2} + \mu\frac{\xi_2}{|\xi|}\right) g'\left(\frac{s\bar{\xi}_2}{|\xi|^2} - \mu\frac{\xi_1}{|\xi|}\right) d\omega_2(\mu). \qquad (9)
\end{aligned}$$

If $\left|\frac{s\bar{\xi}_2}{|\xi|^2} - \mu\frac{\xi_1}{|\xi|}\right| \leq \frac{1}{2}$, then it follows from (8) that $|\xi_1| > 0$ and

$$\begin{aligned}
\left|\frac{s\bar{\xi}_1}{|\xi|^2} + \mu\frac{\xi_2}{|\xi|}\right| &\leq \frac{|s\bar{\xi}_1|}{|\xi|^2} + |\mu|\frac{|\xi_2|}{|\xi|} \\
&\leq \frac{|s\bar{\xi}_1|}{|\xi|^2} + \frac{|s\bar{\xi}_2||\xi_2|}{|\xi|^2|\xi_1|} + \frac{|\xi_2|}{2|\xi_1|} = \frac{|s| + |\xi_2|/2}{|\xi_1|} \leq 1.
\end{aligned}$$

Since $\varphi_1(z_1) = 1$ for $|z_1| \leq 1$, the integral on the right-hand side of (9) equals

$$\frac{1}{|\xi|^2} \int_{\mathbb{C}} g'\left(\frac{s\bar{\xi}_2}{|\xi|^2} - \mu\frac{\xi_1}{|\xi|}\right) d\omega_2(\mu).$$

This integral equals zero because $\xi_1 \neq 0$. Thus, for every $(\xi, s) \notin \hat{K}$, we have $\hat{\varphi}(\xi, s) = 0$. However $\varphi(z) \not\equiv 0$ outside K because $\varphi(0) = 1$. $\square$

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